A new interpretation of the quadratic closure of a field

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with my terrific collaborator Ján Mináč

working at middlesex college on a beautiful day.

Outline

- Motivation
- The Bloch–Kato conjecture
- Capturing Galois cohomology
- The quadratic closure of a field.
- Detecting absolute Galois groups

Motivation

F – field that contains a primitive p–th root of unity.
F_{sep} – the separable closure of F.

The ultimate goal of mankind: understand the structure of the absolute Galois group

$$A_F := \text{Gal}(F_{\text{sep}}/F).$$

A burning problem: How to distinguish absolute Galois groups amongst profinite groups?

Galois cohomology: $\text{H}^*(A_F, \mathbb{F}_p)$
This is the continuous cohomology of the profinite group A_F.

Bloch–Kato conjecture: gives a presentation of the *rather mysterious* Galois cohomology ring $\text{H}^*(A_F, \mathbb{F}_p)$ by generators and relations.

Milnor K–theory

F^*: multiplicative group of F.

Milnor K–theory $K_*(F)$: graded ring defined (1970) as

$$K_*(F) := T(F^*)/\langle a \otimes b \,|\, a + b = 1 \rangle.$$

Bloch–Kato conjecture: $K_*(F)/p \cong \mathrm{H}^*(A_F, \mathbb{F}_p)$!!

(When $p = 2$ this is called the Milnor conjecture — proved by Voevodsky in 1996.)

$\mathrm{H}^*(A_F, \mathbb{F}_p)$ is generated by one-dimensional classes and the relations in the ring are generated by two-dimensional classes.

This provides one answer to the burning problem!

The norm residue homomorphism

Consider the Kummer sequence of A_F–modules:

$$1 \longrightarrow \mu_p \longrightarrow F^*_{\text{sep}} \xrightarrow{x^p} F^*_{\text{sep}} \longrightarrow 1.$$

The boundary map in the long exact sequence in Galois cohomology is a homomorphism $F^* \longrightarrow \mathrm{H}^1(A_F, \mathbb{F}_p)$

Bass and Tate proved (1973) that the Steinberg relations: $(a) \cup (b) = 0$ hold in $\mathrm{H}^2(A_F, \mathbb{F}_p)$ whenever $a + b = 1$ in F^*.

$$\begin{array}{ccccccc}
F^* & \hookrightarrow & T(F^*) & \longrightarrow & K_*(F) & \longrightarrow & K_*(F)/p \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \cong \\
H^1(A_F, \mathbb{F}_p) & \hookrightarrow & H^*(A_F, \mathbb{F}_p) & \xrightarrow{=} & H^*(A_F, \mathbb{F}_p) & \xrightarrow{=} & H^*(A_F, \mathbb{F}_p)
\end{array}$$

The Bloch-Kato conjecture: $K_*(F)/p \xrightarrow{\cong} \mathrm{H}^*(A_F, \mathbb{F}_p)$.

The History of the Bloch–Kato conjecture

The Bloch–Kato conjecture is a huge industry with a long, rich, interesting and convoluted history of over 40 years spreading in areas of geometry, number theory, and homotopy theory.

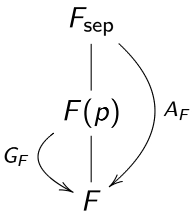
The Bloch–Kato conjecture is now a theorem of Voevodsky and Rost (some details necessary were given by Weibel).

An incomplete list of some other great mathematicians who contributed to the literature on the Bloch–Kato conjecture:

Milnor, Bass, Tate, Bloch, Kato, Lichtenbaum, Beilinson, Suslin, Merkurjev, Izhboldin, Quillen, Swan, Levine, Morel, Orlov, Vishik, Friedlander, Arason, Jacob, Elman, Lam ......

The maximal p–extension

Let $F(p)$ denote the maximal p–extension of F.



The Bloch–Kato conjecture implies that LHS spectral sequence collapses at the E^2-term:

$$\inf\colon \mathrm{H}^*(G_F, \mathbb{F}_p) \xrightarrow{\cong} \mathrm{H}^*(A_F, \mathbb{F}_p).$$

Therefore it is enough to study $\mathrm{H}^*(G_F, \mathbb{F}_p)$.

The Galois Tower

Consider the tower of fields

$$
\begin{array}{c}
F(p) \\
| \\
\vdots \\
| \\
F^{(n)} \\
| \\
\vdots \\
| \\
F^{(3)} \\
| \\
F^{(2)} \\
| \\
F
\end{array}
$$

$$F(p) = \bigcup_i F^{(i)}$$

Inflation maps along the Galois tower

Define Galois groups:

$$G_F^{[n]} := \mathrm{Gal}(F^{(n)}/F)$$
$$G_F^{(n)} := \mathrm{Gal}(F(p)/F^{(n)})$$

These fit in a sequence

$$1 \to G_F^{(n)} \to G_F \to G_F^{[n]} \to 1.$$

$$G_F^{(n)} = \left[G_F, G_F^{(n-1)}\right] \left(G_F^{(n-1)}\right)^p - \text{ } p\text{-central descending series.}$$

Inflation maps: $\quad \mathrm{H}^*(G_F^{[n]}, \mathbb{F}_p) \longrightarrow \mathrm{H}^*(G_F^{[n+1]}, \mathbb{F}_p)$

Capturing $\mathrm{H}^1(G_F, \mathbb{F}_p)$

Lemma

The inflation map $\mathrm{H}^1(G_F^{[2]}, \mathbb{F}_p) \longrightarrow \mathrm{H}^1(G_F, \mathbb{F}_p)$ *is a natural isomorphism.*

Proof.

Since $\mathrm{H}^1(G, \mathbb{F}_p) \cong \mathrm{Hom}(G, \mathbb{F}_p)$, the inflation map in question sends $G_F^{[2]} \xrightarrow{f} \mathbb{F}_p$ to $f\pi\colon G_F \twoheadrightarrow G_F^{[2]} \xrightarrow{f} \mathbb{F}_p$. Injectivity is now clear.

Surjectivity follows from the fact that commutators and p-th powers belong to the kernel of every map $G_F \longrightarrow \mathbb{F}_p$. □

Thus, $\mathrm{H}^1(G_F, \mathbb{F}_p)$ is captured at the level of $F^{(2)}$.

Capturing the entire Galois cohomology

Theorem (C.–Mináč, 2008)

The decomposable part of $\mathrm{H}^*(G_F^{[3]}, \mathbb{F}_p)$ *is naturally isomorphic to* $\mathrm{H}^*(G_F, \mathbb{F}_p)$.

Proof Summary: The isomorphism is given by the inflation map $\inf\colon \mathrm{Dec}(\mathrm{H}^*(G_F^{[3]}, \mathbb{F}_p)) \longrightarrow \mathrm{H}^*(G_F, \mathbb{F}_p)$.

$$\begin{array}{ccccc} \mathrm{Dec}(\mathrm{H}^*(G_F^{[2]},\mathbb{F}_p)) & \xrightarrow{\inf} & \mathrm{Dec}(\mathrm{H}^*(G_F^{[3]},\mathbb{F}_p)) & \overset{\exists}{\twoheadleftarrow\cdots\cdots} & k_*^M(F) \\ & \searrow^{\inf} & \downarrow{\scriptstyle \inf} & \swarrow_{\cong} & \\ & & \mathrm{H}^*(G_F,\mathbb{F}_p) & & \end{array}$$

(The case $p = 2$ was proved by Alejandro, Karagueuzian, and Mináč.)

New interpretation of the quadratic closure of F.

Theorem (C.–Mináč, 2008)

Let H be a proper quotient of G_F such that

$$G_F \twoheadrightarrow H \twoheadrightarrow G_F^{[3]}.$$

Then $\mathrm{H}^*(H, \mathbb{F}_p)$ *cannot be generated by one-dimensional classes.*

Moral: The maximal p-extension $F(p)$ is the smallest Galois extension in the separable closure whose Galois cohomology dissolves completely (i.e., splits into one dimensional classes).

In fact, we shall show that there is an indecomposable class in $H^2(H, \mathbb{F}_p)$.

Sketch proof

- H corresponds to an intermediate field L such that $\mathrm{Gal}(L/F) = H$.
- $(L^*/L^{*p})^H \neq 0$. So we pick $\theta \neq 0$
- Consider the tower of Galois extensions:

$$\begin{array}{c} L(\theta^{1/p}) \\ | \\ L \\ | \\ F \end{array}$$

- This gives an extension of groups:

$$1 \to \mathbb{F}_p \to \mathrm{Gal}(L(\theta^{1/p})/F) \to H \to 1$$

- The above extension corresponds to a class in $H^2(H, \mathbb{F}_p)$ which is shown to be indecomposable.

Detecting absolute Galois groups

The Central Question: Given a profinite group G, when is it an absolute Galois group?

The previous theorem allows us to identify some pro-p-groups which cannot be the Galois group of any maximal p-extension.

Suppose if G is a pro-p-group such that

$$G \twoheadrightarrow H \twoheadrightarrow G^{[3]},$$

and the cohomology of H is decomposable (e.g. free products of Demuškin groups), then G cannot be the Galois group of a maximal p-extension.

A pro–p–group D is a Demuškin group if it has cohomological dimension 2 and Poincaré duality:

$$\langle\ ,\ \rangle :\ \mathrm{H}^1(D, \mathbb{F}_p) \times \mathrm{H}^1(D, \mathbb{F}_p) \xrightarrow{\cup} \mathrm{H}^2(D, \mathbb{F}_p) = \mathbb{F}_p$$

Thank You