Clique-width of Hereditary Graph Classes

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Motivation

Many natural problems in algorithmic graph theory are NP-complete.

Want to find restricted classes of graphs where we can solve some problems in polynomial time.

Best if we can find classes where lots of problems can be solved in polynomial time.

Why Clique-width?

Theorem (Courcelle, Makowsky and Rotics 2000, Kobler and Rotics 2003, Rao 2007, Oum 2008)

Any problem expressible in "monadic second-order logic with quantification over vertices" (and certain other classes of problems) can be solved in polynomial time on any graph class of bounded clique-width.

This includes:

- **Vertex Colouring**
- Maximum Independent Set
- Minimum Dominating Set
- Hamilton Path/Cycle
- Partitioning into Perfect Graphs
- ...

Clique-width

The clique-width is the minimum number of labels needed to construct G by using the following four operations:

(i) creating a new graph consisting of a single vertex v with label i (represented by $i(v)$)

(ii) taking the disjoint union of two labelled graphs G_1 and G_2 (represented by $G_1 \oplus G_2$)

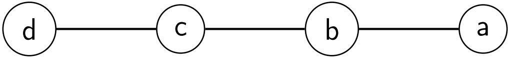
(iii) joining each vertex with label i to each vertex with label j ($i \neq j$) (represented by $\eta_{i,j}$)

(iv) renaming label i to j (represented by $\rho_{i \to j}$)

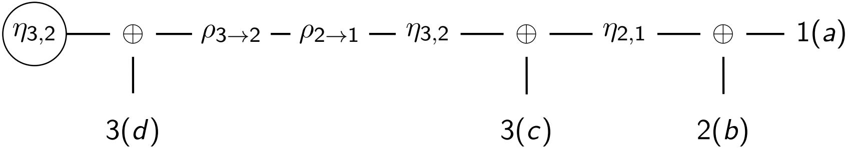
For example, P_4 has clique-width 3.

An expression for a graph can be represented by a rooted tree.

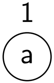
Clique-width



$$\eta_{3,2}(3(d) \oplus \rho_{3\to 2}(\rho_{2\to 1}(\eta_{3,2}(3(c) \oplus \eta_{2,1}(2(b) \oplus 1(a))))))$$



Clique-width



$1(a)$

$1(a)$

Clique-width

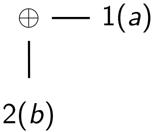
$2(b)$ $1(a)$

$1(a)$

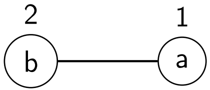
$2(b)$

Clique-width

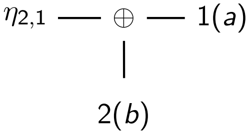
$$2(b) \oplus 1(a)$$



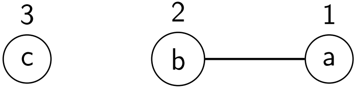
Clique-width



$$\eta_{2,1}(2(b) \oplus 1(a))$$



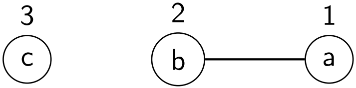
Clique-width



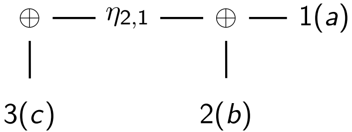
$3(c) \qquad \eta_{2,1}(2(b) \oplus 1(a))$

$$\begin{tikzcd}
 & \eta_{2,1} \arrow[r, dash] & \oplus \arrow[r, dash] \arrow[d, dash] & 1(a) \\
3(c) & & 2(b) &
\end{tikzcd}$$

Clique-width



$$3(c) \color{red}{\oplus} \color{black} \eta_{2,1}(2(b) \oplus 1(a))$$



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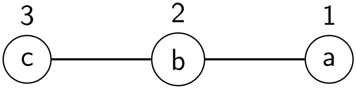
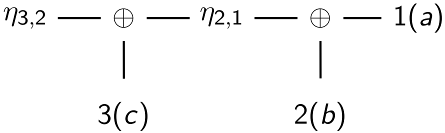


$$\textcolor{red}{\eta_{3,2}}(3(c) \oplus \eta_{2,1}(2(b) \oplus 1(a)))$$



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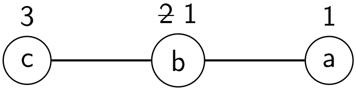
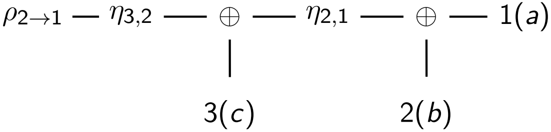
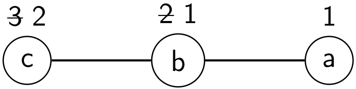


$$\color{red}{\rho_{2\to 1}}(\eta_{3,2}(3(c) \oplus \eta_{2,1}(2(b) \oplus 1(a))))$$



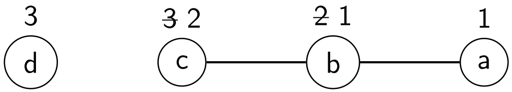
Clique-width



$$\textcolor{red}{\rho_{3\to 2}}(\rho_{2\to 1}(\eta_{3,2}(3(c) \oplus \eta_{2,1}(2(b) \oplus 1(a)))))$$

$$\begin{tikzcd}
\rho_{3\to 2} \arrow[r, dash] & \rho_{2\to 1} \arrow[r, dash] & \eta_{3,2} \arrow[r, dash] & \oplus \arrow[r, dash] \arrow[d, dash] & \eta_{2,1} \arrow[r, dash] & \oplus \arrow[r, dash] \arrow[d, dash] & 1(a) \\
 & & & 3(c) & & 2(b) &
\end{tikzcd}$$

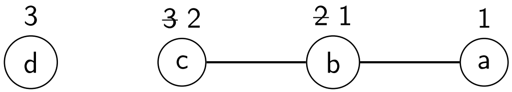
Clique-width



$\color{red}{3(d)}$ $\quad \rho_{3\to2}(\rho_{2\to1}(\eta_{3,2}(3(c) \oplus \eta_{2,1}(2(b) \oplus 1(a)))))$

$$\begin{array}{ccccccccccccc}
 & \rho_{3\to2} & - & \rho_{2\to1} & - & \eta_{3,2} & — & \oplus & — & \eta_{2,1} & — & \oplus & — & 1(a) \\
 & & & & & & & | & & & & | & & \\
3(d) & & & & & & & 3(c) & & & & 2(b) & &
\end{array}$$

Clique-width



$$3(d) \; {\color{red}\oplus} \; \rho_{3\to 2}(\rho_{2\to 1}(\eta_{3,2}(3(c) \oplus \eta_{2,1}(2(b) \oplus 1(a)))))$$

$$\begin{tikzcd}
\oplus \arrow[r, dash] \arrow[d, dash] & \rho_{3\to 2} \arrow[r, dash] & \rho_{2\to 1} \arrow[r, dash] & \eta_{3,2} \arrow[r, dash] & \oplus \arrow[r, dash] \arrow[d, dash] & \eta_{2,1} \arrow[r, dash] & \oplus \arrow[r, dash] \arrow[d, dash] & 1(a) \\
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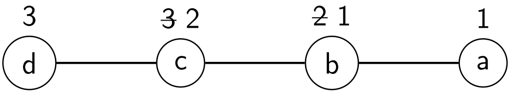
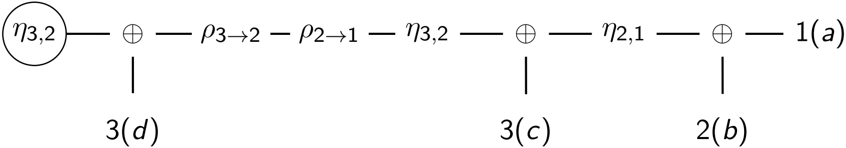
Clique-width



$$\color{red}{\eta_{3,2}}\color{black}(3(d) \oplus \rho_{3\to 2}(\rho_{2\to 1}(\eta_{3,2}(3(c) \oplus \eta_{2,1}(2(b) \oplus 1(a))))))$$



Calculating clique-width

Theorem (Fellows, Rosamond, Rotics, Szeider 2009)

Calculating clique-width is NP-hard.

Theorem (Corneil, Habib, Lanlignel, Reed, Rotics 2012)

Can detect graphs of clique-width at most 3 in polynomial time.

It's not known if this is also the case for graphs of clique-width 4.

Theorem (Oum 2008)

Can find a c-expression for a graph G where $c \leq 8^{\mathrm{cw}(G)} - 1$ in cubic time.

The clique-width of all graphs up to 10 vertices has been calculated (Heule & Szeider 2013).

Why clique-width?

- "Equivalent" to rank-width and NLC-width
- Generalises tree-width
- "Equivalent" to tree-width on graphs of bounded degree

The following operations don't change the clique-width by "too much"

- Complementation
- Bipartite complementation
- Vertex deletion
- Edge subdivision (for graphs of bounded-degree)

Need only look at graphs that are

- prime
- 2-connected

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Aim

Underlying Research Question

What kinds of graph properties ensure bounded clique-width?

By knowing what the bounded cases are, we may be able to reduce other classes down to known cases and get polynomial algorithms.

Hereditary Classes

A graph H is an induced subgraph of G if H can be obtained by deleting vertices of G, written $H \subseteq_i G$.

P_4

$3P_1$

$P_1 + P_2$

So $P_1 + P_2 \subseteq_i P_4$, but $3P_1 \not\subseteq_i P_4$.

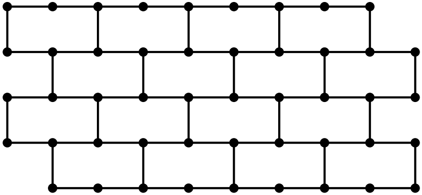
A class of graphs is hereditary if it is closed under taking induced subgraphs.

Let $\mathcal{H}$ be a set of graphs. The class of $\mathcal{H}$-free graphs is the set of graphs that do not contain any graph in $\mathcal{H}$ as an induced subgraph.

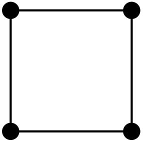
For example: bipartite graphs are the $(C_3, C_5, C_7, \ldots)$-free graphs

We will consider classes defined by finite set of forbidden induced subgraphs.

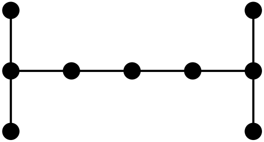
Graphs of large clique-width



Walls are bipartite and have unbounded clique-width, even if we subdivide each edge k times.



C_4



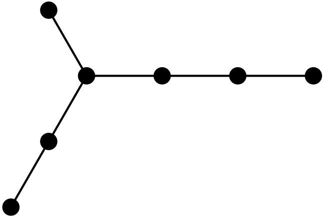
I_4

If H contains a C_k or I_k, then the k-subdivided walls are H-free.

Which classes have bounded clique-width?

If the class of H-free graphs has bounded clique-width then H must contain no cycles and no I_k.

Every component of H must be a subdivided claw, path or isolated vertex. The set of such graphs is called $\mathcal{S}$.



$S_{1,2,3}$

P_5

P_1

H-free graphs

A graph has clique-width at most 2 if and only if it is a P_4-free graph (Courcelle and Olariu. 2004).

Theorem (Dabrowski, P. 2014)

The class of H-free graphs has bounded clique-width if and only if $H \subseteq_i P_4$.

Colouring H-free graphs

Theorem (Král', Kratochvíl, Tuza & Woeginger, 2001)

The Vertex Colouring *problem is polynomial-time solvable for H-free graphs if and only if $H \subseteq_i P_1 + P_3$ or P_4, otherwise it is NP-complete.*

$P_1 + P_3$

P_4

Colouring (H_1, H_2)-free graphs

The Vertex Colouring problem is polynomial-time solvable for (H_1, H_2)-free graphs if

1. H_1 or H_2 is an induced subgraph of $P_1 + P_3$ or of P_4
2. $H_1 \subseteq_i K_{1,3}$, and $H_2 \subseteq_i C_3^{++}$, $H_2 \subseteq_i C_3^*$ or $H_2 \subseteq_i P_5$
3. $H_1 \neq K_{1,5}$ is a forest on at most six vertices or $H_1 = K_{1,3} + 3P_1$, and $H_2 \subseteq_i \overline{P_1 + P_3}$
4. $H_1 \subseteq_i sP_2$ or $H_1 \subseteq_i sP_1 + P_5$ for $s \geq 1$, and $H_2 = K_t$ for $t \geq 4$
5. $H_1 \subseteq_i sP_2$ or $H_1 \subseteq_i sP_1 + P_5$ for $s \geq 1$, and $H_2 \subseteq_i \overline{P_1 + P_3}$
6. $H_1 \subseteq_i P_1 + P_4$ or $H_1 \subseteq_i P_5$, and $H_2 \subseteq_i \overline{P_1 + P_4}$
7. $H_1 \subseteq_i P_1 + P_4$ or $H_1 \subseteq_i P_5$, and $H_2 \subseteq_i \overline{P_5}$
8. $H_1 \subseteq_i 2P_1 + P_2$, and $H_2 \subseteq_i \overline{2P_1 + P_3}$ or $H_2 \subseteq_i \overline{P_2 + P_3}$
9. $H_1 \subseteq_i \overline{2P_1 + P_2}$, and $H_2 \subseteq_i 2P_1 + P_3$ or $H_2 \subseteq_i P_2 + P_3$
10. $H_1 \subseteq_i sP_1 + P_2$ for $s \geq 0$ or $H_1 = P_5$, and $H_2 \subseteq_i \overline{tP_1 + P_2}$ for $t \geq 0$
11. $H_1 \subseteq_i 4P_1$ and $H_2 \subseteq_i \overline{2P_1 + P_3}$
12. $H_1 \subseteq_i P_5$, and $H_2 \subseteq_i C_4$ or $H_2 \subseteq_i \overline{2P_1 + P_3}$.

Colouring (H_1, H_2)-free graphs

The Vertex Colouring problem is polynomial-time solvable for (H_1, H_2)-free graphs if

1. H_1 or H_2 is an induced subgraph of $P_1 + P_3$ or of P_4
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3. $H_1 \neq K_{1,5}$ is a forest on at most six vertices $(H_1 \subseteq_i K_{1,3} + P_2, P_1 + S_{1,1,2}, P_6$ or $S_{1,1,3})$ or $H_1 = K_{1,3} + 3P_1$, and $H_2 \subseteq_i \overline{P_1 + P_3}$
4. $H_1 \subseteq_i sP_2$ or $H_1 \subseteq_i sP_1 + P_5$ for $s \geq 1$, and $H_2 = K_t$ for $t \geq 4$
5. $H_1 \subseteq_i sP_2$ or $H_1 \subseteq_i sP_1 + P_5$ for $s \geq 1$, and $H_2 \subseteq_i \overline{P_1 + P_3}$
6. $H_1 \subseteq_i P_1 + P_4$ or $H_1 \subseteq_i P_5$, and $H_2 \subseteq_i \overline{P_1 + P_4}$
7. $H_1 \subseteq_i P_1 + P_4$ or $H_1 \subseteq_i P_5$, and $H_2 \subseteq_i \overline{P_5}$
8. $H_1 \subseteq_i 2P_1 + P_2$, and $H_2 \subseteq_i \overline{2P_1 + P_3}$ or $H_2 \subseteq_i \overline{P_2 + P_3}$
9. $H_1 \subseteq_i \overline{2P_1 + P_2}$, and $H_2 \subseteq_i 2P_1 + P_3$ or $H_2 \subseteq_i P_2 + P_3$
10. $H_1 \subseteq_i sP_1 + P_2$ for $s \geq 0$ or $H_1 = P_5$, and $H_2 \subseteq_i \overline{tP_1 + P_2}$ for $t \geq 0$
11. $H_1 \subseteq_i 4P_1$ and $H_2 \subseteq_i \overline{2P_1 + P_3}$
12. $H_1 \subseteq_i P_5$, and $H_2 \subseteq_i C_4$ or $H_2 \subseteq_i \overline{2P_1 + P_3}$.

The class of (H_1, H_2)-free graphs has bounded clique-width if:

1. H_1 or $H_2 \subseteq_i P_4$;
2. $H_1 = sP_1$ and $H_2 = K_t$ for some s, t;
3. $H_1 \subseteq_i P_1 + P_3$ and $\overline{H_2} \subseteq_i K_{1,3} + 3P_1,\ K_{1,3} + P_2,\ P_1 + S_{1,1,2},$ P_6 or $S_{1,1,3}$;
4. $H_1 \subseteq_i 2P_1 + P_2$ and $\overline{H_2} \subseteq_i 2P_1 + P_3,\ 3P_1 + P_2$ or $P_2 + P_3$;
5. $H_1 \subseteq_i P_1 + P_4$ and $\overline{H_2} \subseteq_i P_1 + P_4$ or P_5;
6. $H_1 \subseteq_i 4P_1$ and $\overline{H_2} \subseteq_i 2P_1 + P_3$;
7. $H_1, \overline{H_2} \subseteq_i K_{1,3}$.

and it has unbounded clique-width if:

1. $H_1 \notin \mathcal{S}$ and $H_2 \notin \mathcal{S}$;
2. $\overline{H_1} \notin \mathcal{S}$ and $\overline{H_2} \notin \mathcal{S}$;
3. $H_1 \supseteq_i K_{1,3}$ or $2P_2$ and $\overline{H_2} \supseteq_i 4P_1$ or $2P_2$;
4. $H_1 \supseteq_i P_1 + P_4$ and $\overline{H_2} \supseteq_i P_2 + P_4$;
5. $H_1 \supseteq_i 2P_1 + P_2$ and $\overline{H_2} \supseteq_i K_{1,3},\ 5P_1,\ P_2 + P_4$ or P_6;
6. $H_1 \supseteq_i 3P_1$ and $\overline{H_2} \supseteq_i 2P_1 + 2P_2,\ 2P_1 + P_4,\ 4P_1 + P_2,\ 3P_2$ or $2P_3$;
7. $H_1 \supseteq_i 4P_1$ and $\overline{H_2} \supseteq_i P_1 + P_4$ or $3P_1 + P_2$.

Theorem (Dabrowski, P. 2015)

This leaves 13 cases where it is unknown if the clique-width of (H_1, H_2)*-free graphs is bounded or not (up to some equivalence relation).*

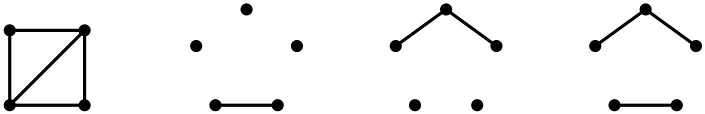
1. $H_1 = 3P_1, \overline{H_2} \in \{P_1 + P_2 + P_3, P_1 + 2P_2, P_1 + P_5, P_1 + S_{1,1,3}, P_2 + P_4, S_{1,2,2}, S_{1,2,3}\}$;
2. $H_1 = 2P_1 + P_2, \overline{H_2} \in \{P_1 + P_2 + P_3, P_1 + 2P_2, P_1 + P_5\}$;
3. $H_1 = P_1 + P_4, \overline{H_2} \in \{P_1 + 2P_2, P_2 + P_3\}$ *or*
4. $H_1 = \overline{H_2} = 2P_1 + P_3$.

There are 15 classes of (H_1, H_2)-free graphs for which both boundedness of clique-width and computational complexity of vertex colouring are open.

The Orange Row

Theorem (Dabrowski, Huang, P., 2015)

The class of (H_1, H_2)*-free graphs has bounded clique-width when* $H_1 = \overline{2P_1 + P_2}$ *and* $H_2 \subseteq_i 2P_1 + P_3,\ 3P_1 + P_2$ *or* $P_2 + P_3$*.*



$\overline{2P_1 + P_2}$ $\quad$ $3P_1 + P_2$ $\quad$ $2P_1 + P_3$ $\quad$ $P_2 + P_3$

Our Technique for the Orange Row

A graph G is perfect if for every induced subgraph H of G, the chromatic number of H is equal to the size of a maximum clique of H.

The clique covering number of a graph G is the smallest number of (mutually vertex-disjoint) cliques such that every vertex of G belongs to exactly one clique.

An alternative definition:

A graph G is perfect if for every induced subgraph H of G, the clique covering number of H is equal to the size of a maximum independent set of H.

Our Technique for the Orange Row

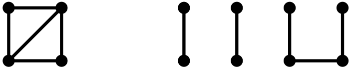
Theorem (Strong Perfect Graph Theorem
Chudnovsky, Robertson, Seymour, Thomas 2006)

Perfect graphs are precisely the $(C_5, \overline{C_5}, C_7, \overline{C_7}, C_9, \overline{C_9}, \ldots)$*-free graphs.*

Our technique can be summarised as follows:

1. Reduce the input graph to a graph that is in some subclass of perfect graphs.
2. While doing so, bound the clique covering number.

The Clique Covering Lemma



Lemma

Let G be a $(\overline{2P_1 + P_2}, 2P_2 + P_4)$-free graph. If the vertex set of G can be partitioned into at most k cliques, then the clique-width of G is bounded by a function depending only on k.

Sketch of proof:

- Divide G into k cliques (k minimal) $X_1, \ldots, X_k$.
- May assume every clique is big (at least $k + 7$ vertices).
- If $x \in X_i$ has two neighbours in X_j then it is complete to X_j.
- If there are two such vertices then X_i is complete to X_j, contradicting minimality of k.
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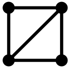
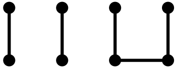
The Clique Covering Lemma



$\overline{2P_1 + P_2}$



$2P_2 + P_4$

Lemma

Let G be a $(\overline{2P_1 + P_2}, 2P_2 + P_4)$-free graph. If the vertex set of G can be partitioned into at most k cliques, then the clique-width of G is bounded by a function depending only on k.

Sketch of proof:

- Divide G into k cliques (k minimal) $X_1, \ldots, X_k$.
- May assume every clique is big (at least $k + 7$ vertices).
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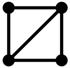

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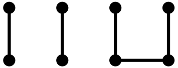

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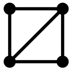

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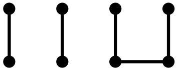

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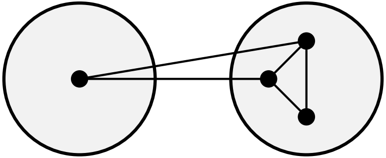

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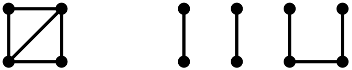

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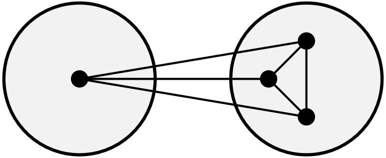

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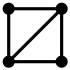

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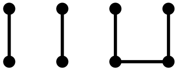

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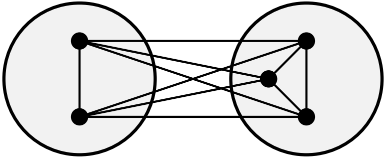

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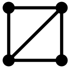

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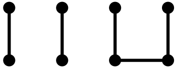

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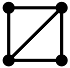

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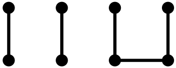

$2P_2 + P_4$

- Every vertex in a clique X_i has at most one neighbour in a different clique X_j and every clique has at least 8 vertices.
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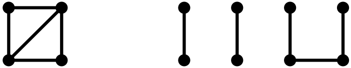

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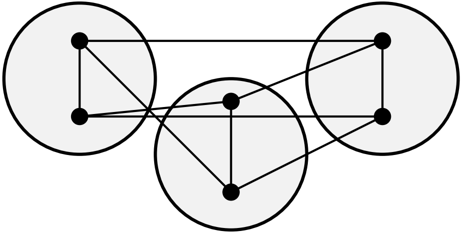

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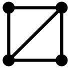 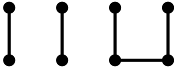

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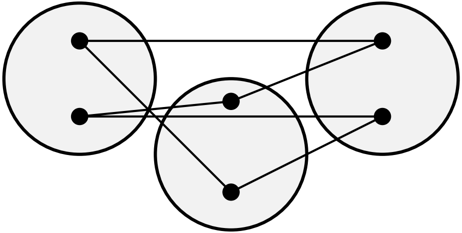

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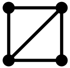

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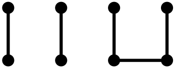

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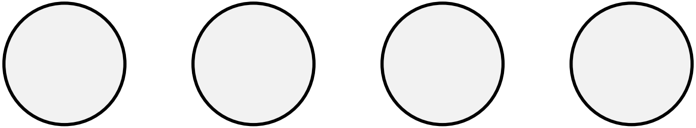

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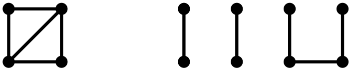

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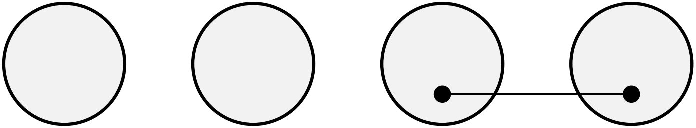

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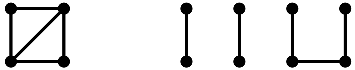

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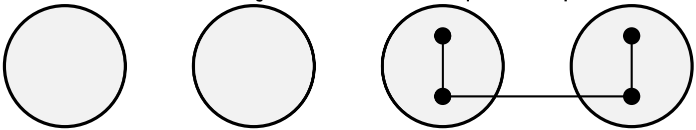

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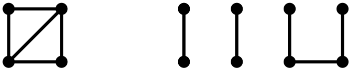

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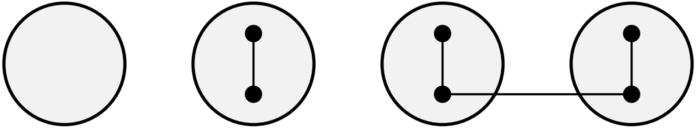

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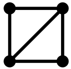

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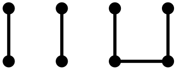

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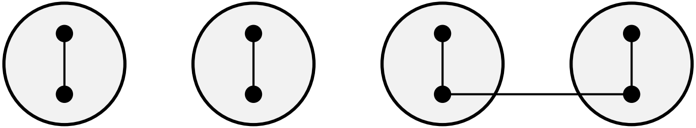

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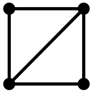
Example



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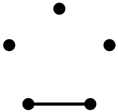


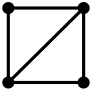
$3P_1 + P_2$

Lemma (Dabrowksi, Lozin, Raman, Ries, 2012)
The class of $(K_3, K_{1,3} + P_2)$*-free graphs has bounded clique-width.*

Lemma (Dabrowski, Golovach, P. 2014)
Let $s \geq 0$ *and* $t \geq 0$*. Then every* $(\overline{sP_1 + P_2}, tP_1 + P_2)$*-free graph is* $(K_{s+1}, tP_1 + P_2)$*-free or* $(\overline{sP_1 + P_2}, (s^2(t-1)+2)P_1)$*-free.*

Consequences: If G is a $(\overline{2P_1 + P_2}, 3P_1 + P_2)$-free graph then it is K_3-free (in which case we use the first lemma) or $10P_1$-free.

Example



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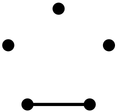

$3P_1 + P_2$

G is a $(\overline{2P_1 + P_2}, 3P_1 + P_2)$-free graph that is $10P_1$-free.

- If G contains an induced C_5 or C_7, partition other vertices according to their neighbourhood in the cycle.
- Each such set is a clique and there are a bounded number of them. Apply Clique Covering Lemma.
- Since G is $3P_1 + P_2$-free, it contains no odd cycles of length 9 or more.
- Since G is $\overline{2P_1 + P_2}$-free it contains no complements of cycles of length 7 or more.
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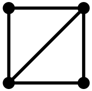

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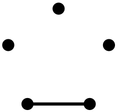

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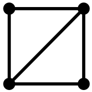

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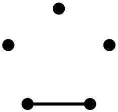

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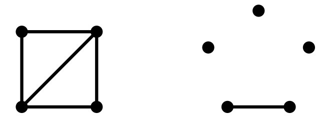

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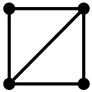
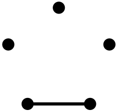
Example



$\overline{2P_1 + P_2}$

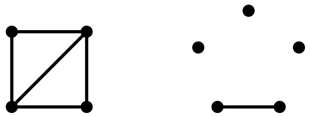


$3P_1 + P_2$

G is a $(\overline{2P_1 + P_2}, 3P_1 + P_2)$-free graph that is $10P_1$-free.

- If G contains an induced C_5 or C_7, partition other vertices according to their neighbourhood in the cycle.
- Each such set is a clique and there are a bounded number of them. Apply Clique Covering Lemma.
- Since G is $3P_1 + P_2$-free, it contains no odd cycles of length 9 or more.
- Since G is $\overline{2P_1 + P_2}$-free it contains no complements of cycles of length 7 or more.
- So G is perfect and $10P_1$-free, so it can be partitioned into at most 9 cliques. Apply Clique Covering Lemma. $\square$

Example



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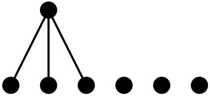
- So G is perfect and $10P_1$-free, so it can be partitioned into at most 9 cliques. Apply Clique Covering Lemma. $\square$

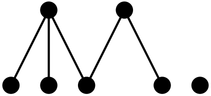
H-Free Bipartite Graphs

Theorem (Dabrowski, P. 2014)

The class of *H-free bipartite* *graphs has bounded clique-width if and only if H is an induced subgraph one of:*

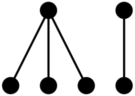


$K_{1,3} + 3P_1$

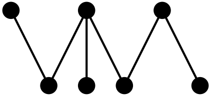


$P_1 + S_{1,1,3}$

sP_1 for some s ($s = 5$ shown)



$K_{1,3} + P_2$



$S_{1,2,3}$

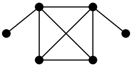
H-Free Weakly Chordal Graphs

A graph G is weakly chordal if both G and $\overline{G}$ are $(C_5, C_6, \ldots)$-free.

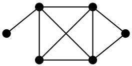
Theorem (Brandstädt, Dabrowski, Huang, P. 2015+)
Let H be a graph. Then the class of H-free weakly chordal graphs has bounded clique-width if and only if $H \subseteq_i P_4$.

A graph G is chordal if G is $(C_4, C_5, \ldots)$-free.

H-Free Chordal Graphs



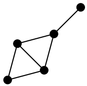
F_1



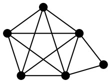
F_2

Theorem (Brandstädt, Dabrowski, Huang, P. 2015)

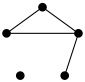
Let H be a graph with $H \notin \{F_1, F_2\}$. The class of H-free chordal graphs has bounded clique-width if and only if H is a an induced subgraph of one of:



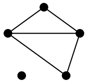
$\overline{S_{1,1,2}}$



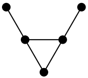
$\overline{K_{1,3} + 2P_1}$



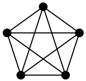
$P_1 + \overline{P_1 + P_3}$



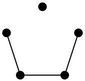
$P_1 + \overline{2P_1 + P_2}$



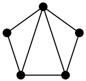
bull



K_r *for some* $r \geq 1$

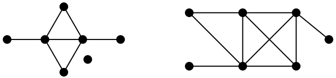


$P_1 + P_4$



$\overline{P_1 + P_4}$

H-Free Split Graphs

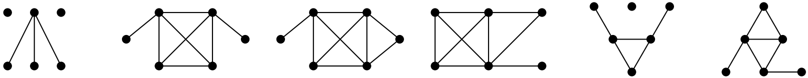


F_4 F_5

Theorem (Brandstädt, Dabrowski, Huang, P. 2015)

Let H be a graph such that neither H nor $\overline{H}$ is in $\{F_4, F_5\}$. The class of H-free split graphs has bounded clique-width if and only if H or $\overline{H}$ is

- *isomorphic to rP_1 for some $r \geq 1$ or*
- *an induced subgraph of one of:*



$K_{1,3} + 2P_1$ F_1 F_2 F_3 $\text{bull} + P_1$ Q

Work in Progress

For which graphs H does the class of H-free chordal graphs have bounded clique-width? (2 open cases)

For which graphs H does the class of H-free split graphs have bounded clique-width? (2 open cases)

For which pairs of graphs (H_1, H_2) does the class of (H_1, H_2)-free graphs have bounded clique-width. (13 open cases)

References

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