An Aronsson type approach to extremal quasiconformal mappings

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Conformal deformations

Notations Let $\Omega \subset \mathbb{R}^n$ and

$$u = (u^1, ..., u^n) : \Omega \to \mathbb{R}^n$$

a $W^{1,n}_{loc}$ orientation-preserving homeomorphism.
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The map u is conformal if there exists a function λ such that at every point of differentiability

$$du^T du = \lambda Id \tag{1}$$

Equivalently, a.e. in Ω, one must have

$$g := \frac{du^T du}{(\det du)^{\frac{2}{n}}} = Id. \tag{2}$$

The function $\sqrt{trace(g)} = |du|^2/(det\ du)^{1/n}$ is called dilation of u.
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In a (rough) comparison with similar problems in elasticity, conformal deformations correspond to isometries. Accordingly, the variational problems we study correspond to finding deformations "closest to isometries" in given classes of competitors.

We call

$$S(g) = g - \frac{trace(g)}{n} Id$$

the distortion tensor of u at x and

$$\mathbb{K}(u, \Omega) = ||\sqrt{trace(g)}||_{L^\infty(\Omega)}$$

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Hence u is conformal if and only if $S(g) = 0$
if and only if $trace(g) = n$ a.e. in Ω.

Denote by $CO_+(n)$ the space of differentials of orientation preserving conformal mappings, then its tangent space $TCO_+(n)$ at the identity is

$$\left\{ A \in \mathbb{R}^{n\times n} s.t.\ S(A) = \frac{A + A^T}{2} - \frac{trace(A)}{n} Id = 0 \right\}.$$

Accordingly we have that the distance of a matrix A from $CO_+(n)$ satisfies

$$d^2(A, CO_+(n)) = c|S(A - I)|^2 + O(|A - I|^4).$$

This shows that the operator S arises naturally when considering distance of a deformation from being conformal.

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- Riemann Mapping Theorem Allows to map any simply connected open planar set conformally to the disc (uniquely if one prescribes target for three points).

Rigidity of conformal deformations

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  Example In mapping one rectangular box into another, sending sides to sides, one can achieve this through a conformal deformation only if the boxes are similar.

Grotsch Problem Among all deformations from one rectangle into another, sending sides to sides, is there one closest to conformal?

Such considerations hint at the need of introducing a more general class of deformations that are less rigid, yet retain some of the useful features of conformal mappings. One also would like to have an instrument to quantify how far a given deformation is from being conformal.

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Definition A $W^{1,n}_{loc}$ homeomorphism is Quasiconformal if

$$\mathbb{K}(u,\Omega) = ||\sqrt{trace(g)}||_{L^\infty} < \infty.$$

The variational problem

Since $K(u, \Omega) = \sqrt{n}$ if and only if u is conformal, we will interpret "closest to conformal" as "minimizing $\mathbb{K}(u, \Omega) = ||\sqrt{trace(g)}||_{L^\infty}$ among all competitors".

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Teichmüller theory has as a starting point this L^∞ variational problem for holomorphisms between Rieman surfaces of same genus ($g > 3$) and where the constraint is given by membership in the same homotopy class. (Teichmueller, Alfhors, Hamilton, Bers, ...). In this setting one can prove existence, uniqueness and some amount of regularity for the minimizers.

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For $n>2$ very little is known. A great amount of recent literature focuses on the L^p problem (Astala, Iwaniec, Onninen, Martin,...).

Difficulties:

- L^∞ functionals are not sensitive to deformations of functions away from their maximum. Unlike L^p averages they are not "local". This makes uniqueness unlikley.
- The problem is vector-valued, and as such not approachable through game theory, viscosity solutions, ...
- There is a topological constraint.

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A minimizing Lipschitz extension is an extension of a Lipschitz scalar function $f : \partial\Omega \to \mathbb{R}$ to $u : \Omega \to \mathbb{R}$ with $u = f$ on $\partial\Omega$ and

$$Lip(u, \Omega) = \sup_{x \neq y, x, y \in \Omega} \frac{|u(x) - u(y)|}{|x - y|} = \mathrm{Lip}(f, \partial\Omega)$$

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Question Is there a special class of extremals for which uniqueness holds?

In the 60's Aronsson proposed a way to "localize" the functional by introducing the formal approximation scheme:

- Consider minimzers u_p of $\int |\nabla u|^p$. They are $p-$harmonic, i.e. $\Delta_p u_p = div(|\nabla u_p|^{p-2} \nabla u_p) = 0$.
- Let $p \to \infty$ and obtain the $\infty-$Laplacian $\Delta_\infty u = u_{ij} u_i u_j = 0$.

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Aronsson proved that C^2 solutions of $\Delta_\infty u = 0$ are Absolute Minimizing Lipschitz Extensions (AMLE), i.e. they minimize $Lip(u, D)$ on EVERY subdomain $D \subset \Omega$. This "locality' is inherited from the L^p problem.

Aronsson also proved that C^2 AMLE are $\infty-$harmonic.

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Transferring Aronsson's approach to the vector valued setting is very challenging: only recently (2010) Sheffield-Smart established an equivalence of solutions to a certain system of PDE with an analogue of the AMLE property for C^2 deformations $u : \Omega \rightarrow \mathbb{R}^m$.

In a joint work with A. Raich (UARK) we used Aronsson's approximation scheme and introduce a notion of absolute minimizers in the quasiconformal setting.

The Euler-Lagrange system for the functional

$$\int \frac{|du|^{np}}{(detdu)^p} dx$$

is

$$(L_p u)^i = np\partial_j \left(|g|^{\frac{np-2}{2}} du^{-1,T} S(g) \right)_{ij}, \text{ for } i = 1, ..., n$$

If we let $p \to \infty$ then formally one obtains

$$(L_\infty u)^i = S(g)_{ij}\partial_j \sqrt{trace(g)} = 0 \text{ for } i = 1, ..., n$$

This PDE tells us that the dilation of the deformation u (i.e. $\sqrt{trace(g)}$) is constant along integral curves of the rows of $S(g)$.

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(iii) There exists $C = C(n) > 0$ such that for every $D \subset \Omega$ and $w : \bar{D} \to \mathbb{R}^n$ C^2 quasiconformal such that $u = w$ on $\partial\, D$ one has $\sup_D \sqrt{trace(g)(u)} \leq C \sup_D \sqrt{trace(g(w))}|$.

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(iv) If $n=2$ the dilation $|g|$ is constant in Ω and if u is affine on $\partial\Omega$ then u must be an affine transformation throughout Ω.

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(iv) If $n = 2$ the dilation $|g|$ is constant in Ω and if u is affine on $\partial\Omega$ then u must be an affine transformation throughout Ω.

Idea of the proof Show that maximum points for $|g|$ propagate along integral curves of rows of $S(g)$ until they reach the boundary. This is achieved by glueing together several of such curves and showing that their total length must be finite.

Theorem B:

If $u : \Omega \to \mathbb{R}^n$ C^2 is a quasiconformal absolute extremal, i.e. for every $D \subset \Omega$ and $w : \bar{D} \to \mathbb{R}^n$ C^2 quasiconformal such that $u = w$ on $\partial\ D$ one has $\sup_D \sqrt{trace(g(w))} \leq \sup_D \sqrt{trace(g(w))}$, then $L_\infty u = 0$ in Ω.

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Idea of the proof Assume there is a ball $B \subset\subset \Omega$ s.t. $L_\infty u \neq 0$ in B. We construct a C^2 quasiconformal diffeomorphism $V : \bar{B} \to \mathbb{R}^n$ with same boundary values as u on ∂B and $\sup_B trace\ g(V) < \sup_B trace\ g(u)$. This is done by perturbing u with a finite number of "bumps" that reduce the dilation near the boundary.

A gradient flow approach

The gradient flow of the quasiconvex functional

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is the system of non-linear parabolic PDE

$$\partial_t u = -L_p u \text{ in } \Omega \times (0, T).$$

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Goal: Global existence of solutions would yield a continuous deformation of an initial quasiconformal mapping into a "closest to conformal" transformation (or at least to a local minimum).

Theorem C

If $u(x,0) \in C^{2,\alpha}$+boundary conditions then there exists a unique $C_1^{2,\alpha}(\Omega \times (0,T), \mathbb{R}^n)$ solutions for small $T = T(p, n, u_0, \Omega) > 0$.

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Moreover, following work of Evans-Gangbo-Savin, if we let $\beta = det\ du^{-1}$ then we show that β solves the scalar PDE

$$\partial_t \beta = [a_{ij}(du)\beta]_{ij}$$

with

$$a_{ij} = p\left(\delta_{ij} - n\frac{du_{jk}du_{ik}}{|du|^2}\right)\sqrt{|g|}^{np}.$$

The lack of a sign in the symbol prevents us from using the maximum principle and establishing global bounds.