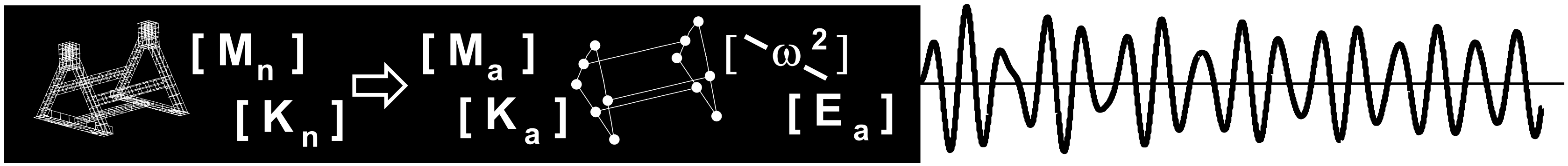

Review
Solid Mechanics
&
Finite Elements

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Solid Mechanics Background

Basic understanding of Solid Mechanics is reviewed

This material is needed for refresher to support finite element model development

Stress

The state of stress in an elemental volume is given as

$$\{\sigma\}^T = \left[\sigma_x \sigma_y \sigma_z \tau_{xy} \tau_{yz} \tau_{xz}\right]$$

If the coordinates are principal axes then

$$\{\sigma\}^T = \left[\sigma_1 \sigma_2 \sigma_3 0 0 0\right]$$

Strain

The state of strain in an elemental volume is given as

$$\{\varepsilon\}^T = \left[\varepsilon_x \, \varepsilon_y \, \varepsilon_z \, \gamma_{xy} \, \gamma_{yz} \, \gamma_{xz}\right]$$

If the coordinates are principal axes then

$$\{\varepsilon\}^T = [\varepsilon_1 \, \varepsilon_2 \, \varepsilon_3 \, 0 \, 0 \, 0]$$

Strain-Displacement Equations

The strain displacement relations are

$$\varepsilon_x = \frac{\partial u}{\partial x} + \frac{1}{2}\left[\left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial v}{\partial x}\right)^2 + \left(\frac{\partial w}{\partial x}\right)^2\right]$$

$$\varepsilon_y = \frac{\partial v}{\partial y} + \frac{1}{2}\left[\left(\frac{\partial u}{\partial y}\right)^2 + \left(\frac{\partial v}{\partial y}\right)^2 + \left(\frac{\partial w}{\partial y}\right)^2\right]$$

$$\varepsilon_z = \frac{\partial w}{\partial z} + \frac{1}{2}\left[\left(\frac{\partial u}{\partial z}\right)^2 + \left(\frac{\partial v}{\partial z}\right)^2 + \left(\frac{\partial w}{\partial z}\right)^2\right]$$

$$\gamma_{xy} = \frac{\partial v}{\partial x} + \frac{\partial u}{\partial y} + \frac{\partial u}{\partial x}\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\frac{\partial v}{\partial y} + \frac{\partial w}{\partial x}\frac{\partial w}{\partial y}$$

$$\gamma_{yz} = \frac{\partial w}{\partial y} + \frac{\partial v}{\partial z} + \frac{\partial u}{\partial y}\frac{\partial u}{\partial z} + \frac{\partial v}{\partial y}\frac{\partial v}{\partial z} + \frac{\partial w}{\partial y}\frac{\partial w}{\partial z}$$

$$\gamma_{zx} = \frac{\partial u}{\partial z} + \frac{\partial w}{\partial x} + \frac{\partial u}{\partial z}\frac{\partial u}{\partial x} + \frac{\partial v}{\partial z}\frac{\partial v}{\partial x} + \frac{\partial w}{\partial z}\frac{\partial w}{\partial x}$$

Strain-Displacement Equations

Retaining only the first order (or linear) terms and neglecting the second order terms gives

$$\varepsilon_x = \frac{\partial u}{\partial x} \qquad \gamma_{xy} = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}$$

$$\varepsilon_y = \frac{\partial v}{\partial y} \qquad \gamma_{xz} = \frac{\partial u}{\partial z} + \frac{\partial w}{\partial x}$$

$$\varepsilon_z = \frac{\partial w}{\partial z} \qquad \gamma_{yz} = \frac{\partial v}{\partial z} + \frac{\partial w}{\partial y}$$

Linear Constitutive Equations

The generalized Hooke's Law can be written as

$$\{\sigma\} = [C]\{\varepsilon\} \Rightarrow \begin{Bmatrix} \sigma_x \\ \sigma_y \\ \sigma_z \\ \tau_{xy} \\ \tau_{xz} \\ \tau_{yz} \end{Bmatrix} = \begin{bmatrix} C_{11} & C_{12} & C_{13} & C_{14} & C_{15} & C_{16} \\ C_{21} & C_{22} & C_{23} & C_{24} & C_{25} & C_{26} \\ C_{31} & C_{32} & C_{33} & C_{34} & C_{35} & C_{36} \\ C_{41} & C_{42} & C_{43} & C_{44} & C_{45} & C_{46} \\ C_{51} & C_{52} & C_{53} & C_{54} & C_{55} & C_{56} \\ C_{61} & C_{62} & C_{63} & C_{64} & C_{65} & C_{66} \end{bmatrix} \begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \varepsilon_z \\ \gamma_{xy} \\ \gamma_{xz} \\ \gamma_{yz} \end{Bmatrix}$$

Linear Constitutive Equations

The generalized Hooke's Law can be written as

$$\{\varepsilon\} = [D]\{\sigma\} \Rightarrow \begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \varepsilon_z \\ \gamma_{xy} \\ \gamma_{xz} \\ \gamma_{yz} \end{Bmatrix} = \begin{bmatrix} D_{11} & D_{12} & D_{13} & D_{14} & D_{15} & D_{16} \\ D_{21} & D_{22} & D_{23} & D_{24} & D_{25} & D_{26} \\ D_{31} & D_{32} & D_{33} & D_{34} & D_{35} & D_{36} \\ D_{41} & D_{42} & D_{43} & D_{44} & D_{45} & D_{46} \\ D_{51} & D_{52} & D_{53} & D_{54} & D_{55} & D_{56} \\ D_{61} & D_{62} & D_{63} & D_{64} & D_{65} & D_{66} \end{bmatrix} \begin{Bmatrix} \sigma_x \\ \sigma_y \\ \sigma_z \\ \tau_{xy} \\ \tau_{xz} \\ \tau_{yz} \end{Bmatrix}$$

Linear Constitutive Equations

The [C] and [D] matrices are symmetric and therefore only 21 constants are required to define a material in general

$$\begin{Bmatrix} \sigma_x \\ \sigma_y \\ \sigma_z \\ \tau_{xy} \\ \tau_{xz} \\ \tau_{yz} \end{Bmatrix} = \begin{bmatrix} C_{11} & C_{12} & C_{13} & C_{14} & C_{15} & C_{16} \\ & C_{22} & C_{23} & C_{24} & C_{25} & C_{26} \\ & & C_{33} & C_{34} & C_{35} & C_{36} \\ & & & C_{44} & C_{45} & C_{46} \\ & & & & C_{55} & C_{56} \\ & & & & & C_{66} \end{bmatrix} \begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \varepsilon_z \\ \gamma_{xy} \\ \gamma_{xz} \\ \gamma_{yz} \end{Bmatrix}$$

Linear Constitutive Equations

Certain materials exibit symmetry with respect to certain planes within the body so that the number of material constants can be reduced from the general number of 21 material constants required for the anisotropic case.

Linear Constitutive Equations

For instance, an orthotropic material can be expressed using only 9 constants

$$\begin{Bmatrix} \sigma_x \\ \sigma_y \\ \sigma_z \\ \tau_{xy} \\ \tau_{xz} \\ \tau_{yz} \end{Bmatrix} = \begin{bmatrix} C_{11} & C_{12} & C_{13} & 0 & 0 & 0 \\ & C_{22} & C_{23} & 0 & 0 & 0 \\ & & C_{33} & 0 & 0 & 0 \\ & & & C_{44} & 0 & 0 \\ & & & & C_{55} & 0 \\ & & & & & C_{66} \end{bmatrix} \begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \varepsilon_z \\ \gamma_{xy} \\ \gamma_{xz} \\ \gamma_{yz} \end{Bmatrix}$$

Linear Constitutive Equations

The stress-strain relations for an orthotropic material may be written in terms of Young's Modulus and Poisson's Ratio as

$$\varepsilon_x = \frac{\sigma_x}{E_x} - \frac{\upsilon_{yx}}{E_y}\sigma_y - \frac{\upsilon_{zx}}{E_z}\sigma_z$$

$$\varepsilon_y = \frac{\sigma_y}{E_y} - \frac{\upsilon_{xy}}{E_x}\sigma_x - \frac{\upsilon_{zy}}{E_z}\sigma_z$$

$$\varepsilon_z = \frac{\sigma_z}{E_z} - \frac{\upsilon_{xz}}{E_x}\sigma_x - \frac{\upsilon_{yz}}{E_y}\sigma_y$$

$$\gamma_{xy} = \frac{\tau_{xy}}{G_{xy}}, \gamma_{yz} = \frac{\tau_{yz}}{G_{yz}}, \gamma_{zx} = \frac{\tau_{zx}}{G_{zx}}$$

Linear Isotropic Elasticity

Simpliest form of the generalized Hooke's Law where the material is linear, elastic and isotropic

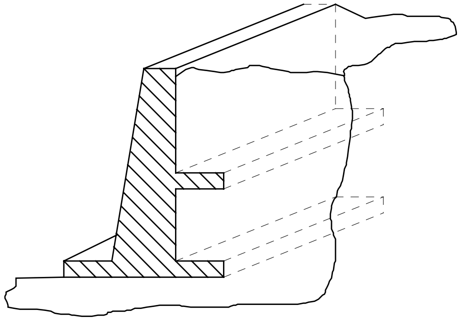
$$
\begin{Bmatrix} \sigma_x \\ \sigma_y \\ \sigma_z \\ \tau_{xy} \\ \tau_{xz} \\ \tau_{yz} \end{Bmatrix} = \frac{E}{(1+\nu)(1-2\nu)} \begin{bmatrix} (1-\nu) & \nu & \nu & & & \\ \nu & (1-\nu) & \nu & & & \\ \nu & \nu & (1-\nu) & & & \\ & & & \left(\frac{1-2\nu}{2}\right) & & \\ & & & & \left(\frac{1-2\nu}{2}\right) & \\ & & & & & \left(\frac{1-2\nu}{2}\right) \end{bmatrix} \begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \varepsilon_z \\ \gamma_{xy} \\ \gamma_{xz} \\ \gamma_{yz} \end{Bmatrix}
$$

$$
\begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \varepsilon_z \\ \gamma_{xy} \\ \gamma_{xz} \\ \gamma_{yz} \end{Bmatrix} = \begin{bmatrix} 1/E & -\nu/E & -\nu/E & & & \\ -\nu/E & 1/E & -\nu/E & & & \\ -\nu/E & -\nu/E & 1/E & & & \\ & & & \frac{2(1+\upsilon)}{E} & & \\ & & & & \frac{2(1+\upsilon)}{E} & \\ & & & & & \frac{2(1+\upsilon)}{E} \end{bmatrix} \begin{Bmatrix} \sigma_x \\ \sigma_y \\ \sigma_z \\ \tau_{xy} \\ \tau_{xz} \\ \tau_{yz} \end{Bmatrix}
$$

Two Dimensional Elasticity - Plane Strain

Typically used for very long bodies where the loading and boundary conditions do not vary in the longitudinal direction and that there are no displacements in the longitudinal direction. For this case,

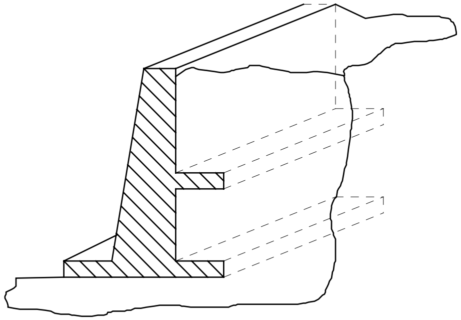
$$\varepsilon_x = \frac{\partial u}{\partial x}, \quad \varepsilon_y = \frac{\partial v}{\partial y}, \quad \gamma_{xy} = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}$$



Two Dimensional Elasticity - Plane Strain

The constitutive law reduces to

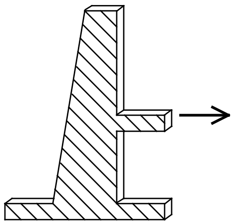
$$\begin{Bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{Bmatrix} = \frac{E}{(1+\nu)(1-2\nu)} \begin{bmatrix} (1-\nu) & \nu & \\ \nu & (1-\nu) & \\ & & \left(\frac{1-2\nu}{2}\right) \end{bmatrix} \begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \end{Bmatrix}$$



Two Dimensional Elasticity - Plane Stress

A condition of plane stress exists when the longitudinal direction is very small in comparison to the other two directions with only inplane loading considered. For this case,

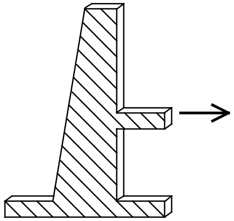
$$\varepsilon_z = -\frac{\upsilon}{1-\upsilon}(\varepsilon_x + \varepsilon_y); \quad \tau_{yz} = \tau_{zx} = 0; \quad \sigma_z = 0$$



Two Dimensional Elasticity - Plane Stress

The constitutive law reduces to

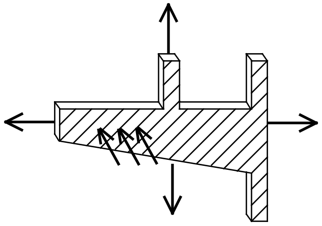
$$\begin{Bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{Bmatrix} = \frac{E}{\left(1-\nu^2\right)} \begin{bmatrix} 1 & \nu & \\ \nu & 1 & \\ & & \left(\frac{1-\nu}{2}\right) \end{bmatrix} \begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \end{Bmatrix}$$



Elementary Plate Theory

When the thickness is small to the other two dimensions assumed displacements can be approximated by

$$u = u^{o}(x, y) - z\frac{\partial w}{\partial x}, \quad v = v^{o}(x, y) - z\frac{\partial w}{\partial y}, \quad w = w^{o}(x, y)$$



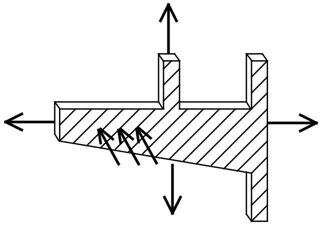
Elementary Plate Theory

The strain-displacement equations becomes

$$\varepsilon_x = \frac{\partial u^o}{\partial x} - z\frac{\partial^2 w}{\partial x^2} = \varepsilon_x - z\frac{\partial^2 w}{\partial x^2}$$

$$\varepsilon_y = \frac{\partial v^o}{\partial y} - z\frac{\partial^2 w}{\partial y^2} = \varepsilon_y - z\frac{\partial^2 w}{\partial y^2}$$

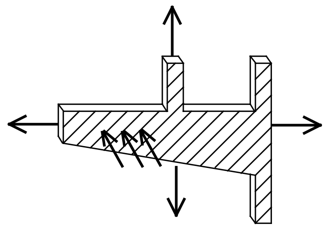
$$\gamma_{xy} = \frac{\partial u^o}{\partial y} + \frac{\partial v^o}{\partial x} - 2z\frac{\partial^2 w}{\partial x \partial y} = \gamma_{xy}^o - 2z\frac{\partial^2 w}{\partial x \partial y}$$



Convention is to express the stress-strain relations as

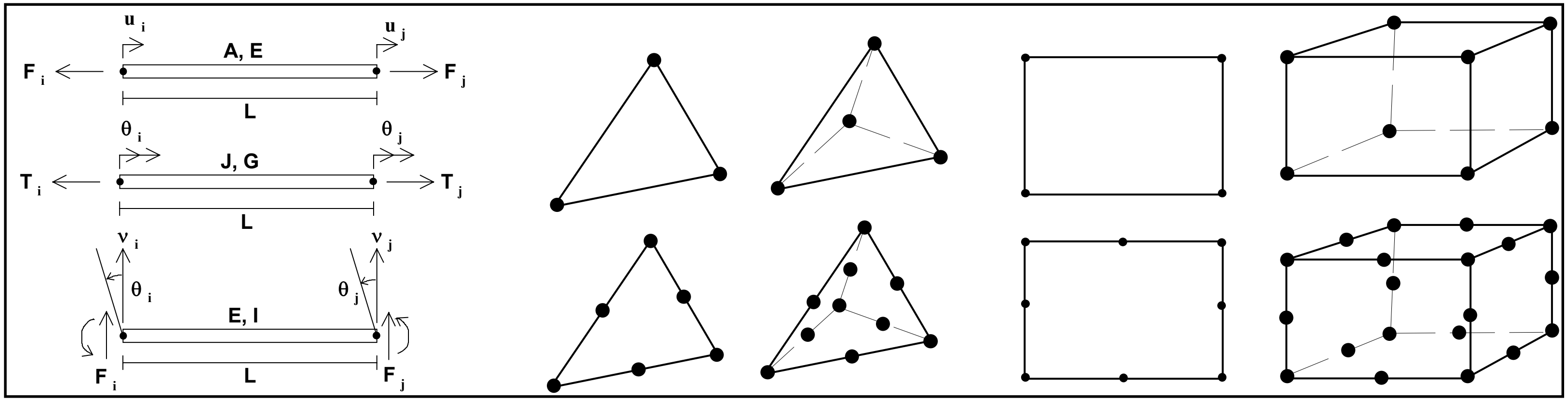
$$N_x = \int_{-t/2}^{t/2} \sigma_x dz, N_y = \int_{-t/2}^{t/2} \sigma_y dz, N_{xy} = \int_{-t/2}^{t/2} \tau_{xy} dz, Q_x = \int_{-t/2}^{t/2} \tau_{xz} dz, Q_y = \int_{-t/2}^{t/2} \tau_{yz} dz$$

$$M_x = -\int_{-t/2}^{t/2} z\sigma_x dz, M_y = -\int_{-t/2}^{t/2} z\sigma_y dz, M_{xy} = \int_{-t/2}^{t/2} \tau_{xy} z dz$$



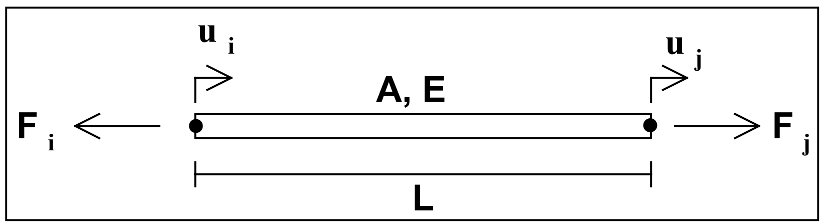
Finite Element Types

An assortment of different element types exist



Finite Element Types - TRUSS

Slender element (length>>area) which supports only tension or compression along its length; essentially a 1D spring



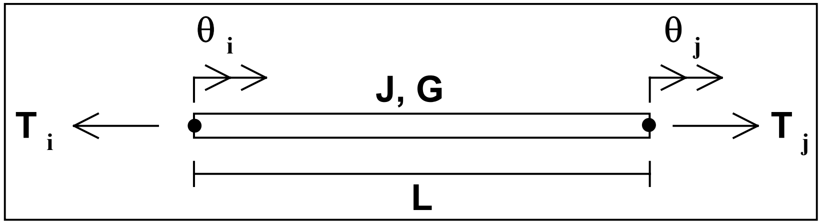
The truss strain is defined as $\varepsilon = {du}/{dx}$

The stiffness and lumped/consistent mass matrices

$$[k] = \frac{AE}{L}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}; \quad [m_l] = \rho AL\begin{bmatrix} 1/2 & \\ & 1/2 \end{bmatrix}; \quad [m_c] = \rho AL\begin{bmatrix} 1/3 & 1/6 \\ 1/6 & 1/3 \end{bmatrix}$$

Finite Element Types - TRUSS

Similar to truss but supports torsion



The torsional stiffness is

$$[k_t] = \frac{JG}{L} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

Finite Element Types - BEAM

Slender element whose length is much greater that its transverse dimension which supports lateral loads which cause flexural bending

Beam assumptions are

- ***constant cross section***
- ***cross section small compared to length***
- ***stress and strain vary linearly across section depth***

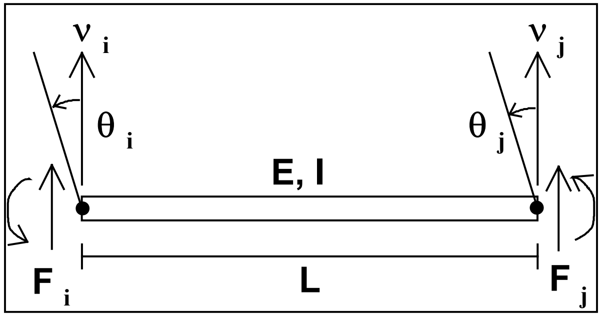
Finite Element Types - BEAM

The beam elastic curvature due to lateral loading is satisfied by

$$EId^4\upsilon / dx^4 = q$$

The longitudinal strain is proportional to the distance from the neutral axis and second derivative of the elastic curvature given as

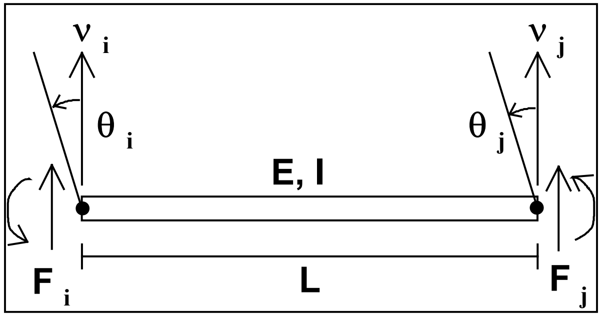
$$\varepsilon = yd^2\upsilon / dx^2$$



Finite Element Types - BEAM

The stiffness and consistent mass matrices are

$$[k] = \frac{EI}{L^3}\begin{bmatrix} 12 & 6L & -12 & 6L \\ 6L & 4L^2 & -6L & 2L^2 \\ -12 & -6L & 12 & -6L \\ 6L & 2L^2 & -6L & 4L \end{bmatrix} \qquad [m] = \frac{\rho AL}{420}\begin{bmatrix} 156 & -22L & 54 & 13L \\ -22L & 4L^2 & -13L & -3L^2 \\ 54 & -13L & 156 & 22L \\ 13L & -3L^2 & 22L & 4L^2 \end{bmatrix}$$

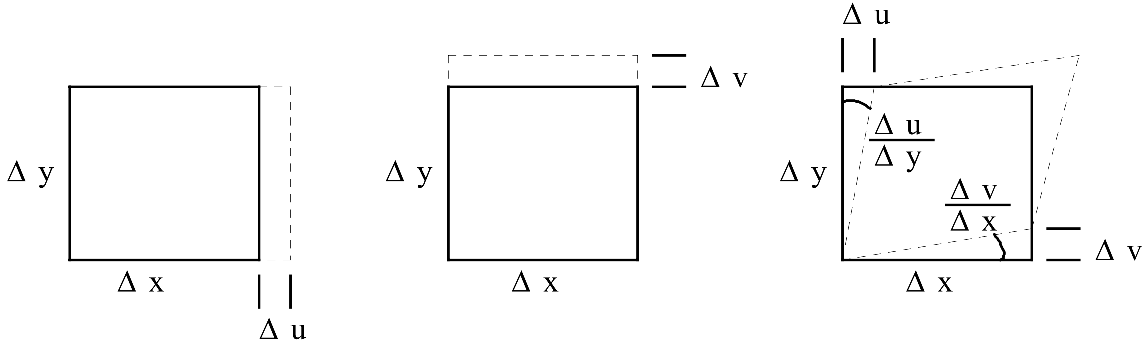


Finite Element Types - BEAM

The full beam stiffness matrix can be assembled using the truss, torsion and two planar beam elements

$$
\begin{bmatrix}
\frac{AE}{L} & & & & & & \frac{-AE}{L} & & & & & \\
& \frac{12EI_Z}{L^3} & & & & \frac{-6EI_Z}{L^2} & & \frac{-12EI_Z}{L^3} & & & & \frac{-6EI_Z}{L^2} \\
& & \frac{12EI_Y}{L^3} & & \frac{-6EI_Y}{L^2} & & & & \frac{12EI_Y}{L^3} & & \frac{-6EI_Y}{L^2} & \\
& & & \frac{JG}{L} & & & & & & \frac{-JG}{L} & & \\
& & \frac{-6EI_Y}{L^2} & & \frac{4EI_Y}{L} & & & & \frac{6EI_Y}{L^2} & & \frac{2EI_Y}{L} & \\
& \frac{6EI_Z}{L^2} & & & & \frac{4EI_Z}{L} & & \frac{-6EI_Z}{L^2} & & & & \frac{2EI_Z}{L} \\
\frac{-AE}{L} & & & & & & \frac{AE}{L} & & & & & \\
& \frac{-12EI_Z}{L^3} & & & & \frac{-6EI_Z}{L^2} & & \frac{12EI_Z}{L^3} & & & & \frac{-6EI_Z}{L^2} \\
& & \frac{-12EI_Y}{L^3} & & \frac{6EI_Y}{L^2} & & & & \frac{12EI_Y}{L^3} & & \frac{6EI_Y}{L^2} & \\
& & & \frac{-JG}{L} & & & & & & \frac{JG}{L} & & \\
& & \frac{-6EI_Y}{L^2} & & \frac{2EI_Y}{L} & & & & \frac{6EI_Y}{L^2} & & \frac{4EI_Y}{L} & \\
& \frac{6EI_Z}{L^2} & & & & \frac{2EI_Z}{L} & & \frac{-6EI_Z}{L^2} & & & & \frac{4EI_Z}{L}
\end{bmatrix}
$$

Element whose geometry definition lies in a plane and applied loads also lie in the same plane



Stress-strain relations are

$$\varepsilon_x = \frac{\partial u}{\partial x}, \quad \varepsilon_y = \frac{\partial v}{\partial y}, \quad \gamma_{xy} = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}$$

$$\begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \end{Bmatrix} = \begin{bmatrix} \partial/\partial x & 0 \\ 0 & \partial/\partial y \\ \partial/\partial y & \partial/\partial x \end{bmatrix} \begin{Bmatrix} u \\ v \end{Bmatrix} \Rightarrow \varepsilon = \partial u$$

Finite Element Types – PLANE STRAIN

Displacements in the finite element are interpolated from nodal displacements using the element shape function

$$\begin{Bmatrix} u \\ v \end{Bmatrix} = \begin{bmatrix} N_1 & 0 & N_2 & 0 & \cdots \\ 0 & N_1 & 0 & N_2 & \cdots \end{bmatrix} \begin{Bmatrix} u_1 \\ v_1 \\ u_2 \\ v_2 \\ \vdots \end{Bmatrix}$$

Finite Element Types – PLANE STRAIN

Strain displacement matrix is

$$\varepsilon = \partial u \Rightarrow \begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \end{Bmatrix} = \partial N \begin{Bmatrix} u_1 \\ v_1 \\ u_2 \\ v_2 \\ \vdots \end{Bmatrix} = B \begin{Bmatrix} u_1 \\ v_1 \\ u_2 \\ v_2 \\ \vdots \end{Bmatrix} \text{where} B = \partial N$$

The element matrices are then given by

$$[M] = \iiint_V [N] \rho [N]^T \partial V$$

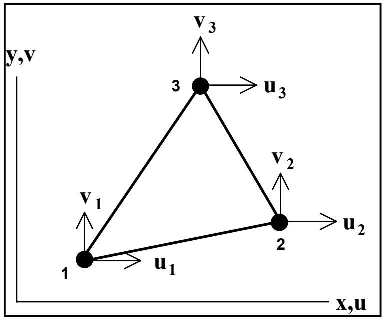
$$[K] = \iiint_V [B]^T [C][B] \partial V$$

Finite Element Types – PLANE STRAIN

__Constant Strain Triangle__ ***- Probably one of the simplest and first of finite elements formulated***

The displacement field is given by

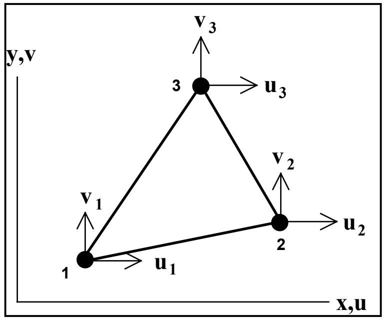
$$u = \beta_1 + \beta_2 x + \beta_3 y$$
$$v = \beta_4 + \beta_5 x + \beta_6 y$$



The resulting strain field is given by

$$\varepsilon_x = \beta_2, \quad \varepsilon_y = \beta_6, \quad \gamma_{xy} = \beta_3 + \beta_5$$

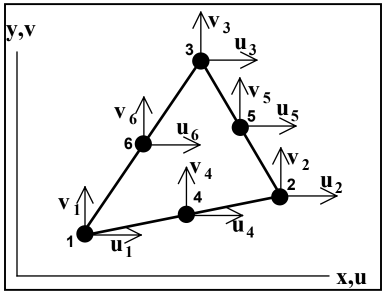
Constant Strain Triangle – The strain field can be expressed in terms of shape functions as



$$\begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \end{Bmatrix} = \frac{1}{2A} \begin{bmatrix} y_2 - y_3 & 0 & y_3 - y_1 & 0 & y_1 - y_2 & 0 \\ 0 & x_3 - x_2 & 0 & x_1 - x_3 & 0 & x_2 - x_1 \\ x_3 - x_2 & y_2 - y_3 & x_1 - x_3 & y_3 - y_1 & x_2 - x_1 & y_1 - y_2 \end{bmatrix} \begin{Bmatrix} u_1 \\ v_1 \\ u_2 \\ v_2 \\ u_3 \\ v_3 \end{Bmatrix}$$

Finite Element Types – PLANE STRAIN

Linear Strain Triangle ***- Adding midside nodes to the constant strain triangle provides for a quadratic displacement field***



$$u = \beta_1 + \beta_2 x + \beta_3 y + \beta_4 x^2 + \beta_5 xy + \beta_6 y^2$$

$$v = \beta_7 + \beta_8 x + \beta_9 y + \beta_{10} x^2 + \beta_{11} xy + \beta_{12} y^2$$

and a resulting strain field is given by

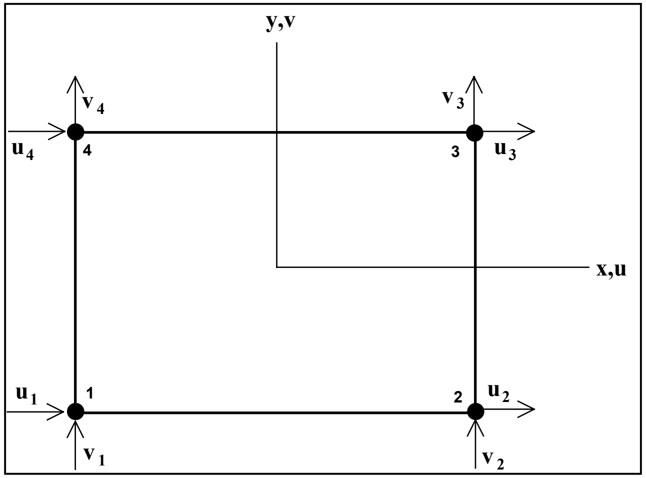
$$\varepsilon_x = \beta_2 + 2\beta_4 x + \beta_5 y$$

$$\varepsilon_y = \beta_9 + \beta_{11} x + 2\beta_{12} y$$

$$\gamma_{xy} = \beta_3 + \beta_8 + (\beta_5 + 2\beta_{10})x + (2\beta_6 + \beta_{11})y$$

Finite Element Types – PLANE STRAIN

***Bilinear Quadrilateral* - Extending from the triangular element to a 4 noded quadrilateral provides for a bilinear displacement field**



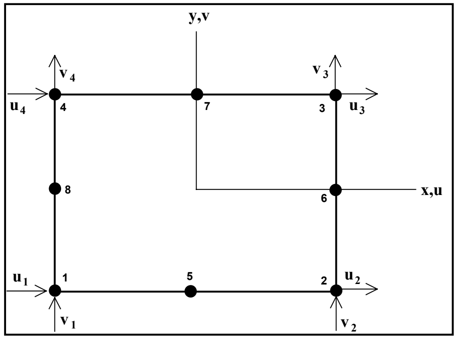
$$u = \beta_1 + \beta_2 x + \beta_3 y + \beta_4 xy$$
$$v = \beta_5 + \beta_6 x + \beta_7 y + \beta_8 xy$$

and a resulting strain field is given by

$$\varepsilon_x = \beta_2 + \beta_4 y$$
$$\varepsilon_y = \beta_7 + \beta_8 x$$
$$\gamma_{xy} = \beta_3 + \beta_6 + \beta_4 x + \beta_8 y$$

Finite Element Types – PLANE STRAIN

<u>Bilinear Quadrilateral</u> - Extending from the 4 noded to 8 noded quadrilateral provides for a bilinear displacement field



$$u = \beta_1 + \beta_2 x + \beta_3 y + \beta_4 x^2 + \beta_5 xy + \beta_6 y^2 + \beta_7 x^2 y + \beta_8 xy^2$$

$$v = \beta_9 + \beta_{10} x + \beta_{11} y + \beta_{12} x^2 + \beta_{13} xy + \beta_{14} y^2 + \beta_{15} x^2 y + \beta_{16} xy^2$$

and a resulting strain field is given by

$$\varepsilon_x = \beta_2 + 2\beta_4 x + \beta_5 y + 2\beta_7 xy + \beta_8 y^2 ; \varepsilon_y = \beta_{11} + \beta_{13} x + 2\beta_{14} y + 2\beta_{15} x^2 + 2\beta_{16} xy$$

$$\gamma_{xy} = \beta_3 + \beta_{10} + (\beta_5 + 2\beta_{12})x + (2\beta_6 + \beta_{13})y + \beta_7 x^2 + 2(\beta_8 + \beta_{15})xy + \beta_{16} y^2$$

Finite Element Types – PLANE STRAIN

Plane Strain Constant Strain Triangle

* good in region where little strain gradient exists

* otherwise element does not behave very well

* poor element for bending applications

* not considered a good general element

Plane Strain Linear Strain Triangle

* better than the constant strain triangle

* but not a particularly good general element

Finite Element Types – PLANE STRAIN

Plane Strain Bilinear Quadrilateral

* does not exactly model a cantilever beam

* can not model a state of pure bending very well due to shear effects

* very stiff in bending

* bending stiffness improved through incompatible displacement effects

Plane Strain Quadratic Quadrilateral

* good for modeling all states of constant strain

* good for modeling pure bending using rectangular elements

Finite Element Types – 3D SOLID

A general 3 dimensional solid that is relativel unrestricted with respect to shape, loading, material properties, boundary conditions, etc.

Displacements are interpolated from nodal displacements from shape function

$$\begin{Bmatrix} u \\ v \\ w \end{Bmatrix} = \begin{bmatrix} N_1 & 0 & 0 & N_2 & 0 & 0 & \cdots \\ 0 & N_1 & 0 & 0 & N_2 & 0 & \cdots \\ 0 & 0 & N_1 & 0 & 0 & N_2 & \cdots \end{bmatrix} \begin{Bmatrix} u_1 \\ v_1 \\ w_1 \\ u_2 \\ v_2 \\ w_2 \\ \vdots \end{Bmatrix}$$

Finite Element Types – 3D SOLID

Strain displacement matrix is

$$\varepsilon = \partial u \Rightarrow \begin{Bmatrix} \varepsilon_x \\ \vdots \\ \gamma_{xy} \\ \vdots \end{Bmatrix} = \partial N \begin{Bmatrix} u_1 \\ v_1 \\ u_2 \\ v_2 \\ \vdots \end{Bmatrix} = B \begin{Bmatrix} u_1 \\ v_1 \\ u_2 \\ v_2 \\ \vdots \end{Bmatrix} \text{where} B = \partial N$$

The element matrices are then given by

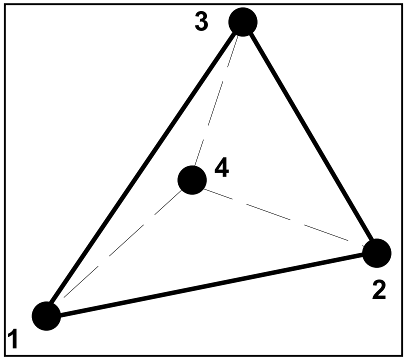
$$[M] = \iiint_V [N] \rho [N]^T \partial V$$

$$[K] = \iiint_V [B]^T [C][B] \partial V$$

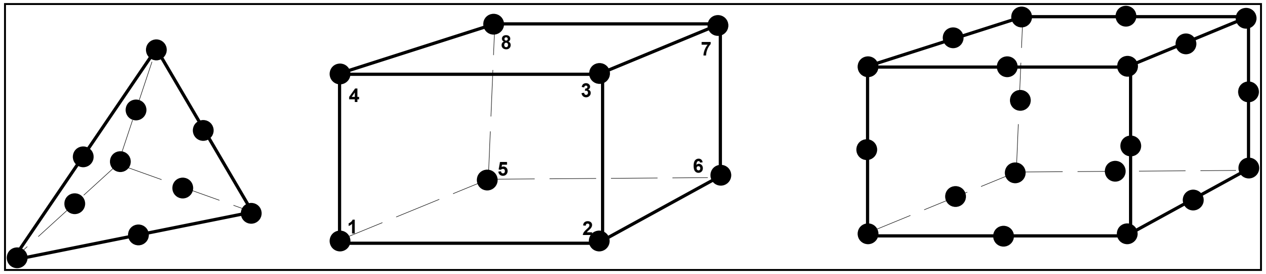
Finite Element Types – 3D SOLID

Constant Strain Tetrahedron ***- Extension of constant strain triangle plain strain element to 3D***

$$u = \beta_1 + \beta_2 x + \beta_3 y + \beta_4 z$$
$$v = \beta_5 + \beta_6 x + \beta_7 y + \beta_8 z$$
$$w = \beta_9 + \beta_{10} x + \beta 11 y + \beta_{12} z$$



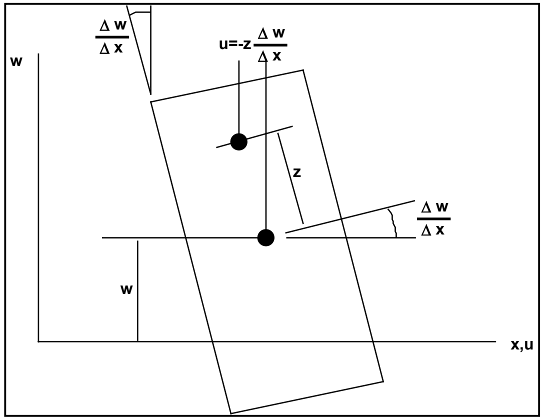
Also, Linear Tetrahedron, Trilinear Tetrahedron and Quadratic Tetrahedron



Finite Element Types – PLATE

Classical Thin Plate – Kirchoff Thin Plate

Very similar to the beam in that flexure occurs - but in two directions. Geometry lies in the plane with loads acting normal to the plane. A two dimensional state of stress exists similar to that of plane stress with the exception that there is a variation of tension to compression across the plate thickness.



Finite Element Types – PLATE

Classical Thin Plate – Kirchoff Thin Plate

Governing equations are

$$u = -z\partial w / \partial x; \quad v = -z\partial w / \partial y$$

The strain is given by

$$\varepsilon_x = \partial u / \partial x = -z\partial^2 w / \partial x^2 ; \varepsilon_y = \partial v / \partial y = -z\partial^2 w / \partial y^2$$

$$\gamma_{xy} = \partial u / \partial y + \partial v / \partial x = -2z\partial^2 w / \partial x \partial y$$

Finite Element Types – PLATE

Classical Thin Plate – Kirchoff Thin Plate

For thin plate theory, the governing partial differential equation is given by

$$\nabla^4 w = q / \left(Et^3 / 12\left(1 - \upsilon^2\right)\right)$$

For an isotropic material,

$$\begin{Bmatrix} \sigma_x \\ \sigma_y \end{Bmatrix} = -z \frac{E}{1-\upsilon^2} \begin{bmatrix} 1 & \upsilon \\ \upsilon & 1 \end{bmatrix} \begin{Bmatrix} \partial^2 w / \partial^2 x \\ \partial^2 w / \partial^2 y \end{Bmatrix} ; \tau_{xy} = -2zG \frac{\partial^2 w}{\partial x \partial y}$$

Finite Element Types – PLATE

Classical Thin Plate – Kirchoff Thin Plate

These stresses are very much like those found in simple beam bending. Flexural stresses vary linearly through the thickness while transverse shear stresses vary quadratically. The plate moments are given by

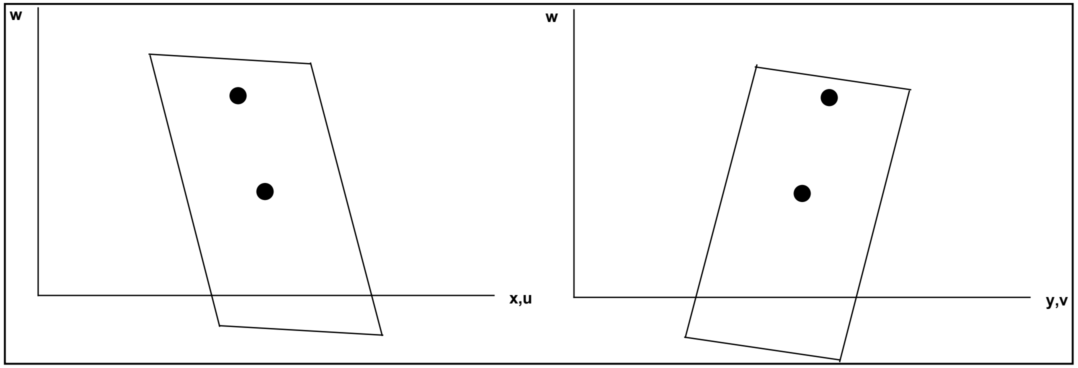
$$M_x = \int_{-t/2}^{+t/2} \sigma_x z\,dz; \quad M_y = \int_{-t/2}^{+t/2} \sigma_y z\,dz; \quad M_{xy} = \int_{-t/2}^{+t/2} \tau_{xy} z\,dz$$

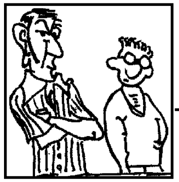
- *transverse shear deformation is neglected*
- *transverse shear can be significant in thick plates (10:1 ratio applies)*
- *cross section is not distorted by deformation*

Finite Element Types – PLATE

Thick Plate - Mindlin Plate

Extension of thin plate theory to account for transverse shear deformations.





Finite Element Types – PLATE

Thick Plate - Mindlin Plate

Combining the displacements for thin plate with those shown constitute the thick plate element.

$$u = z\theta_y \qquad v = -z\theta_x$$

$$\varepsilon_x = z\frac{\partial \theta_y}{\partial x} \qquad \varepsilon_y = -z\frac{\partial \theta_x}{\partial y}$$

$$\gamma_{xy} = z\left(\frac{\partial \theta_y}{\partial y} - \frac{\partial \theta_x}{\partial x}\right)$$

$$\gamma_{yz} = \frac{\partial w}{\partial y} - \theta_x$$

$$\gamma_{yzx} = \frac{\partial w}{\partial x} + \theta_y$$

* transverse shear deformation is included

* cross section does not remain the same due to shear deformation