Nilpotent orbits of a reductive group over a local field

(Raleigh special session)

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Reductive groups: basic examples

G: connected, reductive group over K. Examples:

- $G = \mathrm{GL}(V)$ or $\mathrm{SL}(V)$ for a K-vector space V
- $G = \mathrm{Sp}(V)$ where V has a non degenerate alternating form

I want to suppose G is "D-standard."

- if G is semisimple, D-standard just means the char is "very good" for G.
- Any K-form of GL_n is D-standard; a form of SL_n is D-standard $\iff$ n is invertible in K
- $\mathrm{Sp}(V)$ is D-standard just when $p \neq 2$.

Nilpotent orbits

- Let G be D-standard and $X \in \mathfrak{g}(K)$ nilpotent.
- Recall G-orbits in nilp variety $\mathcal{N}$ are classified *geometrically* by Bala-Carter data ...
- ...in particular, the *geometric* nilpotent orbits depend only on the root datum of G.
- More complicated: $G(K)$-orbits in $\mathcal{N}(K)$.

Nilpotent centralizers

- if char. K is 0, $\mathfrak{sl}_2$-triples containing X are a useful tool; unavailable in general.
- For a general D-standard group, one replaces the ss elt H of a triple by a suitable cocharacter $\phi : \mathbf{G}_m \to G$ "associated with X".
- if $X^{[p]} = 0$, ϕ determs an "optimal" maps $\psi : \mathrm{SL}_2 \to G$ for which X is in the image of $d\psi$, and $\psi \begin{pmatrix} t & 0 \\ 0 & t^{-1} \end{pmatrix} = \phi(t)$.
- following Premet, one knows such a cocharacter to exist by using geometric invariant theory result of Kempf-Rousseau (since nilpotent elements are precisely the unstable vectors in the adjoint representation)
- If ϕ is a cocharacter associated with X, one knows that $M = C \cap C_G(\operatorname{im} \phi)$ is a Levi factor of C (over K).

Structure of a nilpotent centralizer

Theorem (M)

The centralizer $C = C_G(X)$ has a Levi decomposition defined over K. Moreover, the following are independent of p (under our standard hyp):

- *the (geometric) root datum of a Levi factor of C,*
- *the (geometric) component group $C(K_{\mathrm{alg}})/C^0(K_{\mathrm{alg}})$*

- Method of proof: may suppose $K = K_{\mathrm{alg}}$.
  - let $\mathcal{A}$ be a DVR with residues K and fractions of char 0.
  - let $\mathcal{G}_{/\mathcal{A}}$ be split reductive with root datum of G
  - find a nilpotent section $X_1 \in \mathfrak{g}(\mathcal{A})$ specializing to X for which the $\mathcal{A}$ group scheme $C_{\mathcal{G}}(X_1)$ is *smooth* over $\mathcal{A}$.
  - Find the Levi factor "over $\mathcal{A}$".

“Local” fields – notation etc.

- Let $\mathcal{A}$ be a discrete valuation ring with maximal ideal $\pi\mathcal{A}$
- assume $\mathcal{A}$ is π-adically complete,
- assume $k = \mathcal{A}/\pi\mathcal{A}$ is *perfect*,
- and let $K = \mathrm{Frac}(\mathcal{A})$.
- Examples:
  - $K = k((\pi))$, $\mathcal{A} = k[\![\pi]\!]$.
  - K a finite extension of $\mathbf{Q}_p$, $\mathcal{A}$ int. closure of $\mathbf{Z}_p$ in K
- We write p for the characteristic of the residue field k.

Parahoric subgroups

Let G be D-standard reductive over the "local" field K.

- It is useful to consider smooth $\mathcal{A}$-group schemes $\mathcal{P}$ with $\mathcal{P}_{/K} = G$.
- e.g. Bruhat and Tits have attached "parahoric subgroups" to G, which are of this form
- parahorics are determ by *facets* in the *affine building* of G.
- there is an $\mathcal{A}$-split torus S in $\mathcal{P}$ for which $S_{/K}$ is a maximal K-split torus in G.

Theorem (Bruhat-Tits, I think)

The special fiber $\mathcal{P}_{/k}$ of a parahoric $\mathcal{P}$ has a unique Levi factor containing $S_{/k}$.

- the case of SU_3 with L/K totally ramified shows this not to be *too* trivial.

Nilpotent orbits over a local field

- DeBacker gave a description of $G(K)$-orbits on $\mathcal{N}(K)$ provided the residue char. is suff. large.
- his result relates nilp. $G(K)$-orbits with nilpotent orbits for the reductive quotients of special fibers of parahorics.
- Leads to the question: can one find nilpotent sections in $\mathrm{Lie}(\mathcal{P})(\mathcal{A})$ with smooth centralizer lifting classes on the special fiber?
- can one use such nilpotent sections to obtain a better understanding of DeBacker's parametrization of rational nilpotent orbits?

A map in Galois cohomology

Let $\mathcal{P}$ a parahoric, write $\mathfrak{p} = \mathrm{Lie}(\mathcal{P})$.

- Let $X \in \mathfrak{p}(\mathcal{A})$ nilpotent for which $X(k)$ lies in the Lie algebra of a Levi factor of $\mathcal{P}_{/k}$.
- Up to conjugacy on the special fiber we may suppose that a cocharacter ϕ associated with $X(k) \in \mathfrak{p}$ (in a Levi factor) takes values in the fixed split maximal torus S – thus ϕ may be viewed as an $\mathcal{A}$-cocharacter of $\mathcal{P}$.

A map in Galois cohomology, continued

- **Desired condition:** Let $C \subset \mathfrak{p} = \mathfrak{p}(\mathcal{A})$ an $\mathcal{A}$-submodule. Write $C_{/K} = C \otimes K$ and $C_{/k}$ for the image of C in $\mathfrak{p}/\pi\mathfrak{p}$. Suppose that:
  - (C1) C is stable under $\phi(\mathcal{A}^\times)$.
  - (C2) as an $\mathcal{A}$-module, $\mathfrak{p} = \mathfrak{p}(\mathcal{A})$ is the direct sum of C and $[X, \mathfrak{p}(\mathcal{A})]$
  - (C3) $C_{/k} \cap \mathrm{Lie}(R_u \mathcal{P}_{/k})$ is a complement to $[X, \mathrm{Lie}(R_u \mathcal{P}_{/k})]$.
- If there is C satisfying (C1)–(C3), the centralizer $C_{\mathcal{P}}(X)$ is a smooth group scheme over $\mathcal{A}$.
- If $p >> 0$ one may use $C = \mathrm{Lie}(C_G(Y))$ determined by a suitable $\mathfrak{sl}_2$-triple (X, H, Y) for which $H = d\phi(1)$. (This is essentially what is done by DeBacker / Waldspurger)

A map in Galois cohomology, continued

Write $\mathfrak{p}^+$ for the pre-image of $\mathrm{Lie}(R_u\mathcal{P}_{/k})$ under the mapping $\mathfrak{p} \to \mathfrak{p}_{/k} = \mathrm{Lie}(\mathcal{P}_{/k})$. Assume C satisfies (C1)–(C3).

Proposition (adaptation of DeBacker/Waldspurger)

The $G(K)$-orbit of X is the nilpotent orbit of minimal dimension having non-empty intersection with $X + \mathfrak{p}^+$.

The proof depends on viewing $\mathcal{P}(\mathcal{A}/\pi^2\mathcal{A})$ as the k-points of a linear group over $k = \mathcal{A}/\pi\mathcal{A}$ (à la Greenberg), and knowing that the centralizer of X is *smooth*.

A map in Galois cohomology, continued

Corollary

For X as above, there is a natural mapping

$$H^1(k, C_{\mathcal{P}_{/k}}(X)) \to H^1(K, C_G(X)),$$

- Note the H^1 of $C_{\mathcal{P}_{/k}}(X)$ identifies with that of its reductive quotient (since k is perfect!)
- This natural mapping is in some sense *realized* by DeBacker's mapping.

The hope!!

- Let $\mathcal{P}$ be a parahoric of G, and let $M \subset \mathcal{P}_{/k}$ be a Levi factor.
- Let X_0 be a distinguished nilp element of M s.t. $X_0^{[p]} = 0$.
- Suppose that M is D-standard (!?!), and let $\psi : \mathrm{SL}_{2/k} \to M$ an optimal SL_2-mapping for X_0.

Hope

The representation $(\mathrm{Ad} \circ \psi, \mathrm{Lie}(\mathcal{P}_{/k}))$ is a tilting module for SL_2 for which all weights μ satisfy $-2p + 2 \leq \mu \leq 2p - 2$.

- Equivalently: the Lie algebra of the unipotent radical of $\mathcal{P}_{/k}$ is a tilting module for SL_2 under the action determined by $\mathrm{Ad} \circ \psi$ (with the indicated condition on the weights).

interlude on tilting modules for SL_2

View $\lambda \in \mathbf{Z}$ with $0 \leq \lambda \leq 2p-2$, as a character of the stndrd max torus of SL_2.

- The stndrd module $H^0(\lambda)$ has dim $\lambda + 1$ and is isom to $\mathrm{Sym}^\lambda V$ where $V = K^2 = H^0(1)$ is natural rep.
- $H^0(\lambda)$ is simple $\iff \lambda < p$.
- If $\lambda = p + \mu \geq p$, there is a non-split extension

  $$0 \to H^0(p-2-\mu) \to T(\lambda) \to H^0(\lambda) \to 0;$$

  $T(\lambda)$ is an indecomp. *tilting module* (of dim $2p$).
- if $\lambda < p$, then $T(\lambda) = H^0(\lambda)$ is a (simple) tilting module.
- Let T be a tilting module for SL_2. Assume $T_\mu \neq 0 \implies |\mu| \leq 2p-2$. Then T is a $\bigoplus$ of various $T(\lambda)$ for $0 \leq \lambda \leq 2p-2$.

Tilting modules for SL_2, conclusion

Let $\mathcal{A}$ a discrete valuation ring with fractions K and residues $k = \mathcal{A}/\pi\mathcal{A}$.

- Let $\mathcal{L}$ be a free $\mathcal{A}$-mod of finite rank.
- Let $\rho : \mathrm{SL}_{2/\mathcal{A}} \to \mathrm{GL}(\mathcal{L})$ be an $\mathcal{A}$-representation – i.e. a morphism of $\mathcal{A}$-group schemes.
- Write $X_0 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \in \mathfrak{sl}_2(\mathcal{A})$, $X_0(k)$ the image in $\mathfrak{sl}_2(k)$ and $X_0(K)$ the image in $\mathfrak{sl}_2(K)$.
- Assume each weight λ of the representation $\mathcal{L}$ satisfies $-2p + 2 \leq \lambda \leq 2p - 2$.

Proposition

If $\mathcal{L}/\pi\mathcal{L}$ is a tilting module for $\mathrm{SL}_{2/k}$, then $\dim_k \ker \rho(X_0(k))$ coincides with $\dim_K \ker \rho(X_0(K))$.

The utility of hope...

Let again X_0 distinguished nilpotent in $\mathrm{Lie}(M)(k)$ for a Levi factor M of $\mathcal{P}_{/k}$ such that $X^{[p]} = 0$.

- Assume that *the hope* holds (i.e. $\mathrm{Lie}(\mathcal{P}_{/k}$ is a suitable tilting module for an optimal SL_2 determined by X_0)

Proposition

Then there is an $\mathcal{A}$-submodule $C \subset \mathfrak{p}$ for which (C1)–(C3) hold.

- An important point is the following: if ϕ is a cocharacter of M associated with X_0, and if (as before) we arrange that ϕ is "defined over $\mathcal{A}$", one needs to know that ϕ is associated with X for some nipotent element $X \in \mathfrak{p}$ whose image in $\mathfrak{p}_{/k}$ is X_0.

Verifying hope...

- The hope holds for GL_n.
- It also holds for $G = \mathrm{Sp}(V)$.
  - Indeed, it is clear for the reductive parahoric.
  - If $\mathcal{P}$ is a non-reductive parahoric, a Levi factor M of the special fiber $\mathcal{P}_{/k}$ has the form $\mathrm{Sp}(W_1) \times \mathrm{Sp}(W_2)$.
  - And as a module for M, $\mathrm{Lie}(R_u\mathcal{P}_{/k})$ is isomorphic to $(W_1 \otimes W_2) \oplus (W_1 \otimes W_2)$.