The Infinite Can Ramsey Theorem (An Exposition)

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Theorem: For every $COL : \binom{\mathsf{N}}{2} \rightarrow [c]$ there is an infinite homogenous set.

What if the number of colors was infinite?

Do not necc get a homog set since could color EVERY edge differently. But then get infinite *rainbow set*.

Attempt

Theorem: For every $COL : \binom{\mathsf{N}}{2} \rightarrow \omega$ there is an infinite homogenous set OR an infinite rainbow set.
VOTE:

Attempt

Theorem: For every $COL : \binom{\mathsf{N}}{2} \rightarrow \omega$ there is an infinite homogenous set OR an infinite rainbow set.
VOTE:
FALSE:

- $COL(i,j) = \min\{i,j\}$.
- $COL(i,j) = \max\{i,j\}$.

Definition: Let $COL : \binom{N}{2} \rightarrow \omega$. Let $V \subseteq N$.

- V is *homogenous* if $COL(a,b) = COL(c,d)$ iff $TRUE$.
- V is *min-homogenous* if $COL(a,b) = COL(c,d)$ iff $a = c$.
- V is *max-homogenous* if $COL(a,b) = COL(c,d)$ iff $b = d$.
- V is *rainbow* if $COL(a,b) = COL(c,d)$ iff $a = c$ and $b = d$.

One-Dim Can Ramsey Theorem

Lemma: Let V be an countable set. Let $COL : V \rightarrow \omega$. Then there exists either an infinite homog set (all the same color) or an infinite rainbow set (all diff colors).

Definition that Will Help Us

Definition Let $COL : \binom{\mathsf{N}}{2} \rightarrow \omega$. If c is a color and $v \in \mathsf{N}$ then $\deg_c(v)$ is the number of c-colored edges with an endpoint in v.

Needed Lemma

Lemma Let X be infinite. Let $COL : \binom{X}{2} \rightarrow \omega$. If for $x \in X$ and $c \in \omega$, $\deg_c(x) \le 1$ then there is an infinite rainbow set.
PROVE IN GROUPS.

Proof

Let R be a MAXIMAL rainbow set of X.

$$(\forall y \in X - R)[X \cup \{y\} \text{ is not a rainbow set}].$$

Let $y \in X - R$. Why is $y \notin R$?

1. There exists $u \in R$ and $\{a, b\} \in \binom{R}{2}$ such that $COL(y, u) = COL(a, b)$.
2. There exists $\{a, b\} \in \binom{R}{2}$ such that $COL(y, a) = COL(y, b)$. This cannot happen since then y has color degree ≤ 1.

Map $X - R$ to $R \times \binom{R}{2}$: map $y \in X - R$ to $(u, \{a, b\})$ (item 1).
Map is injective: if y_1 and y_2 both map to $(u, \{a, b\})$ then $COL(y_1, u) = COL(y_2, u)$ but $\deg_c(u) \leq 1$.
Injection from $X - R$ to $R \times \binom{R}{2}$. If R finite then injection from an infinite set to a finite set Impossible! Hence R is infinite.

Canonical Ramsey Theorem for Graphs

Theorem: For all $COL : \binom{\mathsf{N}}{2} \rightarrow \omega$ there is either

- an infinite homogenous set,
- an infinite min-homog set,
- an infinite max-homog set, or
- an infinite rainbow set.

Proof of Can Ramsey Theorem for Graphs

Given $COL : \binom{\mathbb{N}}{2} \rightarrow \omega$. We use COL to obtain $COL' : \binom{\mathbb{N}}{3} \rightarrow [4]$
We will use the 3-ary Ramsey theorem.

1. If $COL(x_1, x_2) = COL(x_1, x_3)$ then $COL'(x_1, x_2, x_3) = 1$.
2. If $COL(x_1, x_3) = COL(x_2, x_3)$ then $COL'(x_1, x_2, x_3) = 2$.
3. If $COL(x_1, x_2) = COL(x_2, x_3)$ then $COL'(x_1, x_2, x_3) = 3$.
4. If none of the above occur then $COL'(x_1, x_2, x_3) = 4$.

PROVE THIS WORKS IN CLASS

A Lemma Needed for an "Application"

Need Lemma:
Geom Lemma: Let P be a countable set of points in R^2 Let $COL : \binom{P}{2} \rightarrow \mathrm{R}^+$ be defined by $COL(x, y) = |x - y|$. Then

1. There is no infinite homogenous set.
2. There is no infinite min-homogenous set.
3. There is no infinite max-homogenous set.

PROVE IN GROUPS

An "Application"

Theorem: Let P be a countable set of points in R^2. There exists a countable subset X of P such that all pairs of points in X have different distances.
Proof: Let $COL : \binom{P}{2} \rightarrow \mathrm{R}^+$ be $COL(x, y) = |x - y|$.
Use Can Ramsey Theorem and Geom Lemma to obtain infinite rainbow set, hence our desired set.

Ramsey's Theorem For 3-hypergraphs

Theorem: For every $COL : \binom{\mathsf{N}}{3} \rightarrow [c]$ there is an infinite homogenous set.

What if the number of colors was infinite?

Do not necc get a homog set since could color EVERY edge differently. But then get infinite *rainbow set*.

Discuss with Class what theorem might be.

Definition: Let $COL : \binom{\mathrm{N}}{3} \rightarrow \omega$. Let $I \subseteq \{1,2,3\}$. A set is *I-homog* if, for all $x_1 < x_2 < x_3$, $y_1 < y_2 < y_3$.

$$COL(x_1, x_2, x_3) = COL(y_1, y_2, y_3) \text{ iff } (\forall i \in I)[x_i = y_i].$$

Theorem: For all $COL : \binom{\mathrm{N}}{3} \rightarrow \omega$ there exists $I \subseteq [3]$ and infinite $H \subseteq \mathrm{N}$ such that H is I-homog.

Proof of 3-ary Ramsey Can Theorem

Given $COL : \binom{N}{3} \rightarrow \omega$. We define $COL' : \binom{N}{4} \rightarrow [7]$ We use 4-ary Ramsey.

$COL'(x_1, x_2, x_3, x_4)$:

1. $COL(x_1, x_2, x_3) = COL(x_1, x_2, x_4) \rightarrow COL'(x_1, x_2, x_3, x_4) = 1.$
2. $COL(x_1, x_2, x_3) = COL(x_1, x_3, x_4) \rightarrow COL'(x_1, x_2, x_3, x_4) = 2.$
3. $COL(x_1, x_2, x_3) = COL(x_2, x_3, x_4) \rightarrow COL'(x_1, x_2, x_3, x_4) = 3.$
4. $COL(x_1, x_2, x_4) = COL(x_1, x_3, x_4) \rightarrow COL'(x_1, x_2, x_3, x_4) = 4.$
5. $COL(x_1, x_2, x_4) = COL(x_2, x_3, x_4) \rightarrow COL'(x_1, x_2, x_3, x_4) = 5.$
6. $COL(x_1, x_3, x_4) = COL(x_2, x_3, x_4) \rightarrow COL'(x_1, x_2, x_3, x_4) = 6.$
7. If none of the above occur then $COL'(x_1, x_2, x_3, x_4) = 7.$

PROVE IN GROUPS: The first 6-cases yield I-homog sets.
WHAT ABOUT THE 7th case?

7th Case

The only case left is when

- $COL(x_1, x_2, x_3) \neq COL(x_1, x_2, x_4)$
- $COL(x_1, x_2, x_3) \neq COL(x_1, x_3, x_4)$
- $COL(x_1, x_2, x_3) \neq COL(x_2, x_3, x_4)$
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Summarize this:

7th Case

The only case left is when

- $COL(x_1, x_2, x_3) \neq COL(x_1, x_2, x_4)$
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Summarize this:

$$(\forall c)(\forall x, y)[\deg_c(x, y) \leq 1].$$

NEED!

NEED the following

Statement: Let X be infinite. Let $COL : \binom{X}{3} \rightarrow \omega$. Assume that $(\forall c)(\forall x, y \in X)[\deg_c(x, y) \le 1]$. Then there is an infinite rainbow subset of X.

VOTE: YES or NO or UNKNOWN TO SCIENCE.

NEED!

NEED the following

Statement: Let X be infinite. Let $COL : \binom{X}{3} \rightarrow \omega$. Assume that $(\forall c)(\forall x, y \in X)[\deg_c(x, y) \leq 1]$. Then there is an infinite rainbow subset of X.

VOTE: YES or NO or UNKNOWN TO SCIENCE.

YES- its true. TRY TO PROVE IT IN GROUPS.

Why Fails

Maximal argument does not work.
BILL- DISCUSS ON BOARD.

Ulrich's Solution

Ulrich's solution:

- Solve the problem
- Get Bill to bet \$5.00 you can't solve it.
- Show him solution and collect \$5.00.

Ulrich's Solution

Ulrich's solution: Stop problem before it starts.
$COL : \binom{X}{3} \rightarrow \omega$.
$(\forall c)(\forall x, y)[\deg_c(x, y) \le 1]$.
DEFINE $COL'' : \binom{X}{5} \rightarrow [4]$.
$COL''(x_1 < x_2 < x_3 < x_4 < x_5) =$

- 1 if $COL(x_1, x_2, x_5) = COL(x_3, x_4, x_5)$.
- 2 if $COL(x_1, x_3, x_5) = COL(x_2, x_4, x_5)$.
- 3 if $COL(x_1, x_4, x_5) = COL(x_2, x_3, x_5)$.
- 4 otherwise.

SHOW IN GROUPS- Can't have inf homog set of color 1,2, or 3.

NOW can finish argument

Let Y be infinite homog set. RECAP:

1. $(\forall c)(\forall x, y \in Y)[\deg_c(x, y) \le 1]$.
2. $(\forall x_1 < x_2 < x_3 < x_4 < x_5)[COL(x_1, x_2, x_5) \ne COL(x_3, x_4, x_5)]$.
3. $(\forall x_1 < x_2 < x_3 < x_4 < x_5)[COL(x_1, x_3, x_5) \ne COL(x_2, x_4, x_5)]$.
4. $(\forall x_1 < x_2 < x_3 < x_4 < x_5)[COL(x_1, x_4, x_5) \ne COL(x_2, x_3, x_5)]$.

PROVE IN GROUPS: There is an infinite Rainbow set.

PROS and CONS

Proof is DONE. PROS and CONS.

1. PRO- proof is CLEAN- only ((4 choose 3) choose 2)+1 $= 7$ cases.
2. PRO- Can do 4-ary- only ((5 choose 4) choose 2)+1 $= 11$ cases.
3. PRO- Can do a-ary Can Ramsey- notation can manage the cases.
4. CON- If finitize this proof you have to use
   - 2-ary Can Ramsey
   - 4-ary hypergraph Ramsey
   - 5-ary hypergraph Ramsey

PLAN

For finite version:

- 2-ary Can Ramsey- We will deal with this FIRST.
- 4-ary hypergraph Ramsey- Stuck with that.
- 5-ary hypergraph Ramsey- TWO ways to deal with this!

GETTING RID OF 2-ary CAN RAMSEY

WE WILL GET RID OF USE OF 2-ARY CAN RAMSEY

NEW Proof of 3-ary Ramsey Can Theorem

Given $COL : \binom{\mathbb{N}}{3} \rightarrow \omega$. We define $COL' : \binom{\mathbb{N}}{4} \rightarrow [8]$. We use 4-ary Ramsey.

$COL'(x_1, x_2, x_3, x_4)$: Abbreviate $COL(x_1, x_2, x_3) = COL(x_1, x_3, x_4)$ by 123=124. Abbreviate NOTHING ELSE EQUAL by NEE

1. $123 = 134 \rightarrow COL'(x_1, x_2, x_3, x_4) = 1$.
2. $124 = 234 \rightarrow COL'(x_1, x_2, x_3, x_4) = 2$.
3. $123 = 234 \rightarrow COL'(x_1, x_2, x_3, x_4) = 3$.
4. $123 = 124$, NEE $\rightarrow COL'(x_1, x_2, x_3, x_4) = 4$.
5. $134 = 234$, NEE$\rightarrow COL'(x_1, x_2, x_3, x_4) = 5$.
6. $134 = 124$, NEE$\rightarrow COL'(x_1, x_2, x_3, x_4) = 6$.
7. $123 = 124$, $134 = 234$, $124 \neq 134 \rightarrow COL'(x_1, x_2, x_3, x_4) = 7$.

PROVE IN GROUPS. IF GET DONE THEN LOOK AT REMAINING CASES.

Remaining Cases

What is true of cases that are left?

1. $COL(x_1, x_2, x_3) \neq COL(x_1, x_3, x_4)$ (Shorthand: $123 \neq 134$).
2. $COL(x_1, x_2, x_4) \neq COL(x_2, x_3, x_4)$ (Shorthand: $124 \neq 234$).
3. $COL(x_1, x_2, x_3) \neq COL(x_2, x_3, x_4)$ (Shorthand: $123 \neq 234$).

Need to look at ALL combinations of $(123, 124)$, $(124, 134)$, $(134, 234)$.

Table

$123 = ?124$	$124 = ?134$	$134 = ?234$	Comment
Y	Y	Y	
Y	Y	N	
Y	N	Y	
Y	N	N	
N	Y	Y	
N	Y	N	
N	N	Y	
N	N	N	

PROVE IN GROUPS.

Table Filled in

123 =? 124	124 =? 134	134 =? 234	Comment
Y	Y	Y	123=134
Y	Y	N	123=134
Y	N	Y	COVERED exactly
Y	N	N	An NEE case
N	Y	Y	124=234
N	Y	N	An NEE case
N	N	Y	An NEE case
N	N	N	Color 8–Rainbow

So we are DONE! Got rid of 2-ary Can Ramsey Use!

GETTING RID OF 5-ary RAMSEY

WE WILL GET RID OF USE OF 5-ARY RAMSEY

RECAP

1. We have an infinite set with $\deg_c(x, y) \leq 1$.
2. Want an infinite Rainbow set.
3. CAN obtain using Ulrich Technique of using 5-ary Hypergraph Ramsey. Elegant! Pedagogically great! But two drawbacks:

RECAP

1. We have an infinite set with $\deg_c(x, y) \le 1$.
2. Want an infinite Rainbow set.
3. CAN obtain using Ulrich Technique of using 5-ary Hypergraph Ramsey. Elegant! Pedagogically great! But two drawbacks:
   - Finite version would have enormous bounds.

RECAP

1. We have an infinite set with $\deg_c(x,y) \le 1$.
2. Want an infinite Rainbow set.
3. CAN obtain using Ulrich Technique of using 5-ary Hypergraph Ramsey. Elegant! Pedagogically great! But two drawbacks:
    - Finite version would have enormous bounds.
    - Costs me $5.00 everytime I use it. (Douglas has great copyright lawyer.)

Our Problem

Given $COL : \binom{\mathsf{N}}{3} \rightarrow \omega$ with $(\forall c)(\forall x, y)[\deg_c(x, y) \le 1]$
Show there exists an infinite rainbow set.

Our Biggest Fear

PLAN: build the set W. Have finite W_s. Want to add to it.
WHAT IF

$$(\forall x \notin W_s)(\exists a_1, b_1, a_2, b_2 \in W_s)[COL(a_1, b_a, x) = COL(a_2, b_2, x)]$$

Then can't add anything to W_s.

Naively Bad and Sneaky Bad

Definition: W finite, X infinite, $W < X$. Let $COL : \binom{W \cup X}{3} \rightarrow \omega$. $x \in X$.

1. x is *W-naively bad* if

$$(\exists a_1, b_1, a_2, b_2 \in W)[COL(a_1, b_1, x) = COL(a_2, b_2, x)].$$

2. x is *(W, X)-sneaky bad* if

$$(\forall^{\infty} y \in X)[y \text{ is } (W \cup \{x\})\text{-naively bad}].$$

Avoid Sneaky Bad

Definition: W finite, X infinite, $W < X$. Let $COL : \binom{W \cup X}{3} \to \omega$. $x \in X$. W is X-nice if

1. W is rainbow, and
2. $(\forall x \in X)[x$ is not naively bad $]$.

KEY: While construction W we want to make sure that each W_s is nice.

Key Lemma

Lemma: Let $W, X \subseteq$ N. Let $COL : \binom{W \cup X}{3} \rightarrow \omega$ be such that

- $\forall x, y \in W \cup X)(\forall c)[\deg_c(x, y) \le 1]$.
- $W < X$ (so W is finite). Assume W is X-nice.

Then there exists $x \in X$ and infinite $X' \subseteq X$ such that $W \cup \{x\}$ is X'-nice.

TRY TO PROVE IN GROUPS.

Key Lemma

Proof:
Inductively no $x \in X$ is naively bad.

Remove the finite number of $x \in X$ s.t.
$(\exists a_1, b_1, c_1, a_2, b_2 \in W)[COL(a_1, b_1, c_1) = COL(a_2, b_2, x)]$

Rename set X. Have
$(\forall x \in X)[W \cup \{x\}$ is rainbow].

GOOD NEWS- adding any x keeps rainbow.
CALLENGE: We need an x that is not sneaky bad.

Need x not sneaky bad

If THERE IS an x NOT sneaky bad then great:
W gets $W \cup \{x\}$.
$X = \{y \in X \mid y \text{ is not } (W \cup \{x\})\text{-naively bad}\}$.
X is infinite since x was not W-sneaky bad.
If THERE IS NO SUCH x then goto next slide (This will NOT be a contradiction.)

ALL x are Sneaky Bad

Assume that ALL x are (W, X)-sneaky bad.
$(\forall x \in X)[W \cup \{x\}$ is NOT nice].
WHY?

$(\forall x \in X)(\forall^{\infty} y \in X)[y$ is naively bad].

$(\forall x \in X)|\forall^{\infty} y \in X)(\exists a, b, a' \in W)[COL(a, b, y) = COL(a', x, y)]$

Infinite Sequence of x's

$(\forall x \in X)(\forall^\infty y \in X)(\exists a, b, a' \in W)[COL(a, b, y) = COL(a', x, y)]$
ABBREVIATE by COL by C
$x_1, x_2, x_3, \ldots$ are the elements of X in order.

$(a_1 < b_1), a'_1 \in W^3$ s.t. $(\exists^\infty y \in X)[C(a_1, b_1, y) = C(a'_1, x_1, y)]$
$Y_1 = \{y \mid C(a_1, b_1, y) = C(a'_1, x_1, y)\}$
NOTE: $(\forall y \in Y_1)[C(a_1, b_1, y) = C(a'_1, x_1, y)]$

$(a_2 < b_2), a'_2 \in W^3$ s.t. $(\exists^\infty y \in Y_1)[C(a_2, b_2, y) = C(a'_2, x_2, y)]$
$Y_2 = \{y \mid C(a_2, b_2, y) = C(a'_2, x_2, y)\}$
NOTE: $(\forall y \in Y_2)[C(a_2, b_2, y) = C(a'_2, x_2, y)]$

$(a_3 < b_3), a'_3 \in W^3$ s.t. $(\exists^\infty y \in Y_2)[C(a_3, b_3, y) = C(a'_3, x_3, y)]$
$Y_3 = \{y \mid C(a_3, b_3, y) = C(a'_3, x_3, y)\}$
NOTE: $(\forall y \in Y_3)[C(a_3, b_3, y) = C(a'_3, x_3, y)]$

....
NOTE $Y_1 \supseteq Y_2 \supseteq Y_3 \cdots$ and all infinite.

Infinite Sequence of x's

Look at $((a_1 < b_1), a'_1)$, $((a_2 < b_2), a'_2)$, $\ldots$.

There exists $i < j$ s.t. $(a_i < b_i), a'_i$, $(a_j < b_j), a'_j = (a, b, a')$.
$(\forall y \in Y_i)[COL(a_i, b_i, y) = COL(a'_i, x_i, y)]$
$(\forall y \in Y_j)[COL(a_j, b_j, y) = COL(a'_j, x_j, y)]$

Since $Y_j \subseteq Y_i$ and $a_i = a_j = a$, $b_i = b_j = b$, $a'_i = a'_j = a'$

$(\forall y \in Y_j)[COL(a, b, y) = COL(a', x_i, y)]$
$(\forall y \in Y_j)[COL(a, b, y) = COL(a', x_j, y)]$

So $(\exists c)[\deg_c(a', y) \geq 2]$.
CONTRADICTION!! Hence some x is not sneaky bad.
Note- proof is constructive— do the construction until get a repeat and then you have your X' and any x left will work.

UPSHOT

We have a proof of Inf Can 3-ary Ramsey that only uses:

- 1-ary can Ramsey
- 4-ary Ramsey.

Finite version yields the following:
Theorem: For all k there exists n such that for any $COL : \binom{[n]}{3} \rightarrow \omega$ there exists $I \subseteq \{1,2,3\}$, and a set H of size k, such that H is I-homog. There is a poly p such that $n \le R_4(p(k))$.

Goal

We want:

Theorem: If P is a countably infinite set of points in the plane, no three collinear, then there exists a countably infinite subset such that all of the areas defined by three points are DIFFERENT.

Lemma: Let $P = \{p_1, p_2, \ldots\}$ be a countable set of points in R^2, no three collinear. Define $COL : \binom{\mathrm{N}}{3}$ via $COL(i,j,k) = AREA(p_i, p_j, p_k)$. For $I \subset \{1,2,3\}$ COL has no I-homog set of size 6.

PROOF

Assume, BWOC, there exists an I-homog set of size 6. Can take I-homog set $\{1,2,3,4,5,6\}$.

Case 1: $I = \{1\}$, $\{1,2\}$, or $\{2\}$.

$AREA(p_1,p_2,p_4) = AREA(p_1,p_2,p_5)$. p_4 and p_5: (1) on a line parallel to p_1p_2, or (2) on different sides of p_1p_2. In the later case the midpoint of p_4p_5 is on p_1p_2.

$AREA(p_1,p_3,p_4) = AREA(p_1,p_3,p_5)$. p_4 and p_5: (1) on a line parallel to p_1p_3, or (2) are on different sides of p_1p_3. In the later case the midpoint of p_4p_5 is on p_1p_3.

$AREA(p_2,p_3,p_4) = AREA(p_2,p_3,p_5)$. p_4 and p_5: (1) on a line parallel to p_2p_3, or (2) on different sides of p_2p_3. In the later case the midpoint of p_4p_5 is on p_2p_3.

PROOF

CASES:

- Two of these cases have p_4, p_5 on the same side of the line. We can assume that p_4, p_5 are on a line parallel to both p_1p_2 and p_1p_3. Since p_1, p_2, p_3 are not collinear there is no such line.
- Two of these cases have p_4, p_5 on opposite sides of the line. We can assume that the midpoint of p_4p_5 is on both p_1p_2 and p_1p_3. Since p_1, p_2, p_3 are not collinear the only point on both p_1p_2 and p_1p_3 is p_1. So the midpoint of p_4, p_5 is p_1. Thus p_4, p_1, p_5 are collinear which is a contradiction.

OTHER CASES

For $I = \{1\}$, $\{1,2\}$, or $\{2\}$ we used the line-point pairs

$$\{p_1p_2, p_1p_3, p_2p_3\} \times \{p_4, p_5\}.$$

For the rest of the cases we just specify which line-point pairs to use.

Case 2: $I = \{3\}$ or $\{2,3\}$. Use

$$\{p_4p_5, p_3p_5, p_3p_4\} \times \{p_1, p_2\}.$$

Case 3: $I = \{1,3\}$ Use

$$\{p_1p_4, p_1p_5, p_1p_6\} \times \{p_2, p_3\}.$$

This is the only case that needs 6 points.

Theorem: If P is a countably infinite set of points in the plane, no three collinear, then there exists a countably infinite subset such that all of the areas defined by three points are DIFFERENT.
Proof: Use Geom Lemma and 3-can Ramsey!

What about 3-d?

For 3-d the Can Ramsey Theory is fine, but we need Geom Lemma.
KNOWN:
Lemma: Let C_1, C_2, C_3 be three cylinders with no pair of parallel axis. Then $C_1 \cap C_2 \cap C_3$ consists of at most 8 points.

Geom Lemma

Lemma: Let $P = \{p_1, p_2, \ldots\}$ be a countably infinite set of points in R^3, no three collinear. Color $\binom{\mathsf{N}}{3}$ via $COL(i,j,k) = AREA(p_i, p_j, p_k)$. This coloring has no homog set of size 13.

PROOF

Assume, BWOC, that there exists an I-homog set of size 13. We take $\{1, \ldots, 13\}$.

Case 1: $I = \{1\}$, $\{1,2\}$, or $\{2\}$.

$AREA(p_1, p_2, p_4) = AREA(p_1, p_2, p_5) = \cdots = AREA(p_1, p_2, p_{12})$.

So $p_4, \ldots, p_{12}$ are on a cylinder with axis $p_1 p_2$.

$AREA(p_1, p_3, p_4) = AREA(p_1, p_3, p_5) = \cdots = AREA(p_1, p_3, p_{12})$.

So $p_4, \ldots, p_{12}$ are on a cylinder with axis $p_1 p_3$.

$AREA(p_2, p_3, p_4) = AREA(p_2, p_3, p_5) = \cdots = AREA(p_2, p_3, p_{12})$.

so $p_4, \ldots, p_{12}$ are on a cylinder with axis $p_2 p_3$.

p_1, p_2, p_3 not collinear, so 3 cylinders have intersection ≤ 8.

However, we just showed 9. Contradiction.

PROOF

For $I = \{1\}$, $\{1,2\}$, or $\{2\}$ we used the line-point pairs

$$\{p_1 p_2, p_1 p_3, p_2 p_3\} \times \{p_4, \ldots, p_{12}\}.$$

For the rest of the cases we just specify which line-point pairs to use.

Case 2: $I = \{3\}$ or $\{2,3\}$. Use

$$\{p_{11} p_{12}, p_{10} p_{12}, p_{10} p_{11}\} \times \{p_1, \ldots, p_9\}.$$

Case 3: $I = \{1,3\}$ Use

$$\{p_1 p_{11}, p_1 p_{12}, p_1 p_{13}\} \times \{p_2, \ldots, p_{10}\}.$$

This is the only case that needs 13 points.

Theorem: If P is a countably infinite set of points in the R^3, no three collinear, then there exists a countably infinite subset such that all of the areas defined by three points are DIFFERENT.
Proof: Use Geom Lemma and 3-can Ramsey!

Generalize to d dimensions?

To get a similar theorem in R^d for $d \geq 3$ need Geometric Lemmas.
OPEN!