Reconfigurable Computing

The CORDIC algorithm

"Simplicity is the ultimate sophistication" - daVinci

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Introduction

- COordinate Rotation DIgital Computer
- Efficient method to compute sin, cos, tan, $\sin^{-1}$, $\cos^{-1}$, $\tan^{-1}$, multiplication, division, $\sqrt{}$, sinh, cosh, tanh
  - Only uses shifts, additions and a very small lookup table

Theory

Rotations

Rotating [x y] by ϕ

$$
\begin{aligned}
x' &= x\cos(\phi) - y\sin(\phi) \\
y' &= y\cos(\phi) + x\sin(\phi).
\end{aligned}
$$

Rearranging

$$
\begin{aligned}
x' &= \cos(\phi)(x - y\tan(\phi)) \\
y' &= \cos(\phi)(y + x\tan(\phi)).
\end{aligned}
$$

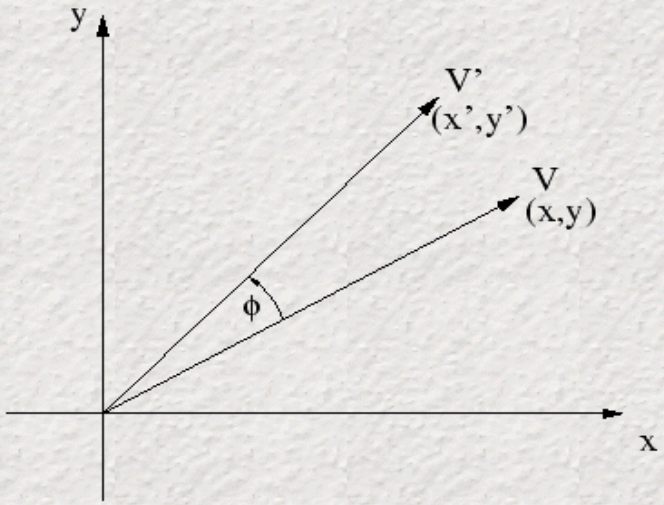

Figure 1.2: *Rotation of a vector V by the angle ϕ*

Key idea

$$
\begin{aligned}
x' &= \cos(\phi)(x - y\tan(\phi)) \\
y' &= \cos(\phi)(y + x\tan(\phi)).
\end{aligned}
$$

Can compute rotation ϕ in steps where each step is of size

$$\tan(\phi) = \pm 2^{-i}.$$

Iterative rotations

$$x_{i+1} = K_i(x_i - (y_i d_i 2^{-i}))$$
$$y_{i+1} = K_i(y_i + (x_i d_i 2^{-i})).$$

where $d_i = \pm 1$ and $K_i = \cos(\tan^{-1} 2^{-i})$

Choose d_i so that after n iterations the rotated angle is ϕ

K_i values

$$\cos(\tan^{-1} 2^{-i}) = 1/\sqrt{(1 + 2^{-2i})}.$$

$$K = \prod_{i=1}^{n} \frac{1}{\sqrt{(1 + 2^{-2i})}},$$

As n → ∞, K → 0.6073 (constant factor which needs to be corrected for)

Actually it's easier to omit it and fix it later!

d_i decision (rotation mode)

Z_i is introduced to keep track of the angle that has been rotated (z0 = ϕ)

$$x_{i+1} = x_i - (y_i d_i 2^{-i})$$

$$y_{i+1} = y_i + (x_i d_i 2^{-i})$$

$$z_{i+1} = z_i - d_i \tan^{-1}(2^{-i})$$

$$d_i = \begin{cases} -1 & if\ z_i < 0 \\ +1 & otherwise \end{cases}$$

Notice we dropped the K! Rotated value is hence (Kx_n,Ky_n)

After n iterations

$$
\begin{aligned}
x_n &= \frac{1}{K}(x_0 cos(z_0) - y_0 sin(z_0)) \\
y_n &= \frac{1}{K}(y_0 cos(z_0) + x_0 sin(z_0)) \\
z_n &\approx 0.
\end{aligned}
$$

Question: What is the procedure to compute sin and cos?

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Computing sin(a) and cos(a)

1. Initialize (x,y,z)=(1,0,a)
2. Iterate through cordic
3. cos(a)=Kx and sin(a)=Ky

An easier way for this example is to change

step 1 to (x,y,z)=(k,0,a)

Sin(75)

0: xi=1.000000 yi=0.000000 zi=1.308997 k=1.000000 kx=1.000000 ky=0.000000

1: xi=1.000000 yi=1.000000 zi=0.523599 k=0.707107 kx=0.707107 ky=0.707107

2: xi=0.500000 yi=1.500000 zi=0.059951 k=0.632456 kx=0.316228 ky=0.948683

3: xi=0.125000 yi=1.625000 zi=-0.185027 k=0.613572 kx=0.076696 ky=0.997054

4: xi=0.328125 yi=1.609375 zi=-0.060673 k=0.608834 kx=0.199774 ky=0.979842

5: xi=0.428711 yi=1.588867 zi=0.001746 k=0.607648 kx=0.260505 ky=0.965472

6: xi=0.379059 yi=1.602264 zi=-0.029494 k=0.607352 kx=0.230222 ky=0.973138

7: xi=0.404094 yi=1.596342 zi=-0.013870 k=0.607278 kx=0.245397 ky=0.969423

d_i decision (vectoring mode)

$$d_i = \begin{cases} +1 & if\ y_i < 0 \\ -1 & otherwise. \end{cases}$$

$$\begin{aligned} x_n &= \frac{1}{K}\sqrt{(x_0^2 + y_0^2)} \\ y_n &\approx 0 \\ z_n &= z_0 + \tan^{-1}(y_0/x_0). \end{aligned}$$

- y_n minimized use to compute $\tan^{-1}$ and magnitude

Linear functions instead of trig

$$
\begin{aligned}
x_{i+1} &= x_i - 0(y_i d_i 2^{-i}) = x_i \\
y_{i+1} &= y_i + (x_i d_i 2^{-i}) \\
z_{i+1} &= z_i - d_i 2^{-i}
\end{aligned}
$$

$$
d_i = \begin{cases} -1 & if\ z_i < 0 \\ +1 & otherwise. \end{cases}
$$

After n iterations

$$
\begin{aligned}
x_n &= x_0 \\
y_n &= y_0 + x_0 z_0 \\
z_n &= 0,
\end{aligned}
$$

No need for K_n correction.

Division

$$d_i = \begin{cases} +1 & if\ y_i < 0 \\ -1 & otherwise. \end{cases}$$

$$x_n = x_0$$

$$y_n = y_0$$

$$z_n = z_0 - y_0/x_0.$$

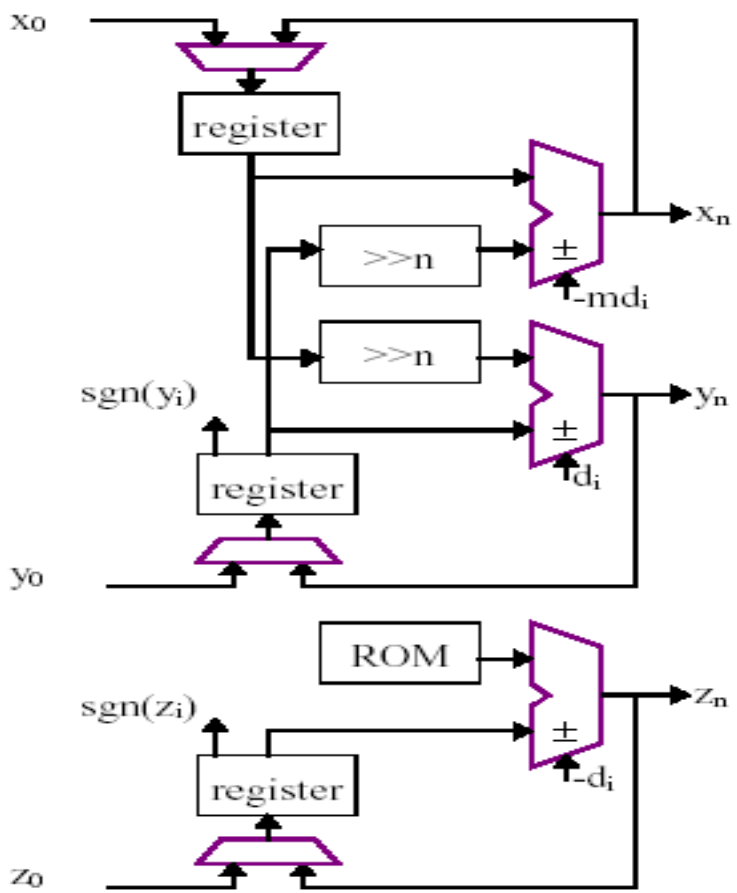
No need for K_n correction.

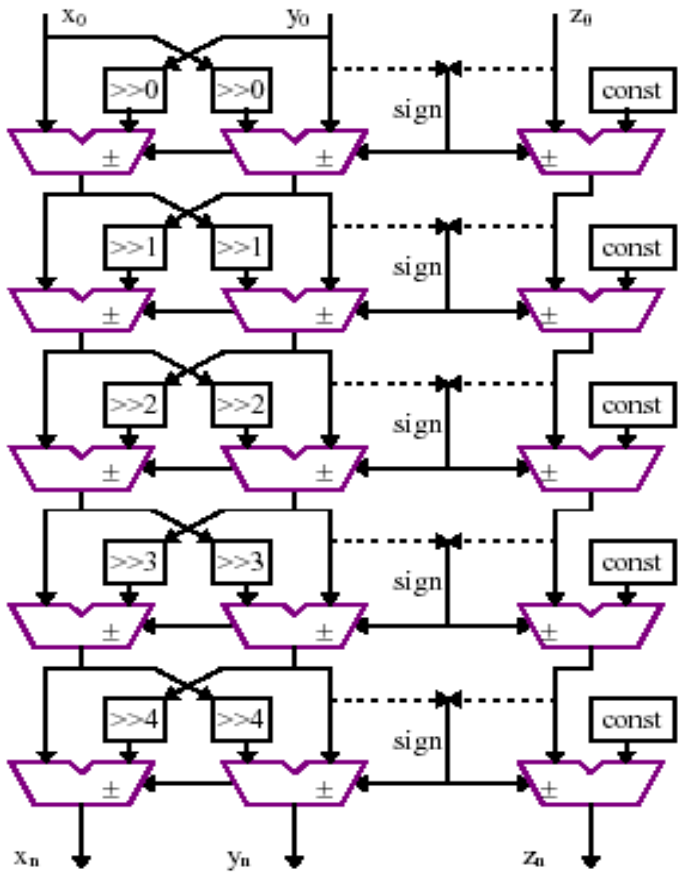
Hyperbolic functions

- Similarly, can get cosh and sinh using $\tanh^{-1}$ instead of $\tan^{-1}$
- Can also get ln and exp easily

Implementation

Andraka's iterative and unrolled cordic structure





Implementation

- Can develop generalized cordic processors which can compute many different functions using similar hardware
- Implementations can be bit serial and/or pipelined as well

Precision

- Need n iterations for n bits
- Converges for $-99.7 \leq z \leq 99.7$ (sum of all the angles $\tan^{-1}(2^{-i})$, $i = 0 .. n$)
  - must convert to this range first

Conclusion

- CORDIC algorithms are an efficient method to compute many different functions
- Low area, high speed
- Used in calculators, DSPs, math coprocessors and supercomputers.

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References

- Ray Andraka, “A survey of CORDIC algorithms for FPGAs”, FPGA '98. Proceedings of the 1998 ACM/SIGDA sixth international symposium on Field programmable gate arrays, Feb. 22-24, 1998, Monterey, CA. pp191-200 (http://www.andraka.com/cordic.htm)

Review Question

- Calculate $\sqrt{2}/K$ using the CORDIC algorithm (4 iterations)
- Hint: use vectoring mode