On the Epistemological Analysis of Modeling and Computational Error in the Mathematical Sciences

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But in real (not "**in principle**") science, things don't work that way. (And this is what philosophically matters.)

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In simple cases, we have this possibility:

$$\boxed{\text{Problem}} \xrightarrow{\text{solve}} \boxed{\text{Expression}} \xrightarrow{\text{evaluate}} \boxed{\text{Number(s)}}$$

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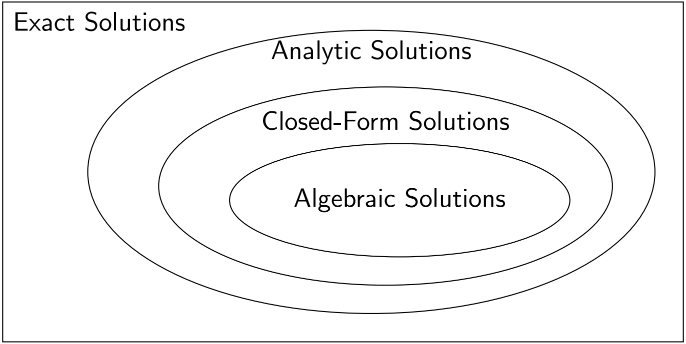
But what can be done depends on the type of solution we have:



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Because of that, in real science, the **connection between theory and evidence is rarely logical or probabilitistic**, despite what philosophers will tell you.

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We'll see how it works, and that it differs from logic & probability.

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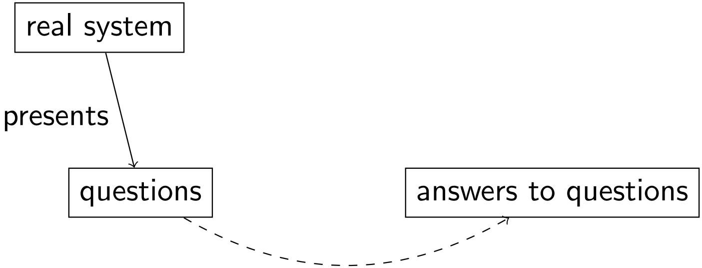

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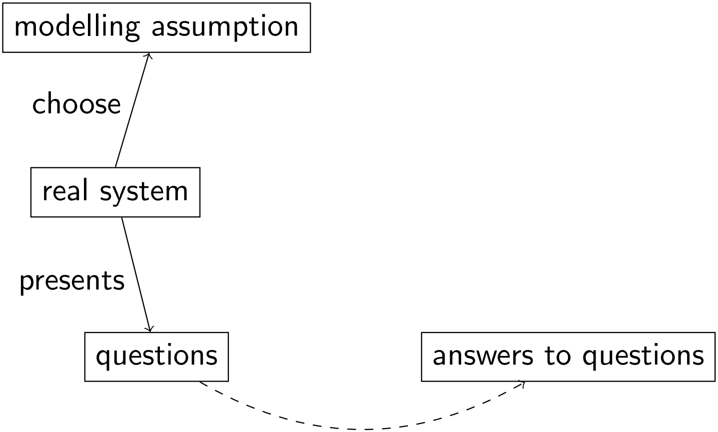

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(When not in mechanics, we use other parameters that are consitutive of the dynamical system, *e.g.*, fitness in evolutionary biology.)

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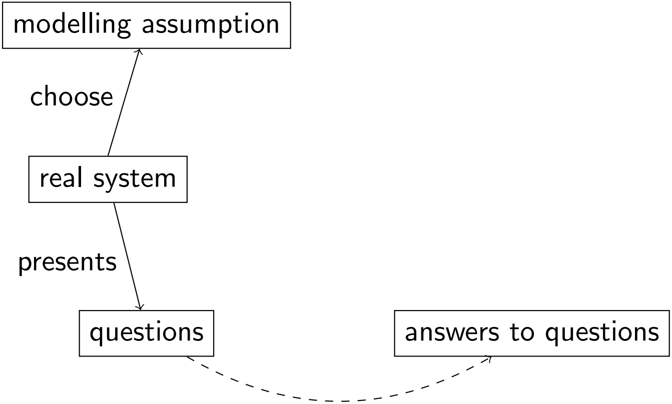

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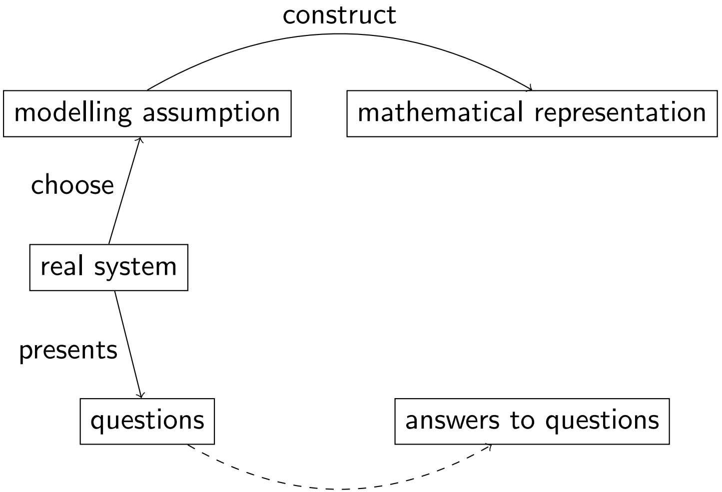

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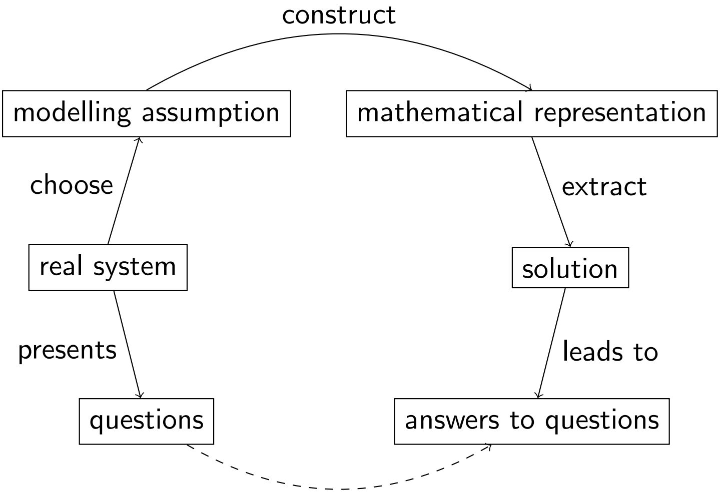

A classical particular case of this is Euler's recipe:

1. Delineate the bodies whose behaviour one wishes to study.
2. Determine what specific forces act between these bodies.
3. Choose Cartesian coordinates and decompose each of the forces along the axes of this coordinate system.
4. Sum up force components along the axes.
5. Set this sum of forces equal to $m\frac{\mathrm{d}^2\mathbf{x}}{\mathrm{d}t^2}$ (Newton's Second Law). It results in a differential equation of motion in the state space ($\ddot{\mathbf{x}} = f(\dot{\mathbf{x}}, \mathbf{x}, t)$).
6. Solve for $\mathbf{x}(t)$ somehow.

So, there are additional types of errors to consider:

1. Systemic Error
2. Experimental Error

3. **Discretization** Error
4. **Rounding** Error

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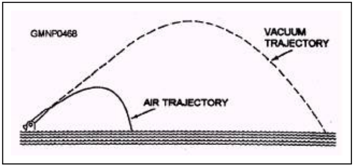

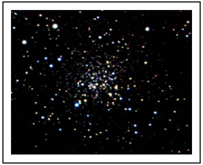

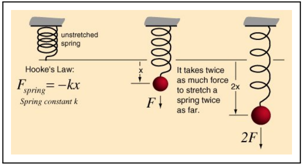

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fcts, integrals, etc. $\xrightarrow{\text{truncate}}$ truncated asymptotic series

$f(x)$, $\int g(x)$, etc. $\qquad y(x,\varepsilon) = \sum_{k=0}^{N} y_k(x)\phi_k(\varepsilon)$

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flow
$\dot{\mathbf{x}} = \mathbf{f}(t, \mathbf{x}(t); \mu)$

$\xrightarrow{\text{discretization}}$

discrete functions (maps)
$x_{k+1} = \Phi(t_k, x_k, \ldots, x_0, h, \mathbf{f})$

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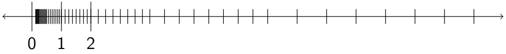
Operations on $\mathbb{R}, \mathbb{C}$ $\xrightarrow{\text{rounding}}$ floating-point arithmetic $\mathbb{F}$



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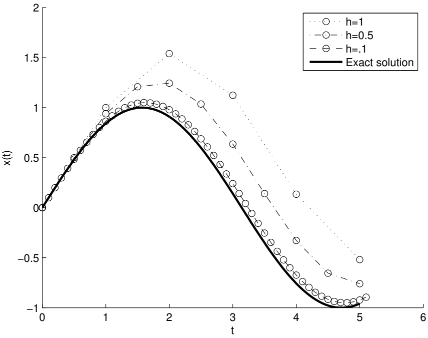
Sources of error in mathematical modelling

Example. Take a differential equation $\frac{dx}{dt} = f(x,t)$. Write it's solution $x(t)$ as an asymptotic power series about t_0:

$$x(t) = x(t_0) + x'(t_0)(t - t_0) + \frac{x''(t_0)}{2}(t - t_0)^2 + \frac{x'''(t_0)}{6}(t - t_0)^3 + \cdots$$



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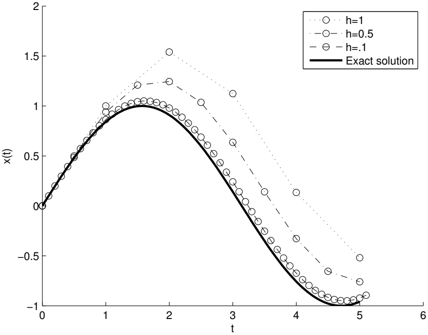

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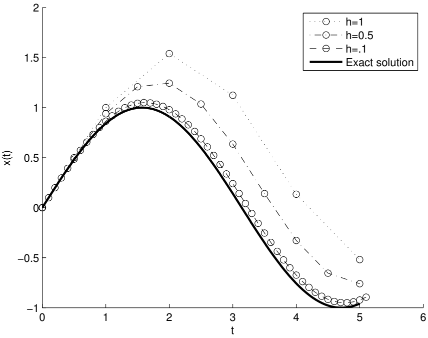

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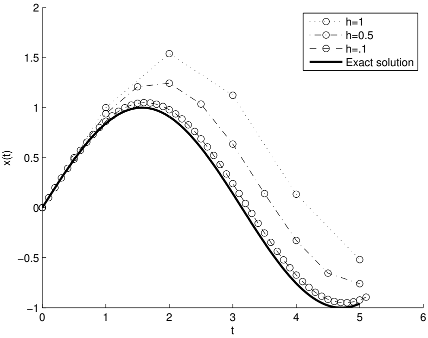

Truncate after **first order terms** and use this as a "marching method" through the vector field."

Then, should we just pray it's good enough? No, of course!

Here, the **crucial epistemological question** is:

When we don't know the exact solution of a model, how do we determine if our "approximate" solution is indeed approximately true (or accurate)?

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Even if it might seem counter-intuitive, **it is generally easier to determine whether we're close to the truth than to know what the truth is**!

The framework of **backward error analysis** explains this insight.

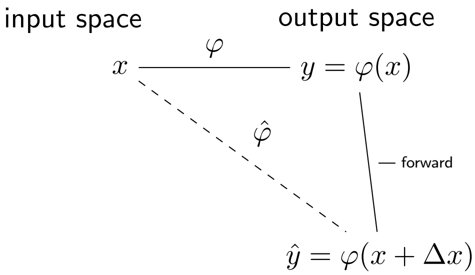

Figure: Backward error analysis: The general picture.

We write $\hat{y} = \hat{\varphi}(x)$. Instead of saying that $\hat{y}$ is the **approximate solution to φ**, we say that it is the **exact solution to $\hat{\varphi}$**.

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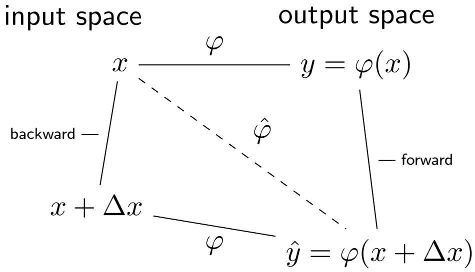

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The central question is now:

⇒ *When we modified the reference problem φ to get the engineered problem $\hat{\varphi}$, for what set of data have we actually solved the problem φ (exactly)?*

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If solving the problem $\hat{\varphi}(x)$ **amounts to having solved the problem $\varphi(x + \Delta x)$ for a Δx smaller than the perturbations inherent in the modeling context**, then our solution $\hat{y}$ must be considered **completely satisfactory**.

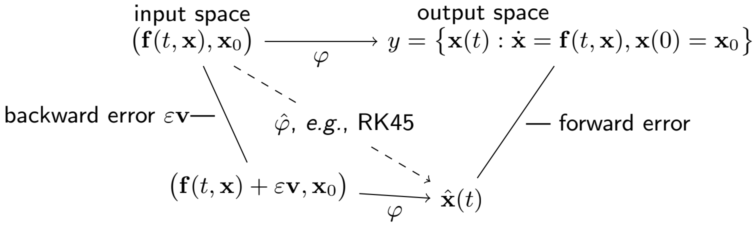

Figure: Commutative diagram for the backward error analysis of initial value problems. Note that we can also perturb $\mathbf{x}_0$, or both $\mathbf{x}_0$ and $\mathbf{f}$. In some cases, this diagram will be implicitly replaced by an "almost commutative diagram."

If we consider perturbations of the functional $\mathbf{f}$, the backward error allows us to find **to which perturbed vector field our computed solution is tangent**!

$$\Delta(t) = \dot{\hat{\mathbf{x}}}(t) - \mathbf{f}(t, \hat{\mathbf{x}}(t)) \quad \Rightarrow \quad \dot{\hat{\mathbf{x}}}(t) = \mathbf{f}(t, \hat{\mathbf{x}}(t)) + \Delta(t)$$

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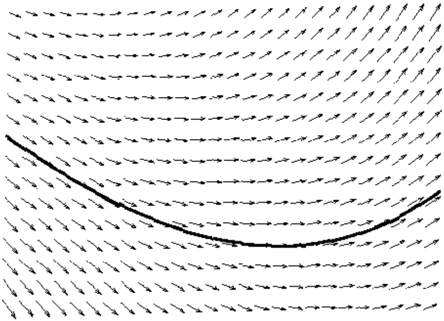

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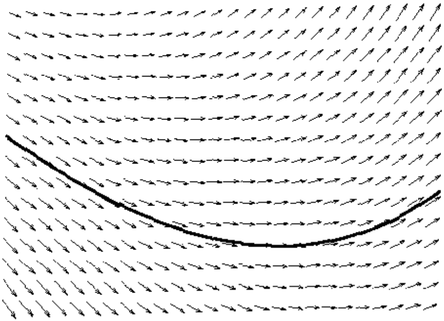

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Assessing computational error is thereby reduced to assessing modelling error.

So, what should we do about modelling error?

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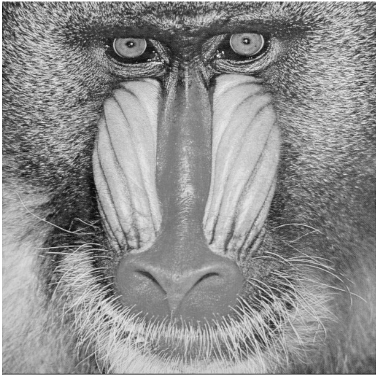
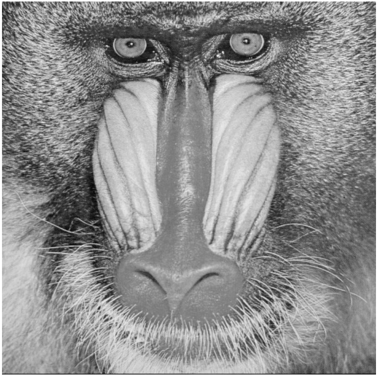
Consider the bitmap image of a mandrill/baboon face.

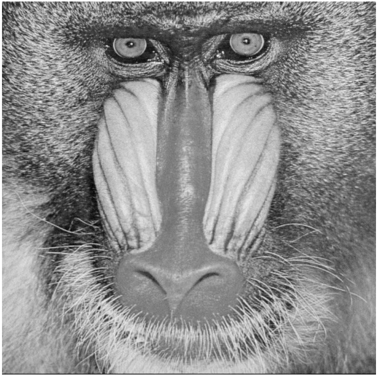
rank=506, stored with 524800 numbers, use 200% of space



We need about $250,000$ numbers to encode this image.

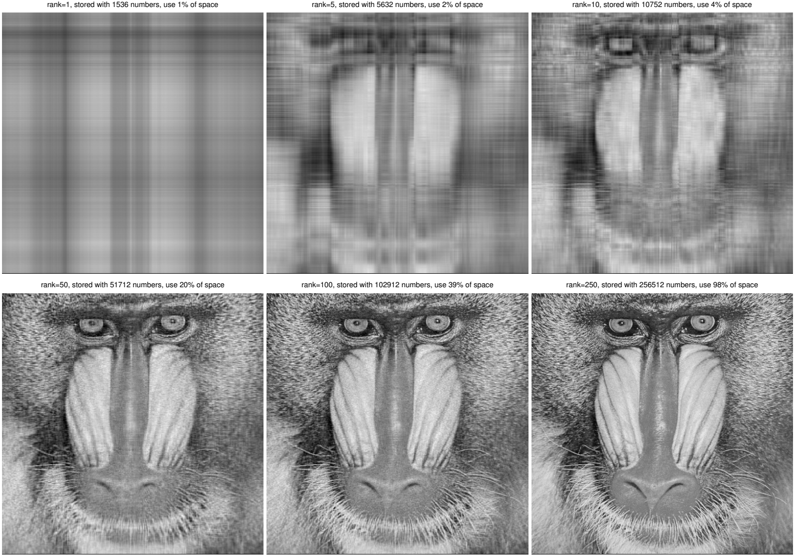
Now, let's throw away 50% of this information:





We barely see a difference!

Assessing the Impact of Modelling Errors



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It is about identifying the most important aspects of a representation, handling those carefully, while throwing away the rest of the information.

The criteria for assessing the consequences of errors are thus the following:

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The general method to determine this is **perturbation analysis**.

Perturbation analysis examines the **effects of small changes** of an aspect of a representation.

Some systems are sensitive to changes in some aspects, others are **robust under perturbation**.

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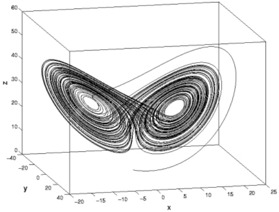

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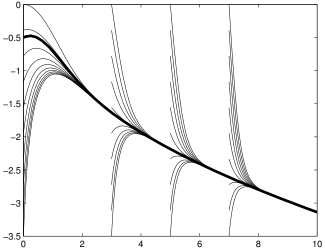

The idea of perturbation analysis

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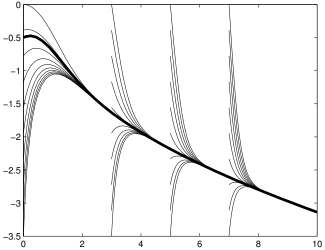

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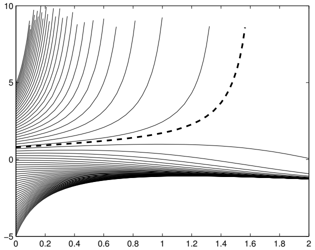
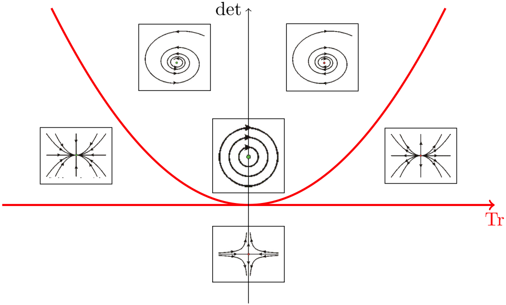


near $x(0) = \sqrt[3]{1/2}$ (which is the dotted line).

Classification of all 2D linear diff. equations $d\mathbf{x}/dt = \mathbf{A}\mathbf{x}$ by two parameters.



Knowing critical and bifurcation points is typically easier than working with a "perfectly accurate and complete model," so there is an **epistemic gain**.

A few concluding remarks...

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But the successes of science are to be explained in terms of perturbations, sensitivity, and robustness.

Thus, **we need to add the methods of perturbation analysis to our rational reconstruction toolbox**. What does that mean? What's the significance?

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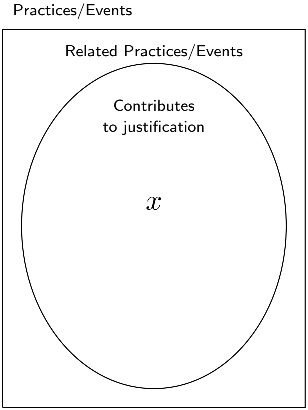

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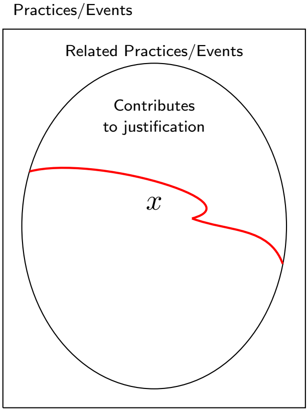
We'll draw the distinction differently depending on **what tools for reconstruction we admit**. Determining where to draw the line is a delicate matter, subject to philosophical argument.



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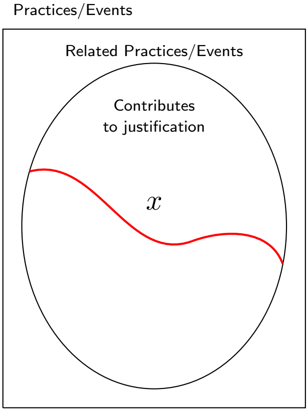

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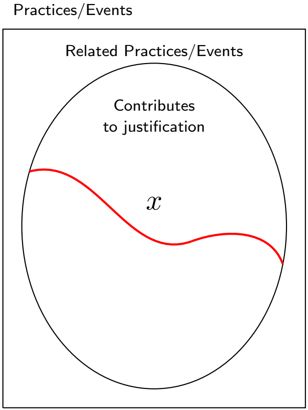

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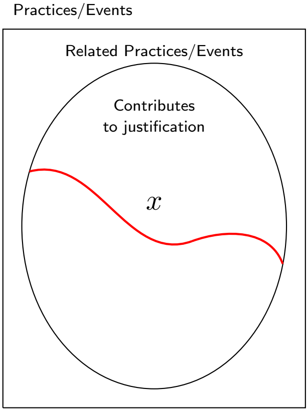
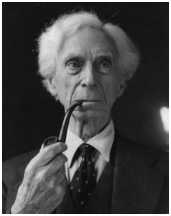


If we are too restrictive, **successful practices** might appear irrationally so, or even **miraculous**. My point is that we need to draw the line between the contexts of discovery and justification in a **more inclusive way**.

The ideal image of science that results from this work is in line with that of the late Russell:

"Although this may seem a paradox, all exact science is dominated by the idea of approximation."



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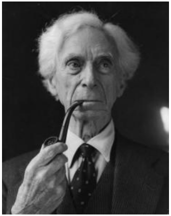

"Although this may seem a paradox, all exact science is dominated by the idea of approximation."

Accordingly, **the point of the epistemology of sciences is not to try to understand how science would be without errors and uncertainty, but rather the point is to understand how we can live with them**.

Thank you!