

# CMPSCI 611: “Advanced Algorithms”

## Lecture 23: Simplex Algorithm

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## Recap: Example Linear Program

Let

$x_1$  = number of bars of Choco ordered

$x_2$  = number of bars of Choco Deluxe ordered

Objective:

$$\max x_1 + 6x_2$$

Constraints:

$$x_1 \leq 200$$

$$x_2 \leq 300$$

$$x_1 + x_2 \leq 400$$

$$x_1, x_2 \geq 0$$

## Variants of Linear Programming

Different variants:

- ▶ Objective function can be maximization or minimization.
- ▶ Constraints can be inequalities or equalities
- ▶ Variables can be restricted to be non-negative or unrestricted

Can reduce between different forms:

- ▶ Max problem to min problem: Multiply objective function by  $-1$ .
- ▶ Inequality constraints to equality constraints: Add slack variables,

$$(x_1 + x_2 + x_3 \leq 400, \quad x_2 + 3x_3 \leq 600, \quad x_1, x_2, x_3 \geq 0)$$

$$\longrightarrow (x_1 + x_2 + x_3 + s_1 = 400, \quad x_2 + 3x_3 + s_2 = 600, \quad x_1, x_2, x_3, s_1, s_2 \geq 0)$$

- ▶ Equality constraints to inequality constraints:

$$(x_1 + x_2 + x_3 = 400) \longrightarrow (x_1 + x_2 + x_3 \leq 400, \quad x_1 + x_2 + x_3 \geq 400)$$

# Concepts

## Definition

An LP is **infeasible** if the constraints are so tight that it is impossible to satisfy all of them. An LP is **unbounded** if the constraints are so loose that it is possible to achieve arbitrarily high objective values.

## Definition

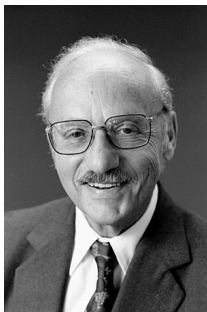
A **vertex** is specified by making a set of  $n$  inequalities tight. Two vertices  $u, v$  are **neighbors** if they have  $n - 1$  defining inequalities in common.

If the linear program is feasible and bounded, the optimum is achieved at a vertex of the feasible region.

For this lecture, assume no two sets of  $n$  inequalities give the same vertex.

# Simplex Algorithm

Simplex Algorithm was devised by George Dantzig in 1947...



## Algorithm

*Pick arbitrary vertex of the feasible region. Move to adjacent vertex with better objective value. If no such vertex exists, terminate.*

Not known to be polynomial time but very quick in practice. Polynomial time algorithms do exist but are less used in practice.

# Outline

Simplex in more detail

# Formalizing the Simplex Algorithm

Objective:

$$\max \mathbf{c}^T \mathbf{x}$$

Subject to:

$$\mathbf{Ax} \leq \mathbf{b}$$

$$\mathbf{x} \geq \mathbf{0}$$

## Algorithm

*Pick arbitrary vertex of the feasible region. Move to adjacent vertex with better objective value. If no such vertex exists, terminate.*

At each iteration there are two tasks:

- ▶ Task 1: Determine if current vertex is optimal
- ▶ Task 2: If not, determine where to move next.

## Completing the tasks: Easy if current vertex is origin

Consider generic LP:

$$\max \mathbf{c}^T \mathbf{x}$$

$$\mathbf{Ax} \leq \mathbf{b}$$

$$\mathbf{x} \geq \mathbf{0}$$

If the origin  $\mathbf{0}$  is feasible, it's a vertex: The  $n$  inequalities  $\mathbf{x} \geq \mathbf{0}$  are tight.

### Lemma

*The origin is optimal if and only if all  $c_i \leq 0$ .*

### Proof.

If some  $c_i > 0$  then the origin is not optimal since we can increase the objective function by raising  $x_i$ . Other direction is clear.  $\square$

Hence Task 1 is easy. For Task 2, consider raising  $x_i$  as much as possible until another inequality becomes tight!



## Example

Objective:

$$\max 2x_1 + 5x_2$$

Constraints:

$$2x_1 - x_2 \leq 4$$

$$x_1 + 2x_2 \leq 9$$

$$-x_1 + x_2 \leq 3$$

$$x_1 \geq 0$$

$$x_2 \geq 0$$

## If current vertex is not origin, shift coordinate system...

- ▶ Consider LP:

$$\max \mathbf{c}^T \mathbf{x}$$

$$\mathbf{Ax} \leq \mathbf{b}$$

$$\mathbf{x} \geq \mathbf{0}$$

and suppose current vertex  $\mathbf{v} \neq \mathbf{0}$ .

- ▶ Rewrite the LP using the variables

$$y_i = b_i - \mathbf{a}_i \cdot \mathbf{x}$$

$$y_j = x_j$$

for all tight equations  $x_j = 0$  or  $\mathbf{a}_i \cdot \mathbf{x} = b_i$  where  $\mathbf{a}_i$  are the rows of  $A$

- ▶ In new coordinate system, current vertex is origin!

## Step 1: Initial LP

Objective:

$$\max 2x_1 + 5x_2$$

Constraints:

$$2x_1 - x_2 \leq 4 \quad (1)$$

$$x_1 + 2x_2 \leq 9 \quad (2)$$

$$-x_1 + x_2 \leq 3 \quad (3)$$

$$x_1 \geq 0 \quad (4)$$

$$x_2 \geq 0 \quad (5)$$

- ▶ Current Vertex: (4), (5)
- ▶ Objective Value: 0
- ▶ Increase  $x_2$ : (5) becomes loose, and (3) becomes tight at  $x_2 = 3$
- ▶ New variables:  $y_1 = x_1$ ,  $y_2 = 3 + x_1 - x_2$

## Step 2: Rewritten LP

Objective:

$$\max 15 + 7y_1 - 5y_2$$

Constraints:

$$y_1 + y_2 \leq 7 \quad (1)$$

$$3y_1 - 2y_2 \leq 3 \quad (2)$$

$$y_2 \geq 0 \quad (3)$$

$$y_1 \geq 0 \quad (4)$$

$$-y_1 + y_2 \leq 3 \quad (5)$$

- ▶ Current Vertex: (4), (3)
- ▶ Objective Value: 15
- ▶ Increase  $y_1$ : (4) becomes loose, and (2) becomes tight at  $y_1 = 1$
- ▶ New variables:  $z_1 = 3 - 3y_1 + 2y_2$ ,  $z_2 = y_2$

## Step 3: Rewritten LP

Objective:

$$\max 22 - 7z_1/3 - z_2/3$$

Constraints:

$$-z_1/3 + 5z_2/3 \leq 6 \quad (1)$$

$$z_1 \geq 0 \quad (2)$$

$$z_2 \geq 0 \quad (3)$$

$$z_1/3 - 2z_2/3 \leq 1 \quad (4)$$

$$z_1/3 + z_2/3 \leq 4 \quad (5)$$

- ▶ Current Vertex: (2), (3)
- ▶ Objective Value: 22
- ▶ Optimal: all  $c_i < 0$  **Hurray!**
- ▶ Relate  $z_i$  values back to original  $x_i$  values.