The Systematic Descent from Order to Chaos

COURTNEY DARVILLE

The Quadratic Iterator

$$x_{n+1} = f_a(x_n), \quad f_a(x) = ax(1-x)$$

$$x_0 \in [0,1], a \in [1,4]$$

Final State Diagrams

A method for examining the long-term behaviour of a system…

1. Choose an initial value at random

2. Take 600 iterations

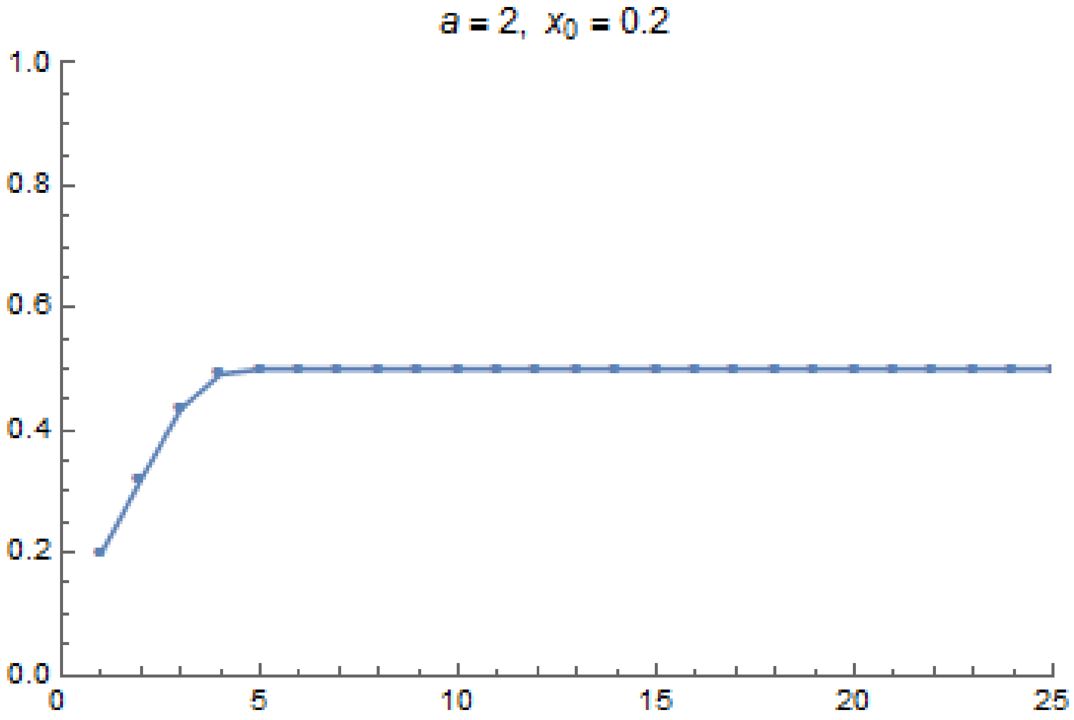
3. Drop the first 500 iterations

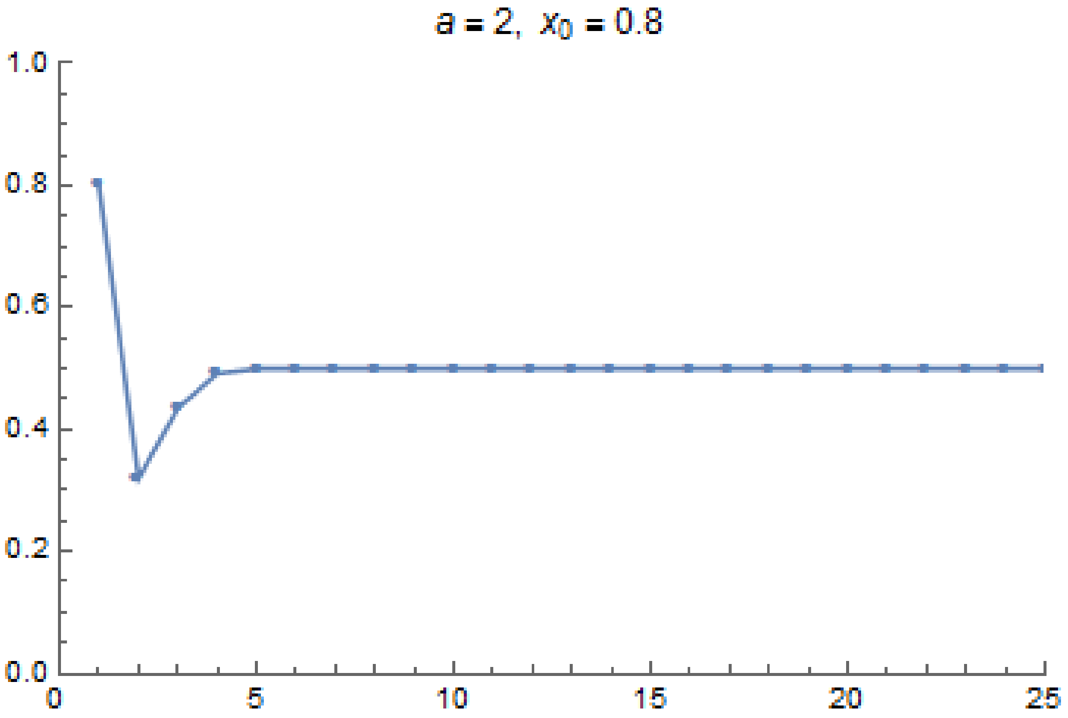
4. Plot the remaining iterations

We can define the attractor, $A(a)$

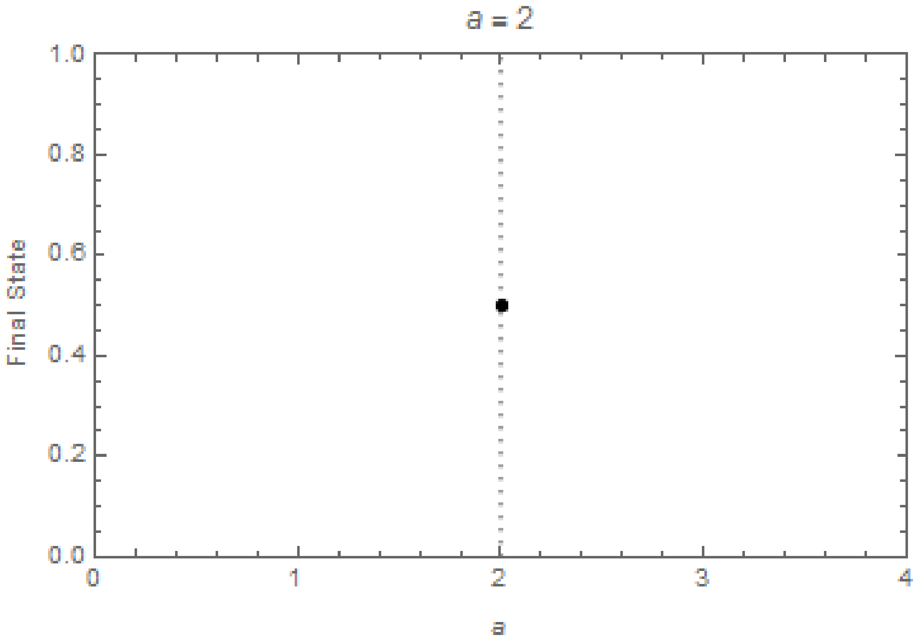
"A set of numerical values towards which the system tends to evolve, for a wide variety of starting conditions of the system" - http://en.wikipedia.org/wiki/Attractor

Final State: A Single Point

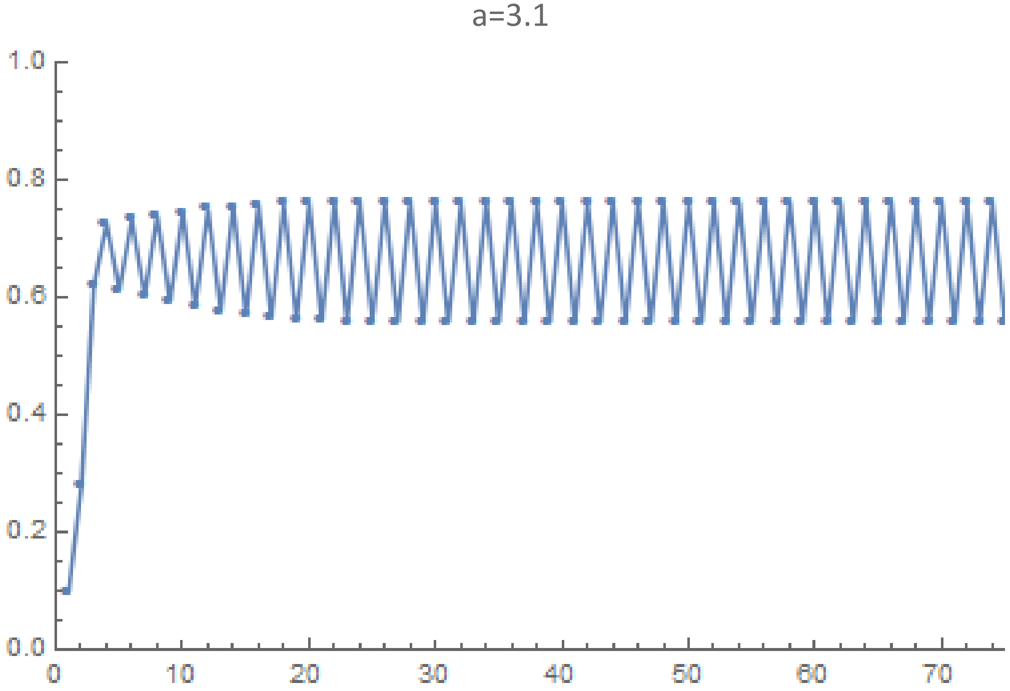


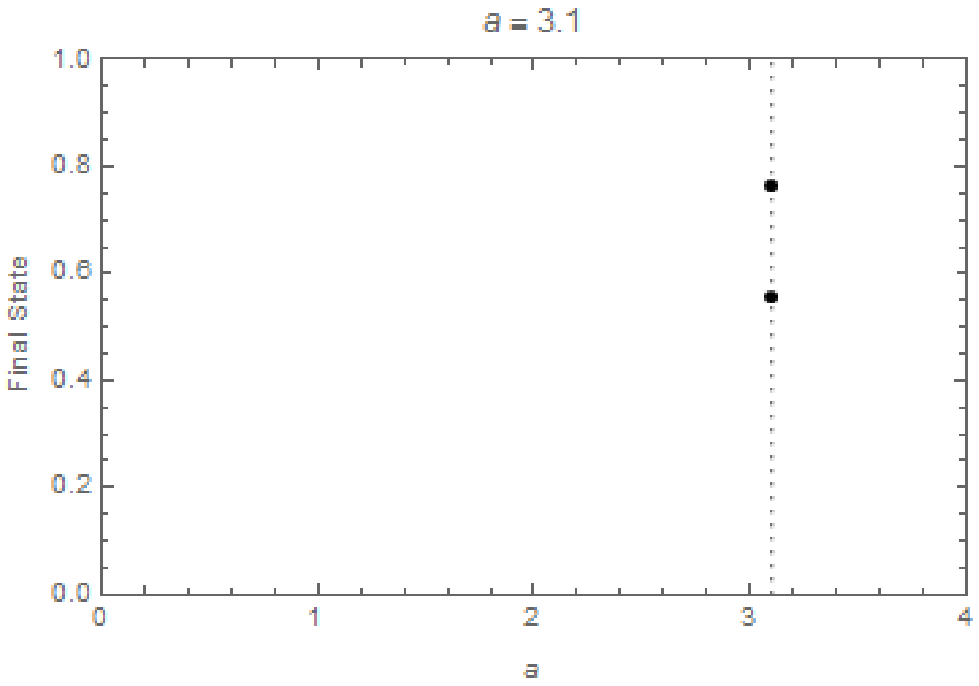


Final State: A Single Point



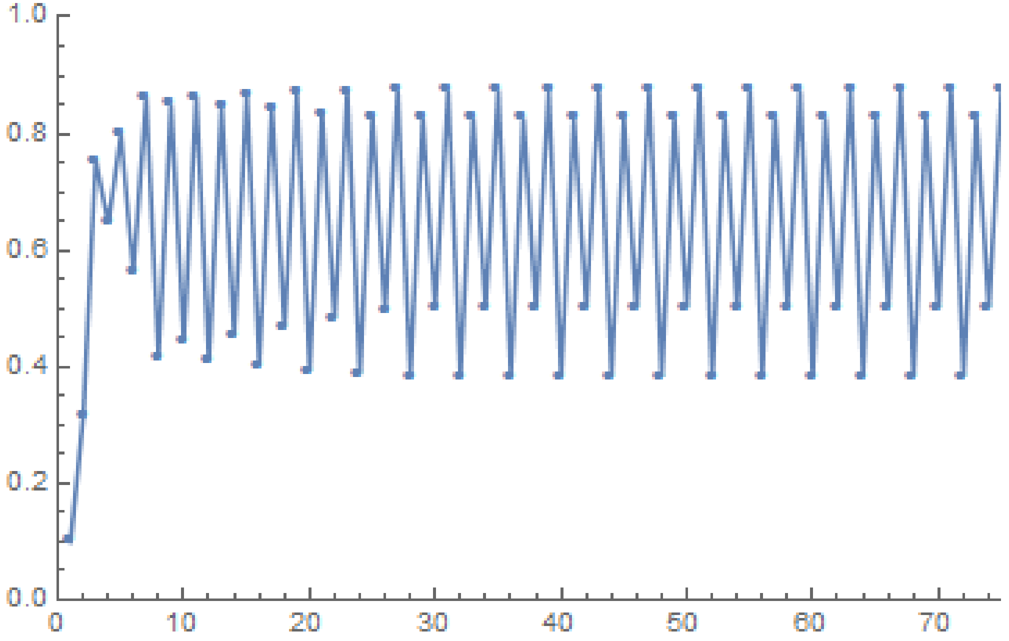
Final State: Periodic Behaviour

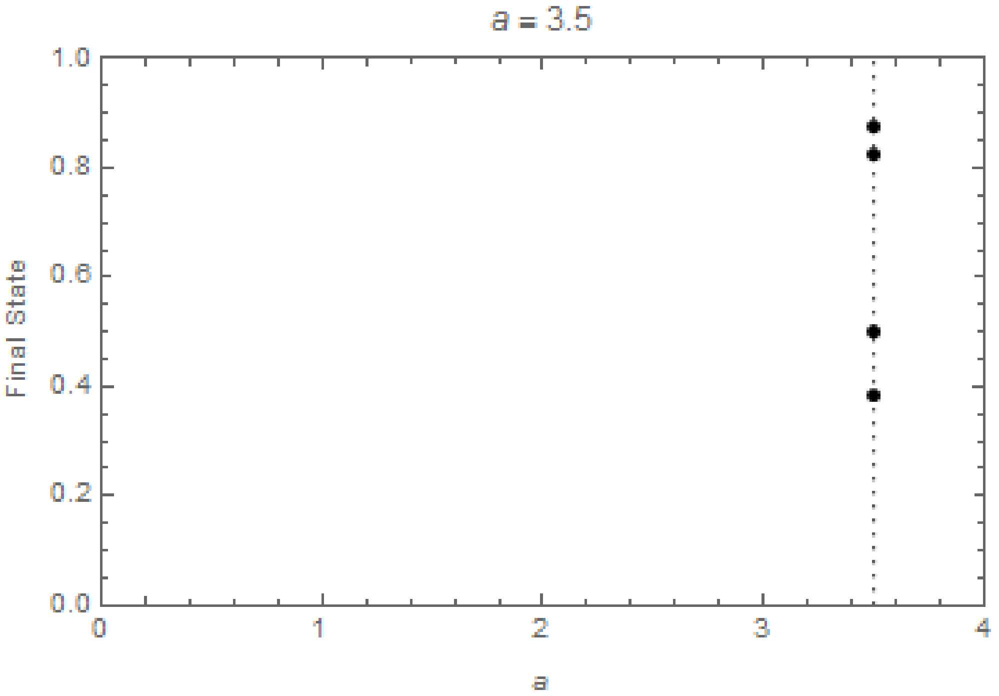




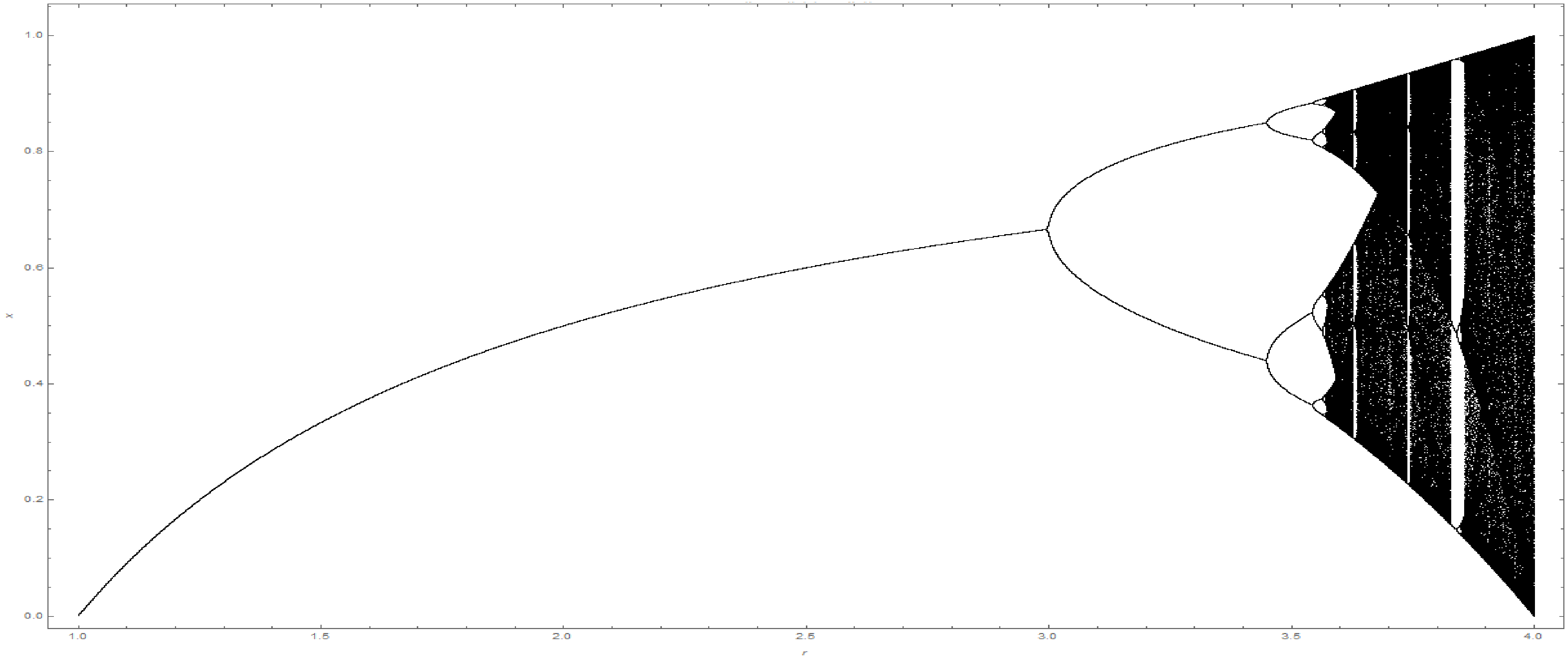
Final State: Periodic Behaviour

a=3.5

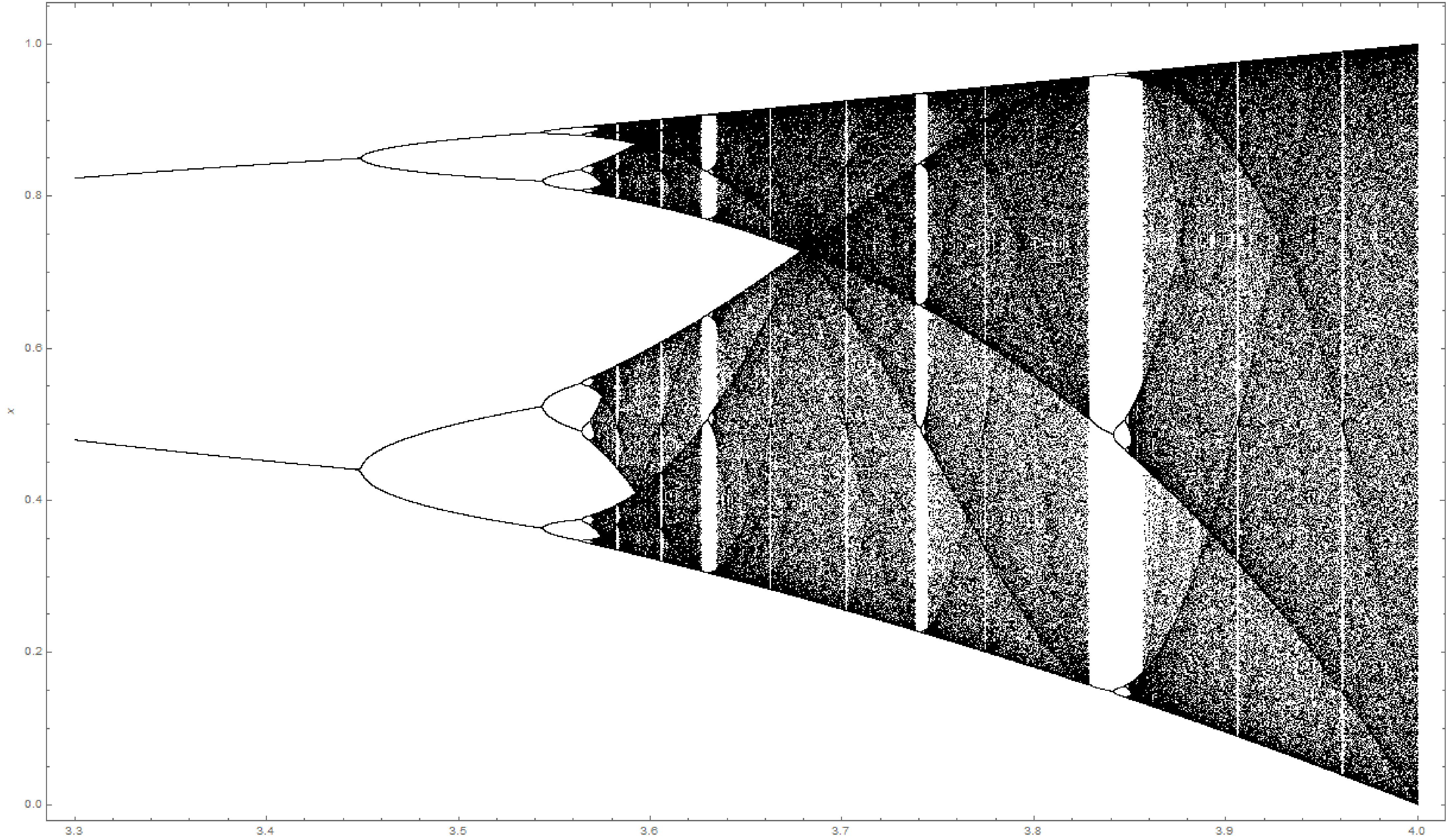




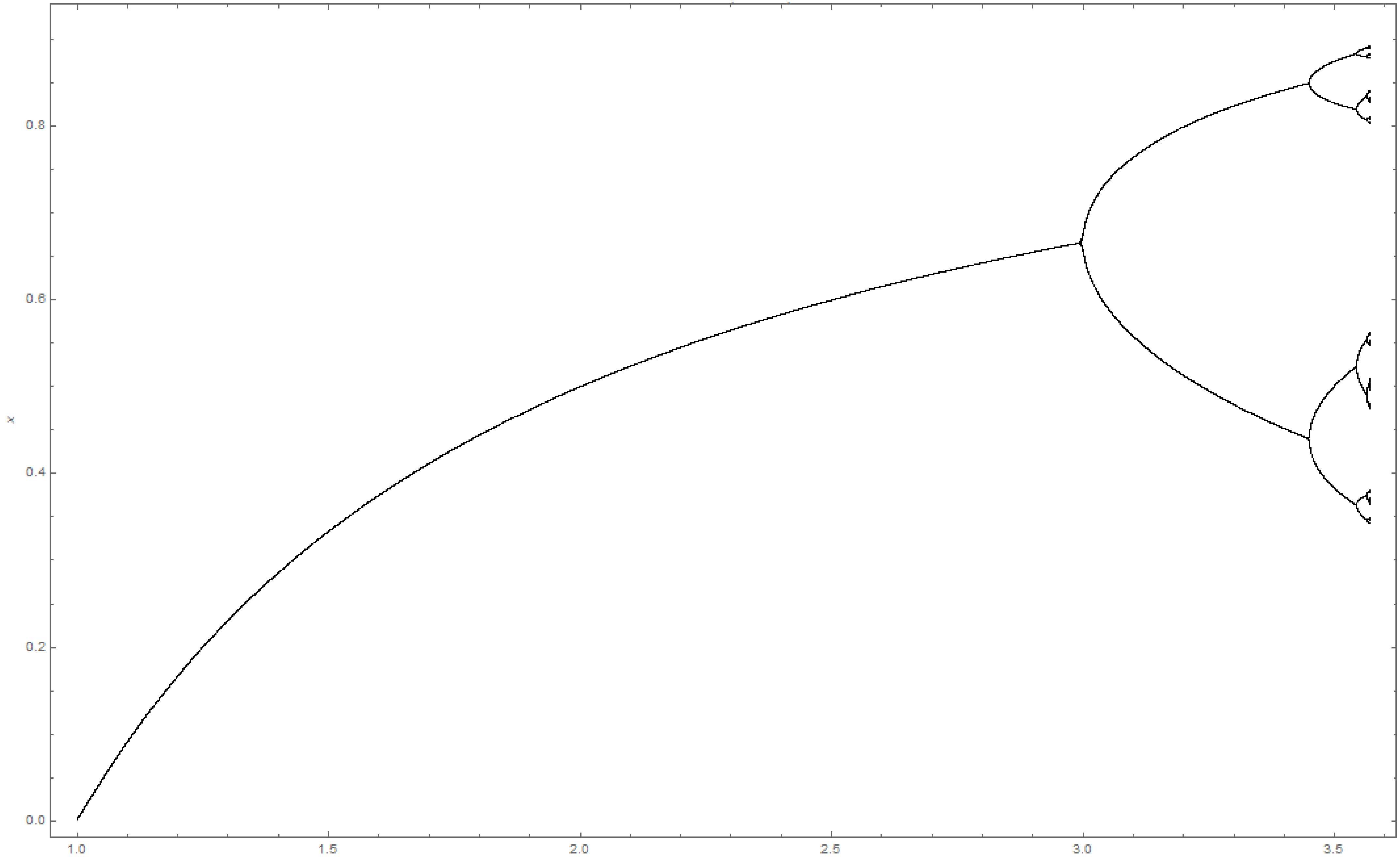
Feigenbaum/Final State Diagram



Code for image from: http://mathworld.wolfram.com/Bifurcation.html



Code for image from: http://mathworld.wolfram.com/Bifurcation.html



Code for image from: http://mathworld.wolfram.com/Bifurcation.html

Fixed Points

We can see a fixed point of the quadratic iterator should satisfy:

$$x = f_a(x) = ax(1 - x)$$

We also have stable and unstable fixed points:

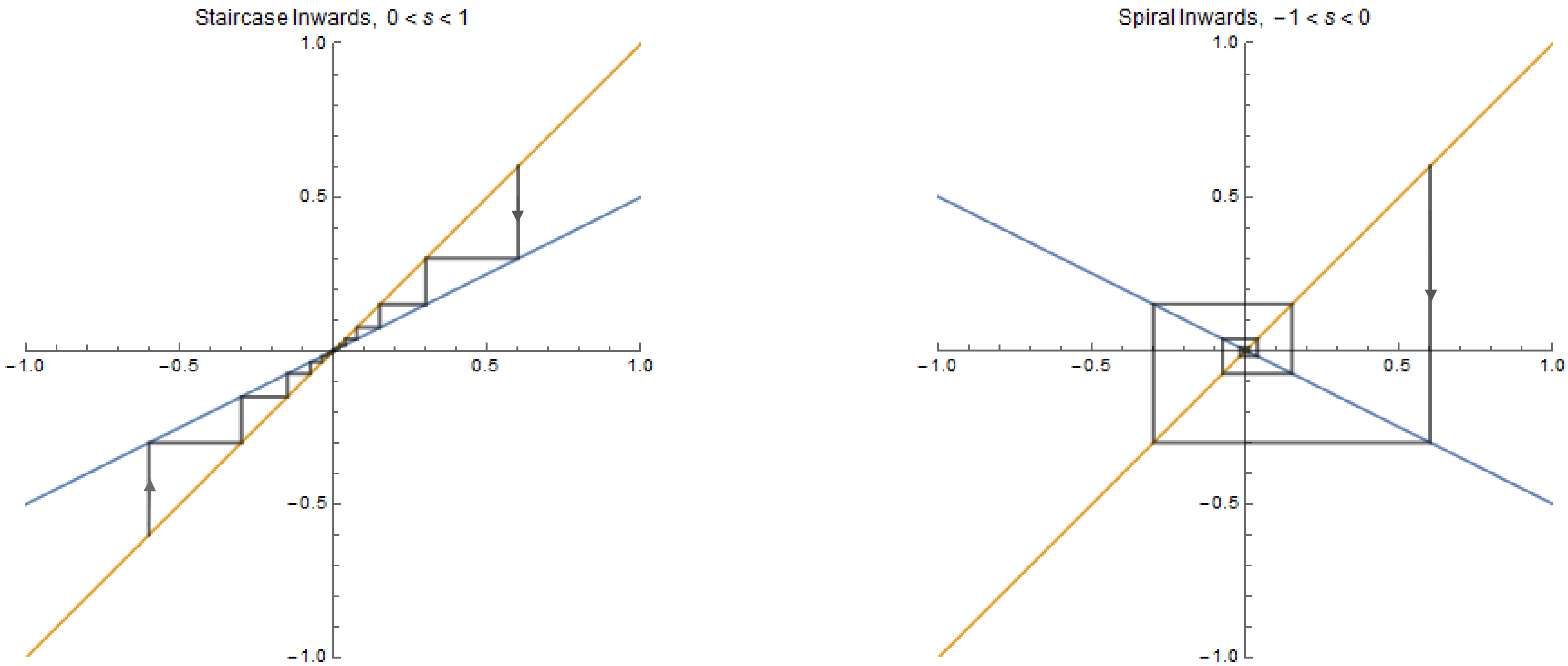
$|f'(x^*)| < 1$ → The fixed point is attractive

$|f'(x^*)| > 1$ → The fixed point is repelling

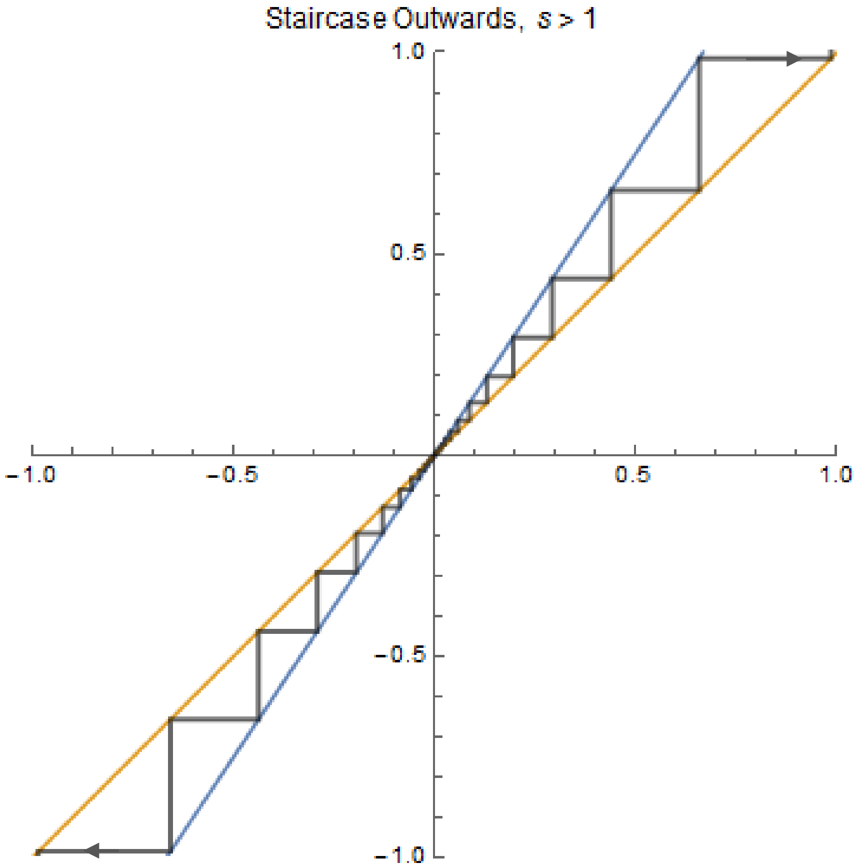
These results can be visualised and further understood using the linear iterator

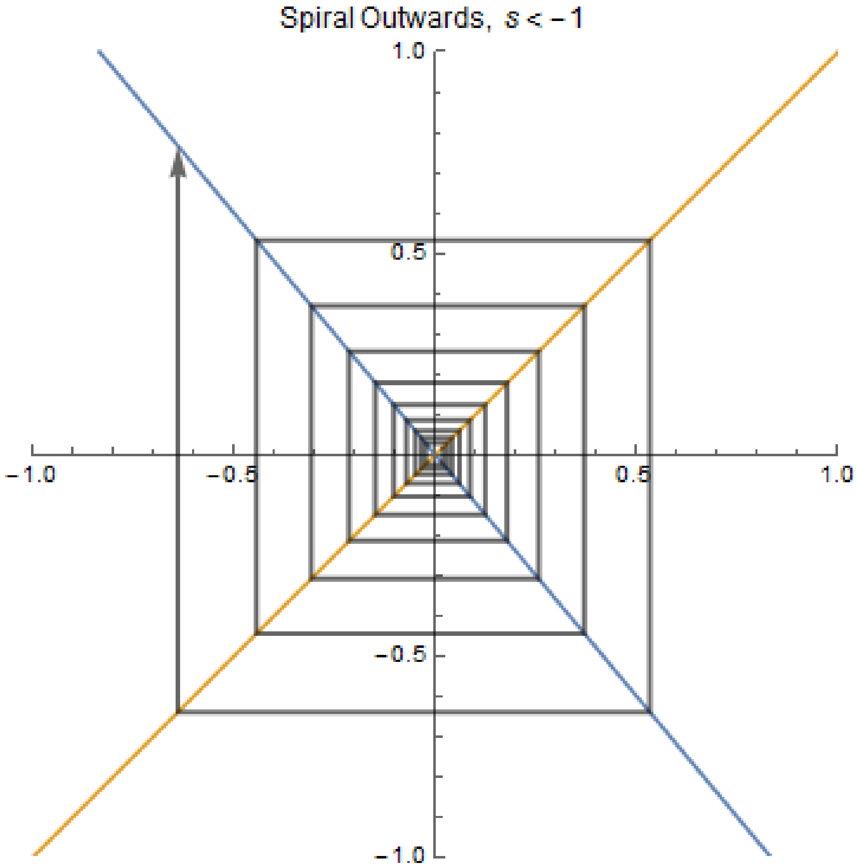
$$x_{n+1} = sx_n$$

Behaviour at Stable Fixed Points

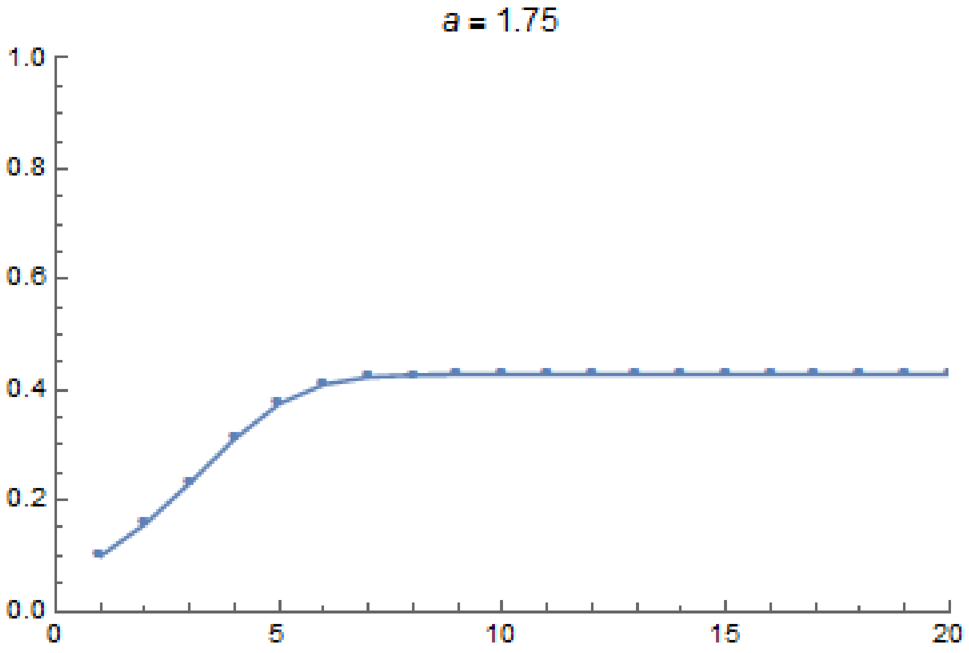


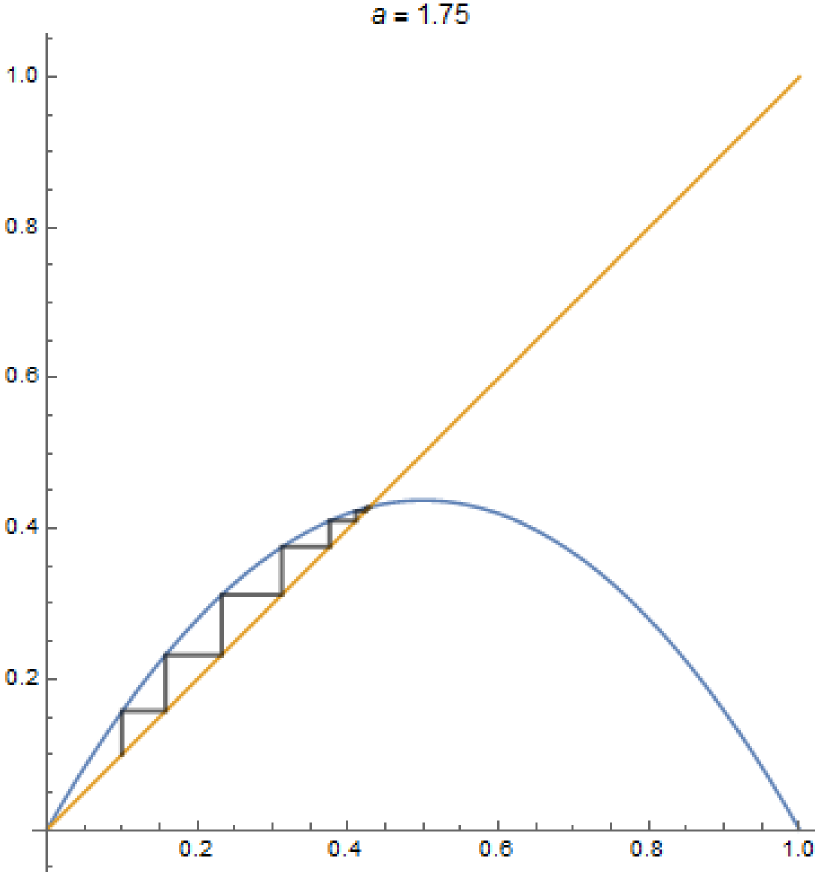
Behaviour at Unstable Fixed Points



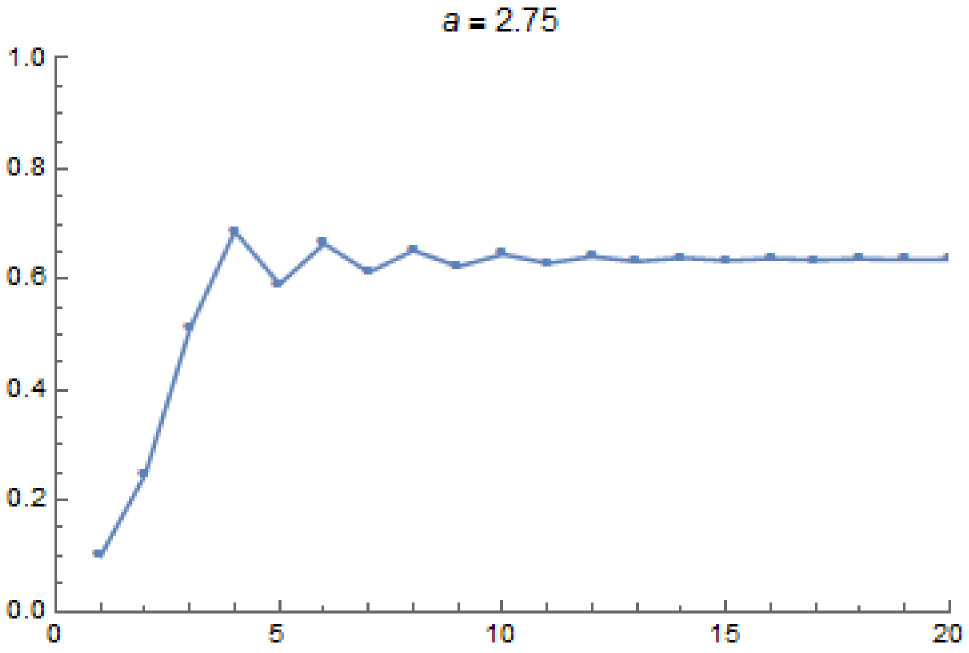


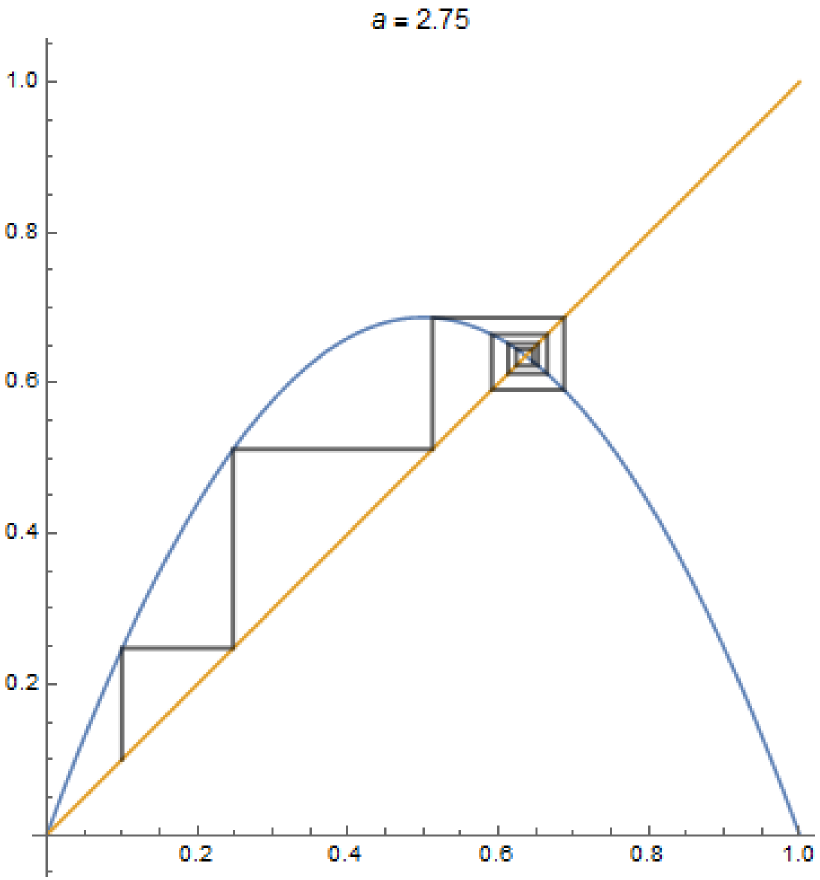
Monotonic Approach



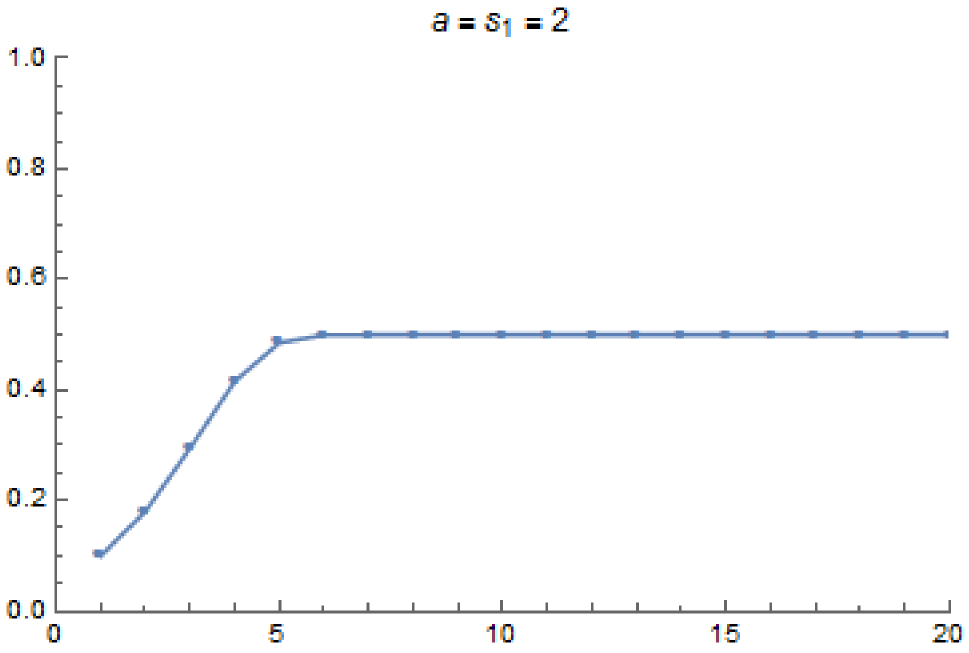


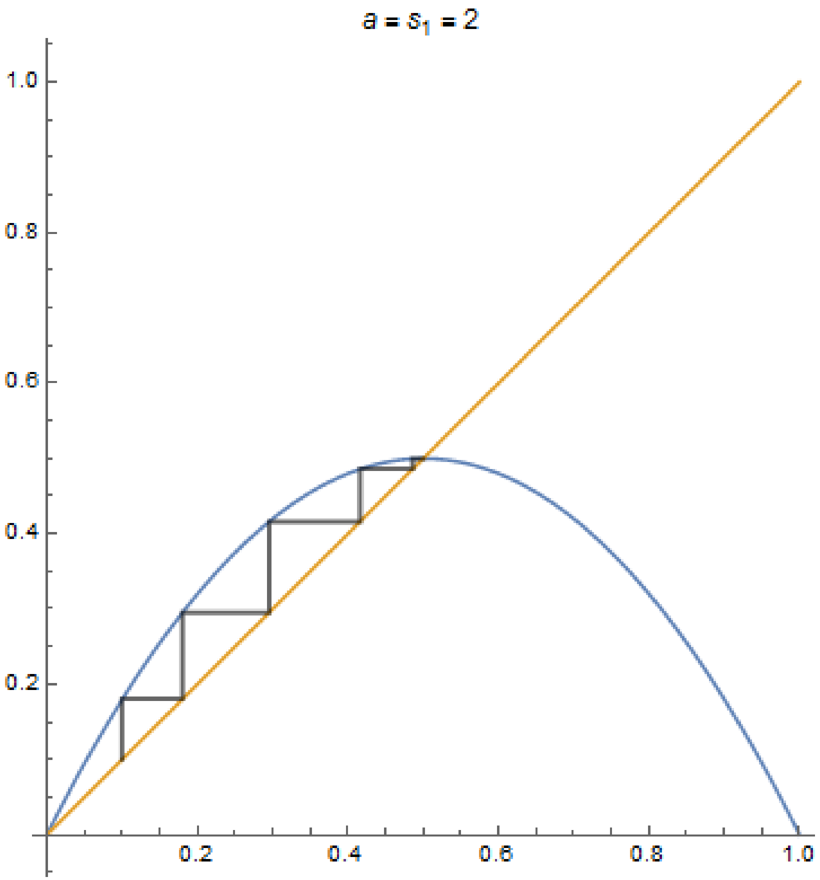
Spiral Approach





The Super-Attractive Case





The General Case

Table 1: $x = p_a = \frac{a-1}{a}$

Parameter Value	Derivative Value	Type of Fixed Point	Behaviour at Fixed Point
$1 < a < 2$	$0 < f'(p_a) < 1$	Stable	Staircase in
$2 < a < 3$	$-1 < f'(p_a) < 0$	Stable	Spiral in
$3 < a \leq 4$	$-2 \leq f'(p_a) < -1$	Unstable	Spiral out

Our Fixed Point Loses Stability…

- At the point a=3 our fixed point becomes unstable
- We call this a bifurcation point

"a *small smooth change made to the parameter values of a system causes a sudden 'qualitative' or topological change in its behaviour*" - *http://en.wikipedia.org/wiki/Bifurcation_theory*

- In particular, a period-doubling bifurcation

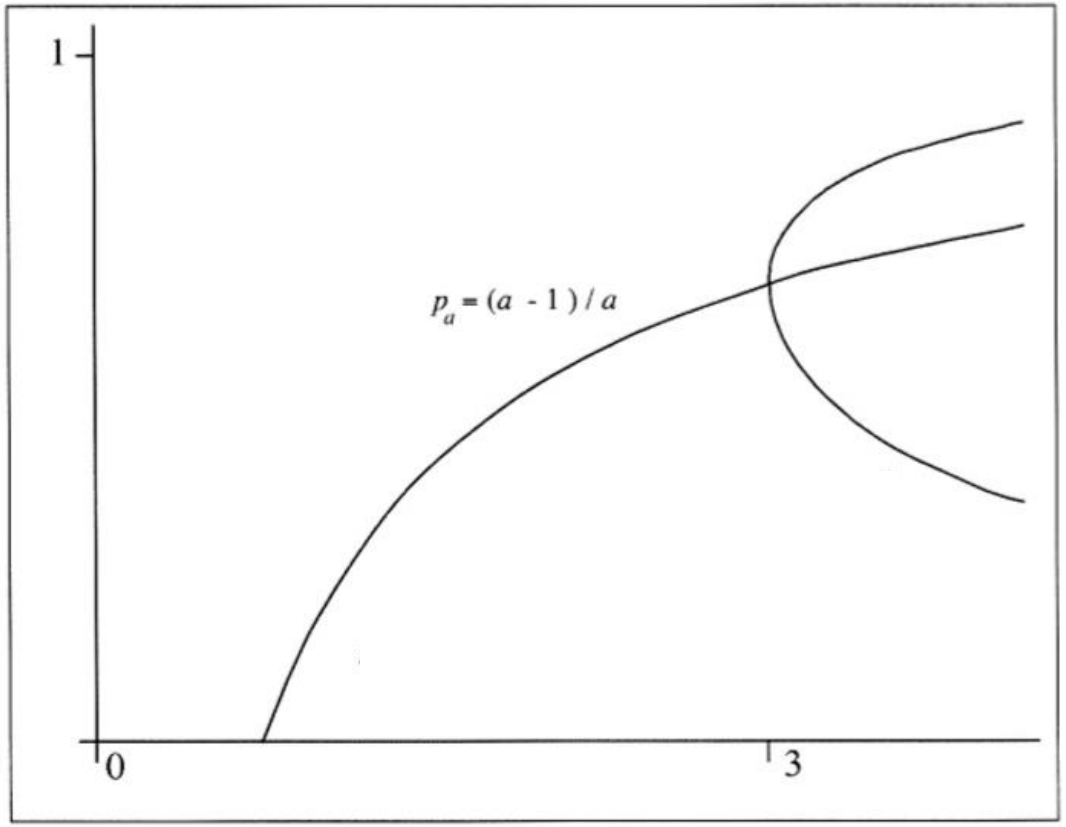


Image: Heinz-Otto Peitgen, Hartmut Jürgens, and Dietmar Saupe. *Chaos and fractals: new frontiers of science.* Springer Science & Business Media, 2004.

The First Bifurcation Point

- We can describe the attractor of the quadratic iterator after the first bifurcation point $a = b_1 = 3$

$$x = f_a(f_a(x)) = f_a^2(x)$$

- Expanding and solving

$$x = a(ax(1-x))(1-(ax(1-x))$$

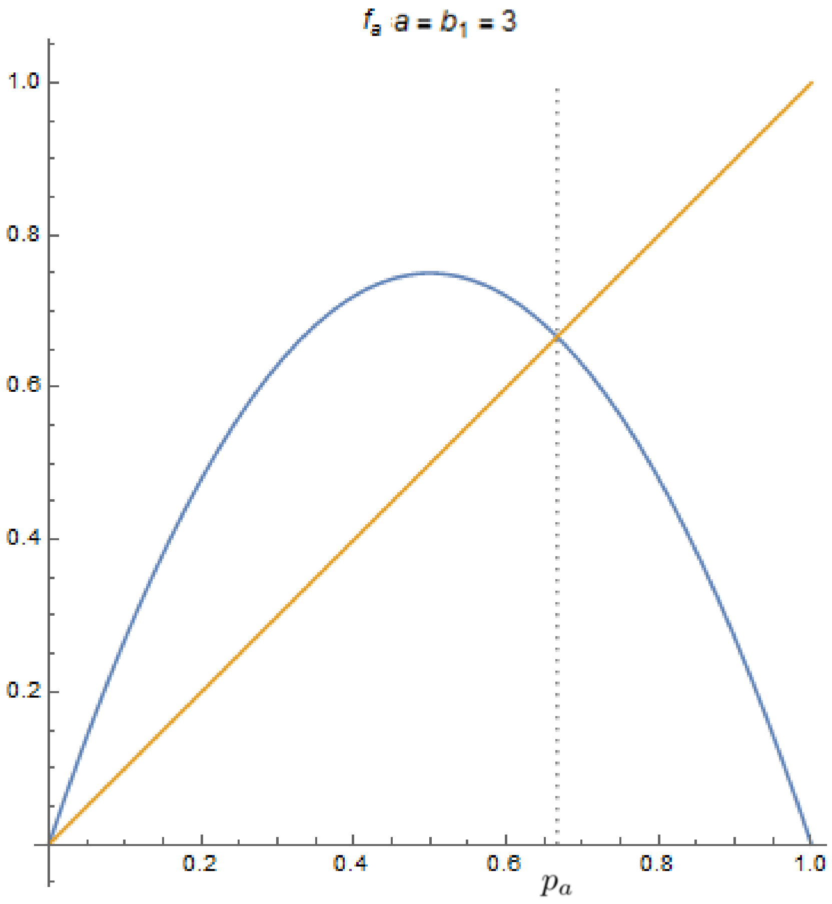
$$-a^3x^4 + 2a^3x^3 - (a^2 + a^3)x^2 + (a^2 - 1)x = 0$$

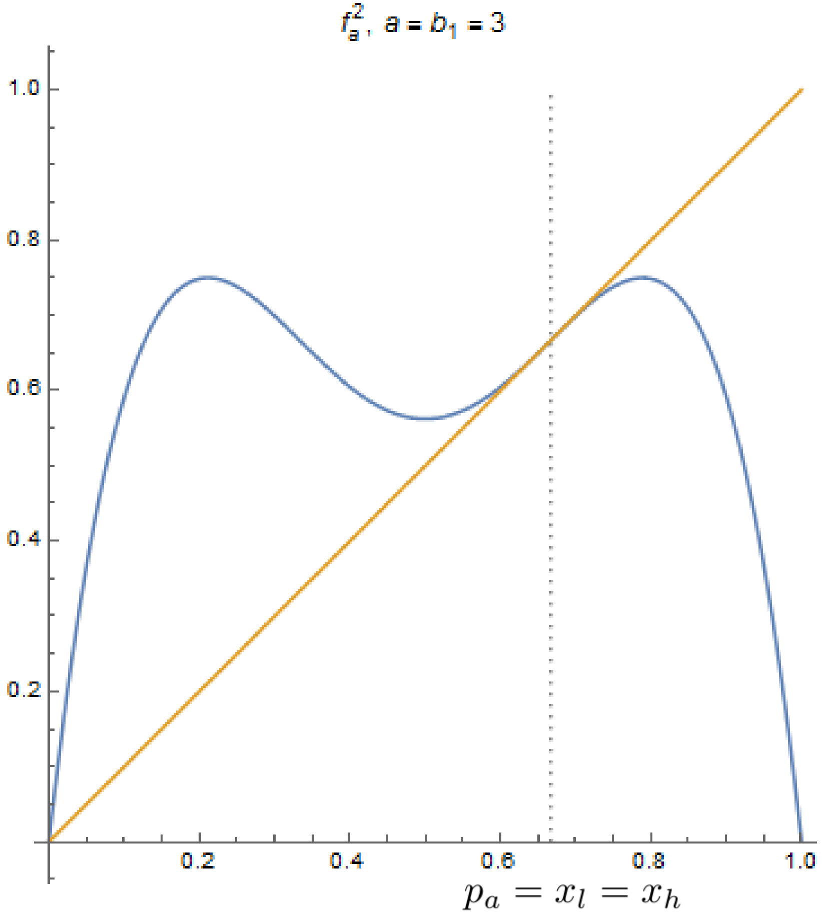
$$-a^3x(x - \frac{a-1}{a})(x^2 - \frac{a+1}{a} + \frac{a+1}{a^2}) = 0$$

- Using the quadratic formula we attain the upper and lower values of the orbit

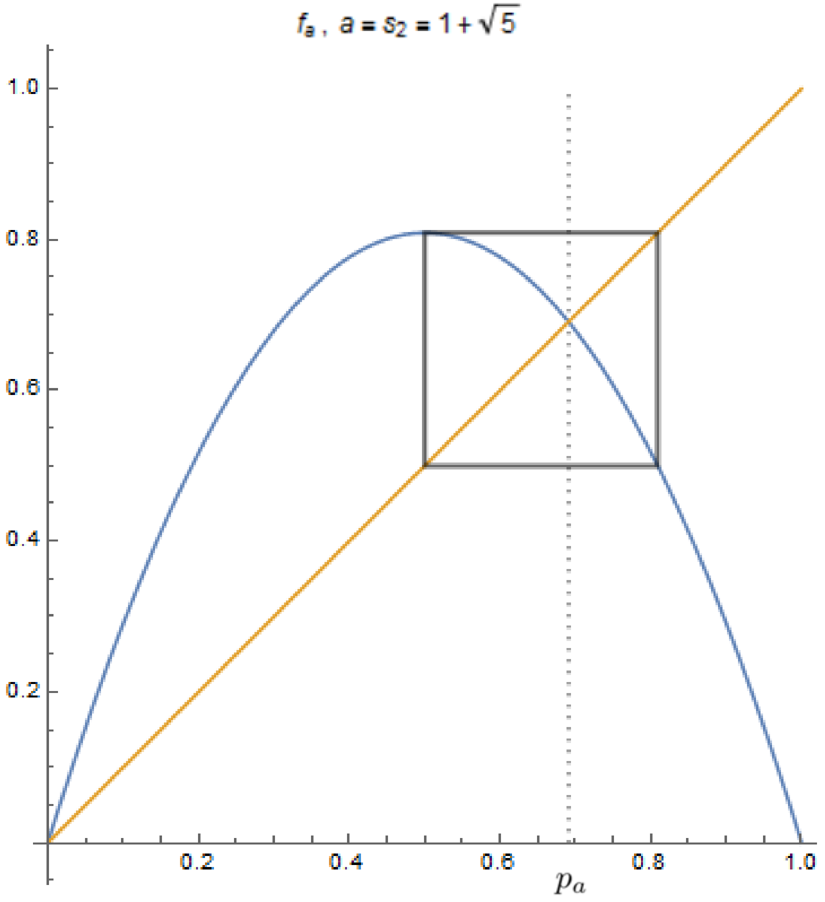
$$x_l(a) = \frac{a+1-\sqrt{a^2-2a-3}}{2a} \qquad x_h(a) = \frac{a+1+\sqrt{a^2-2a-3}}{2a}$$

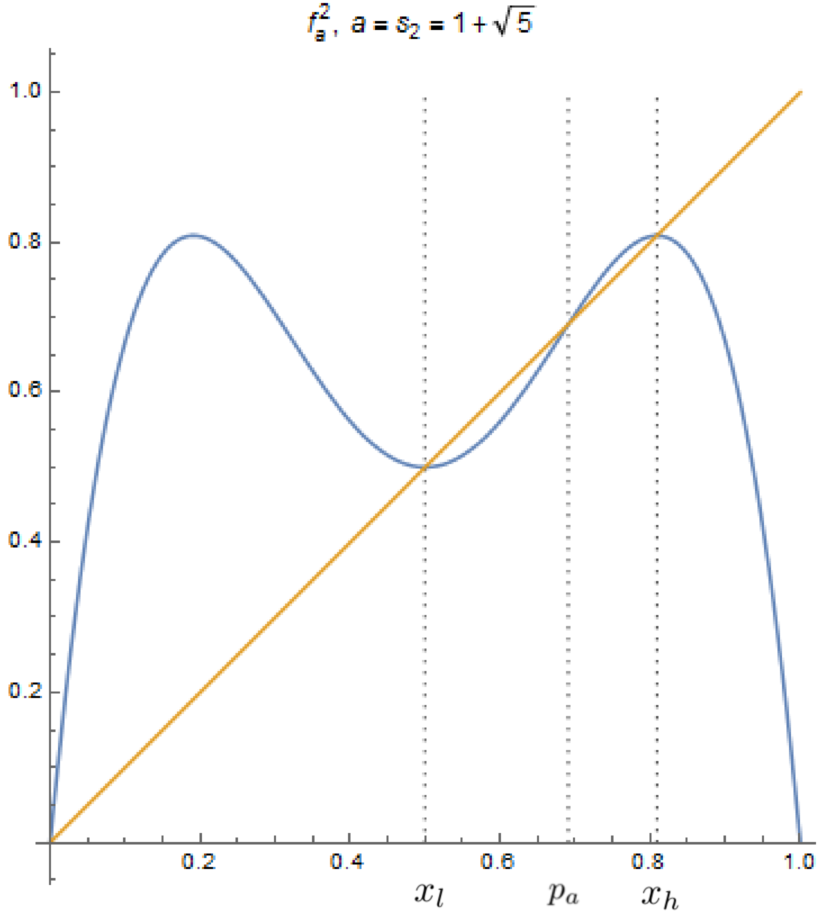
An Orbit of Period 2



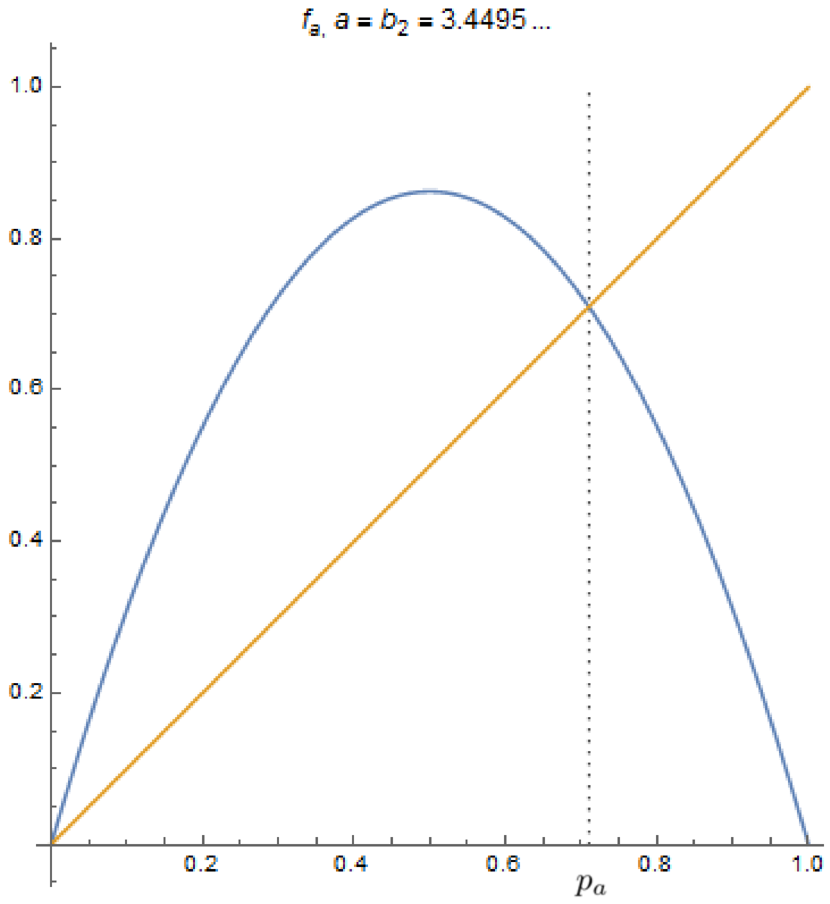


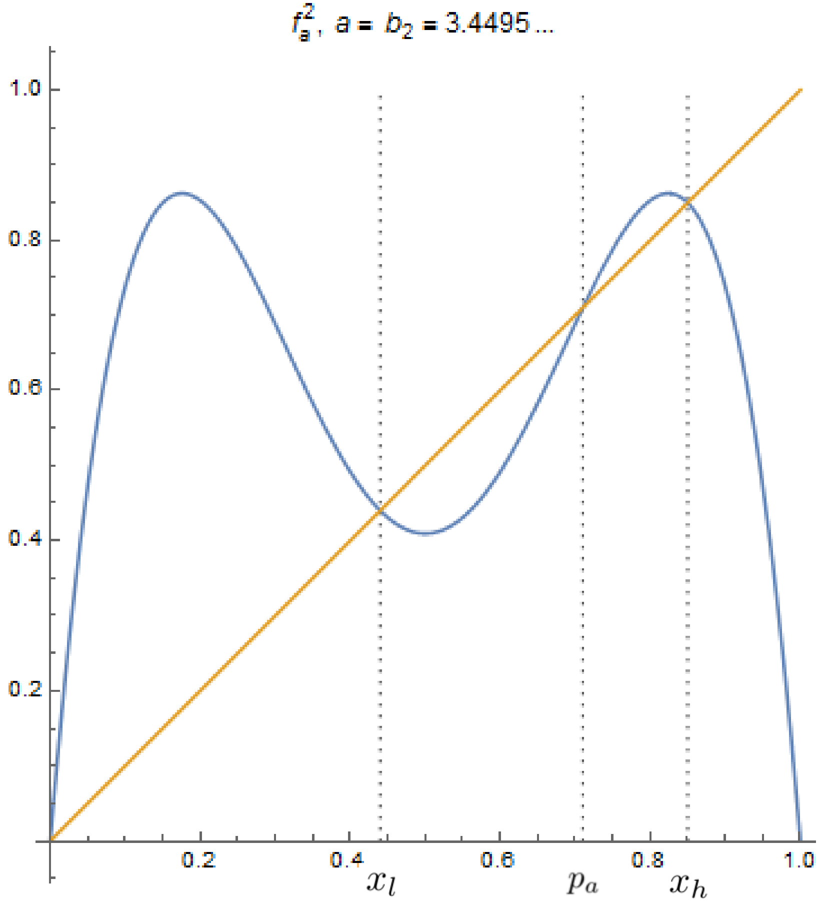
An Orbit of Period 2





An Orbit of Period 2





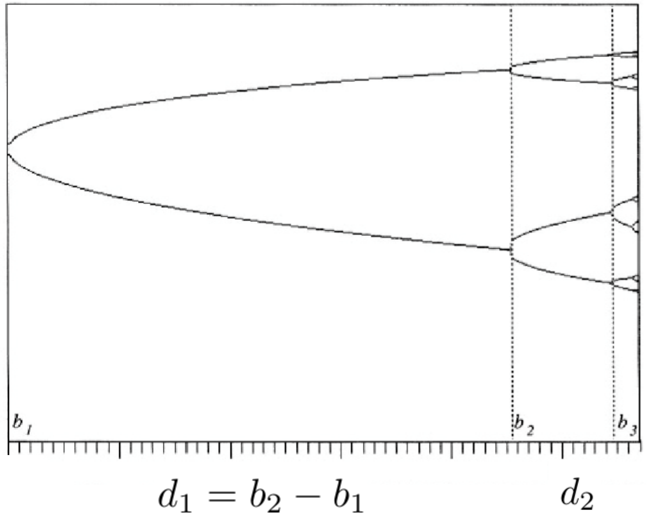
Self-Similarity Across Bifurcation Points

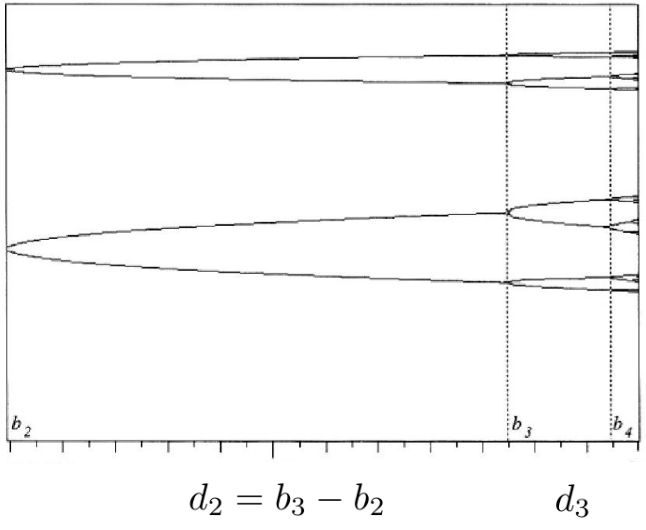
Parameter Value	$f_a(x)$	$f_a^2(x)$
2	super attractive case; $p_a = x_{crit}$	
3	p_a becomes unstable	bifurcation of x_l and x_h
$3.236 = 1 + \sqrt{5}$	$f_a(f_a(x_{crit})) = x_{crit}$	super attractive case; $x_l = x_{crit}$
3.449		x_l becomes unstable

- If we continue increasing the parameter beyond the second bifurcation point, we see the same patterns of behaviour occurring → *Qualitative approach to chaos*
- (Of course we would have to examine $f_a^2(f_a^2(x)) = f_a^4(x)$ in the same way as above…)

As the Period of the Orbit Increases..

- As a increases, the rate at which orbits become unstable and bifurcates increases





- That is, $d_k = b_{k+1} - b_k$ decreases as a increases
- Is the rate at which d_k is decreasing geometric? Or in other words can we find an r for which

$$d_{k+1} = r d_k$$

Images: Heinz-Otto Peitgen, Hartmut Jürgens, and Dietmar Saupe. *Chaos and fractals: new frontiers of science.* Springer Science & Business Media, 2004.

The Feigenbaum Constant

- It turns out the sequence d_k is not *exactly* geometric
- However,

$$\lim_{k \to \infty} \delta_k = \lim_{k \to \infty} \frac{d_k}{d_{k+1}}$$

$$= \lim_{k \to \infty} \frac{b_{k+1} - b_k}{b_{k+2} - b_{k+1}}$$

$$= \delta = 4.669202...$$

- We call this value the Feigenbaum constant, and it is in fact a universal constant – independent of the quadratic iterator!

Universality of the Feigenbaum Constant

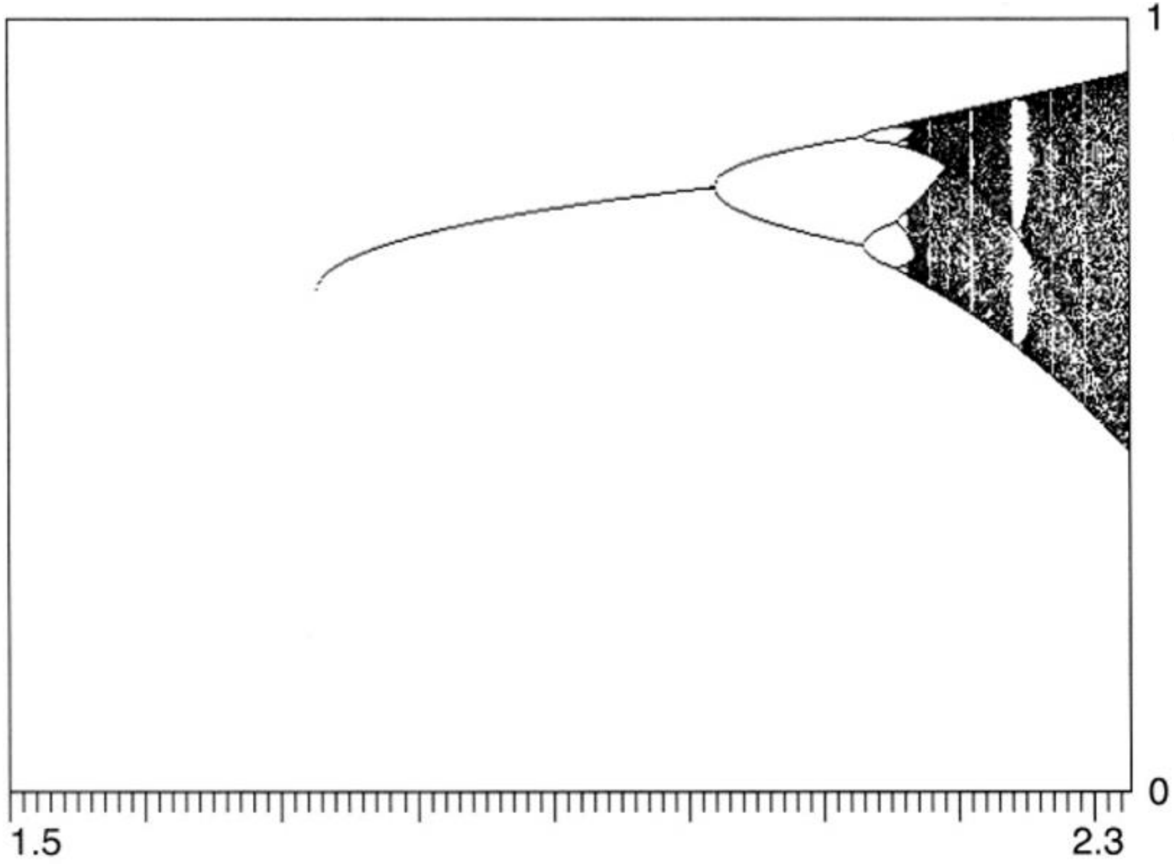
The Feigenbaum constant can be derived similarly from a whole class of functions satisfying the following 4 conditions:

1. The function is a smooth function from [0,1] into the real numbers
2. The function attains a maximum at x_m which is at least quadratic; that is $f''(x_m) \neq 0$
3. The function is monotone in $[0, x_m)$ and in $(x_m, 1]$
4. The function has a negative Schwarzian derivative; i.e. $S_f(x) < 0 \quad \forall x \in [0,1]$ where

$$S_f(x) = \frac{f'''(x)}{f'(x)} - \frac{3}{2}\left(\frac{f'''(x)}{f'(x)}\right)^2$$

(don't ask)

A Similar Feigenbaum Diagram…



$$g_a(x) = ax^2 \sin(\pi x)$$

Image: Heinz-Otto Peitgen, Hartmut Jürgens, and Dietmar Saupe. *Chaos and fractals: new frontiers of science.* Springer Science & Business Media, 2004.

The Feigenbaum Point

- The Feigenbaum point is the point at which the series of bifurcation points converge
- Thus the Feigenbaum point marks the parameter value at which the period –doubling regime ceases
- At the Feigenbaum point the Attractor is an infinite set
- We denote the Feigenbaum point s_∞
- For the quadratic iterator, $s_\infty = 3.569946...$
- Note that the value of the Feigenbaum point is different for different systems

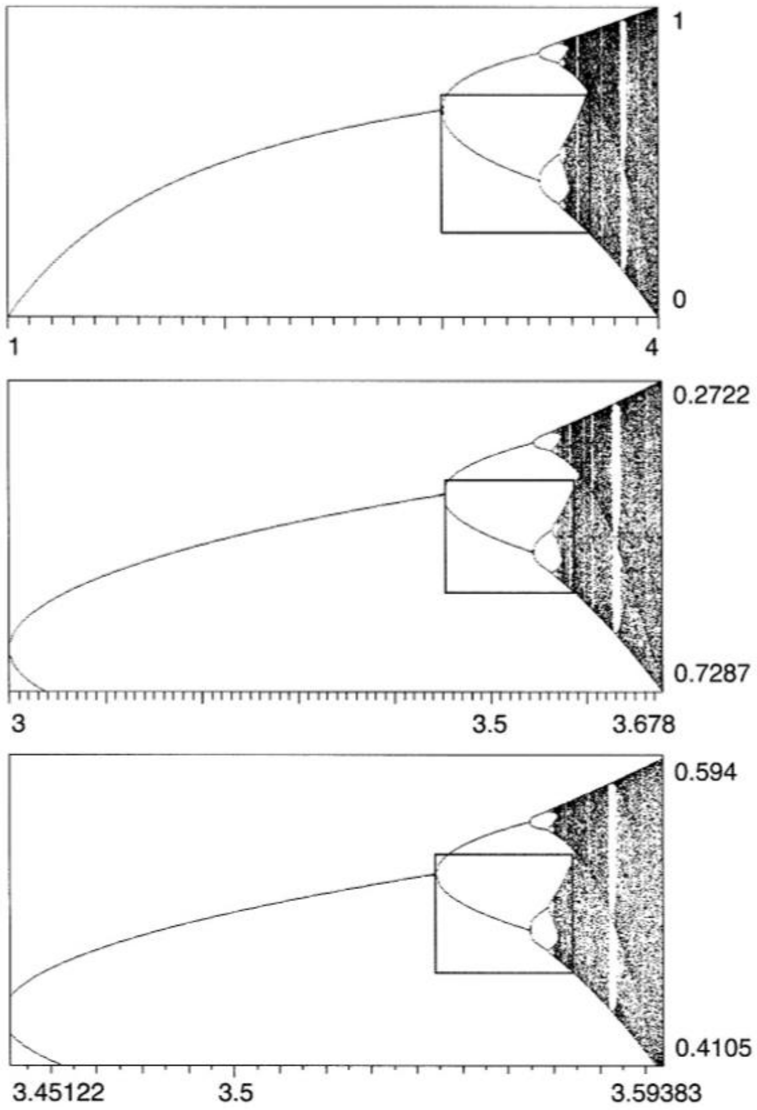
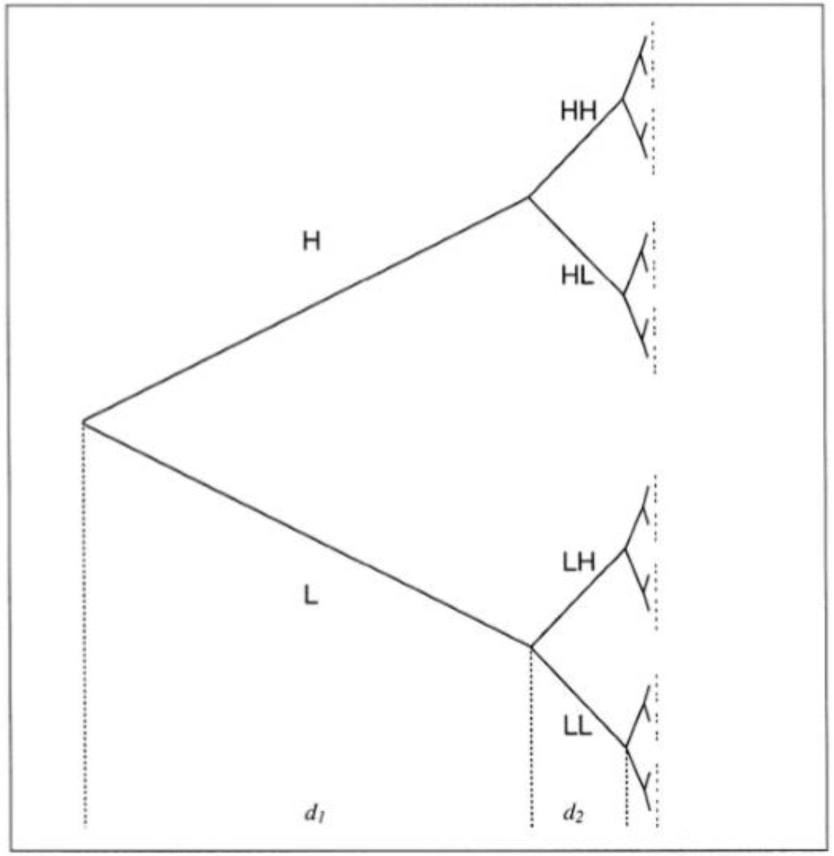


Image: Heinz-Otto Peitgen, Hartmut Jürgens, and Dietmar Saupe. *Chaos and fractals: new frontiers of science.* Springer Science & Business Media, 2004.

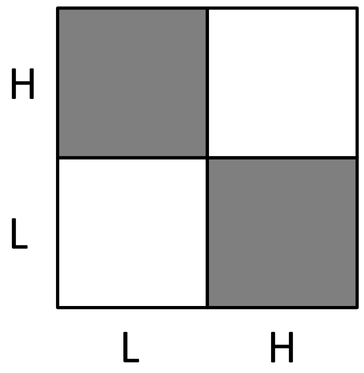
At the Feigenbaum Point

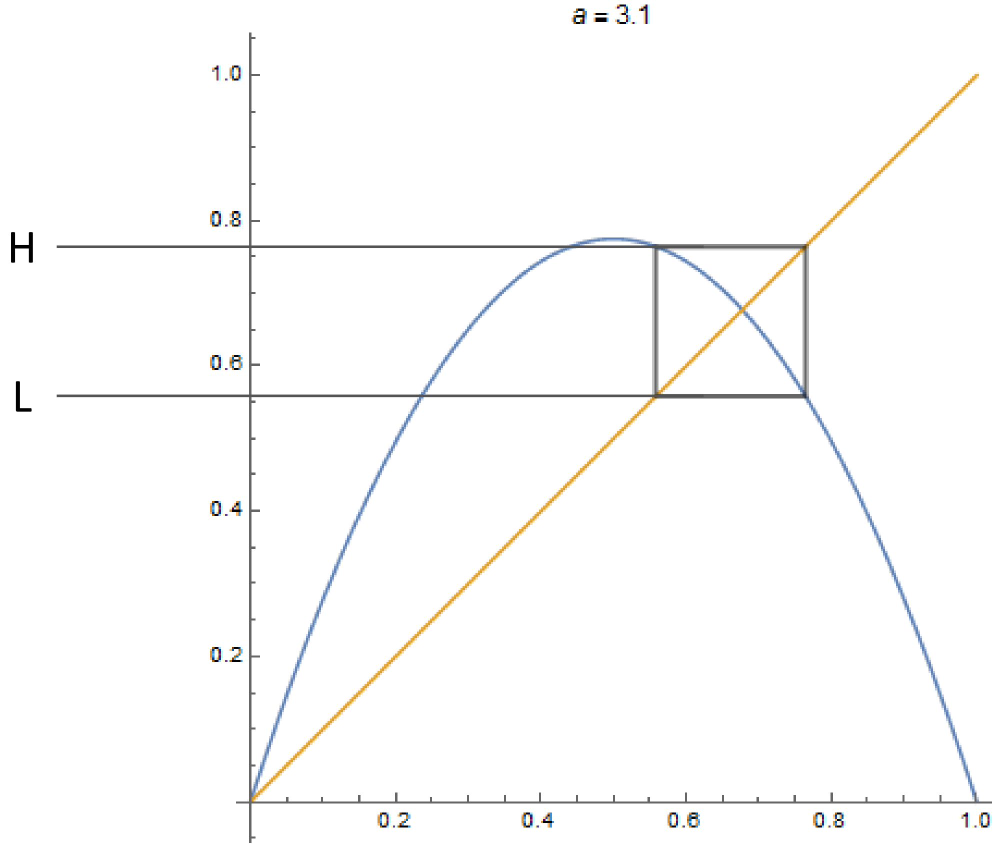


- To the left is a schematic representation of the final state diagram ending at the Feigenbaum point
- Thus the 'leaves' of the diagram represent the attractor of the iterator at the Feigenbaum point
- We give each branch (or point in successive stable orbits) an address
- Since the tree has bifurcated infinitely many times by the Feigenbaum point, the addresses of the leaves are infinite strings of Hs and Ls

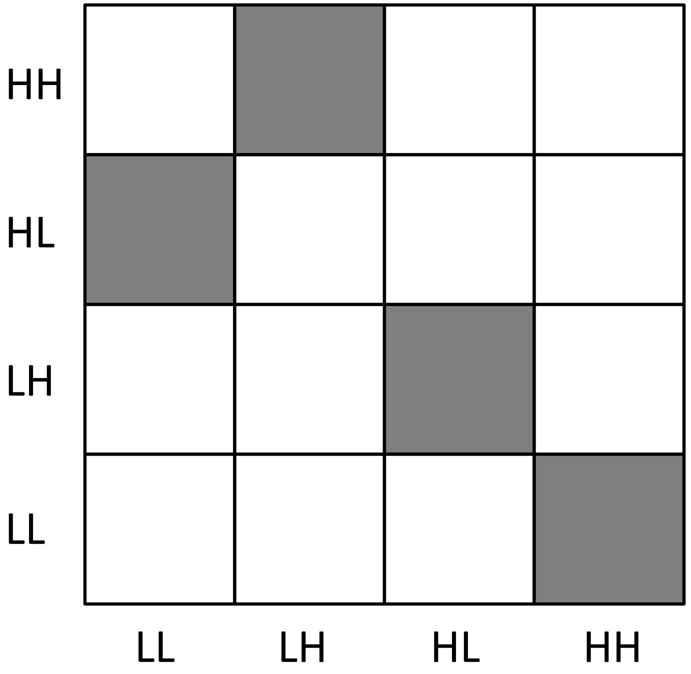
Image: Heinz-Otto Peitgen, Hartmut Jürgens, and Dietmar Saupe. *Chaos and fractals: new frontiers of science.* Springer Science & Business Media, 2004.

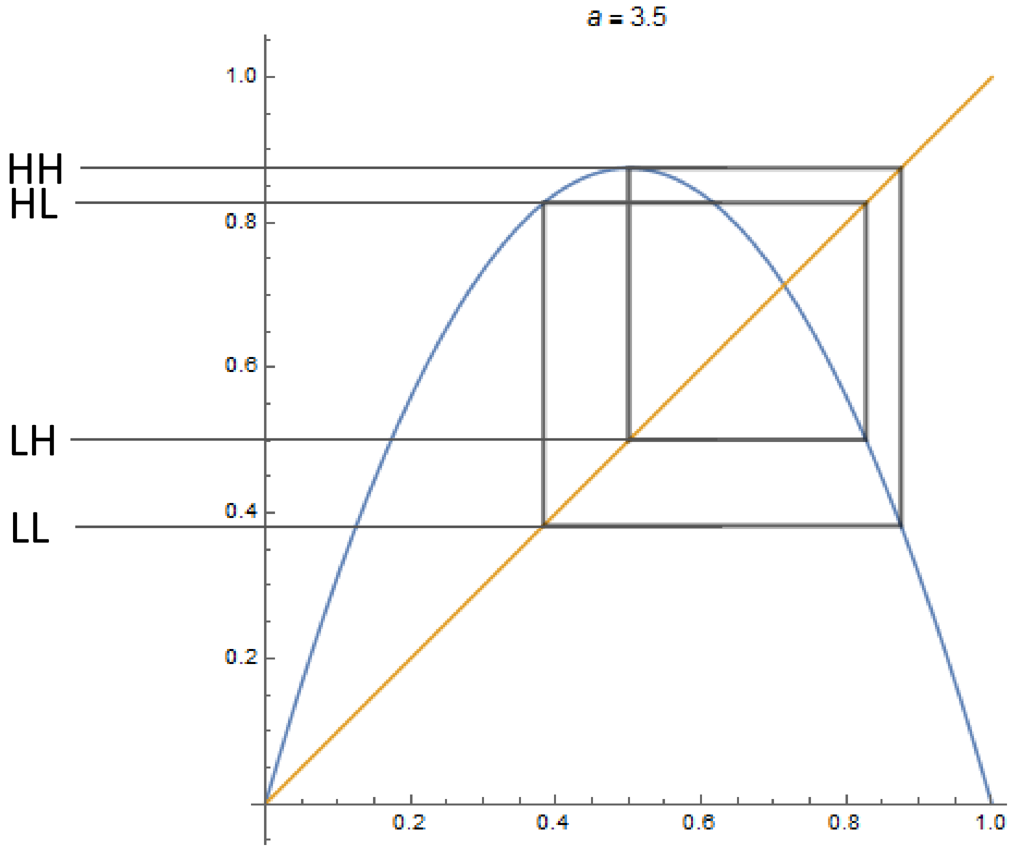
Orbital Dynamics: Period 2



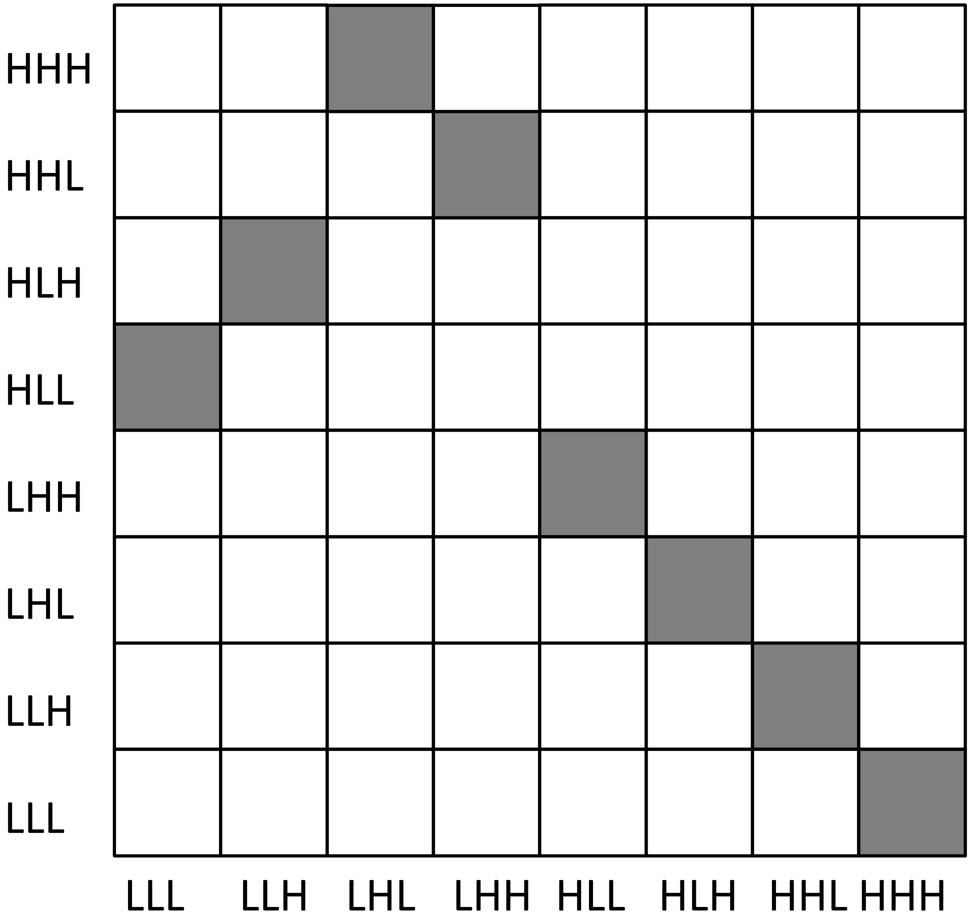


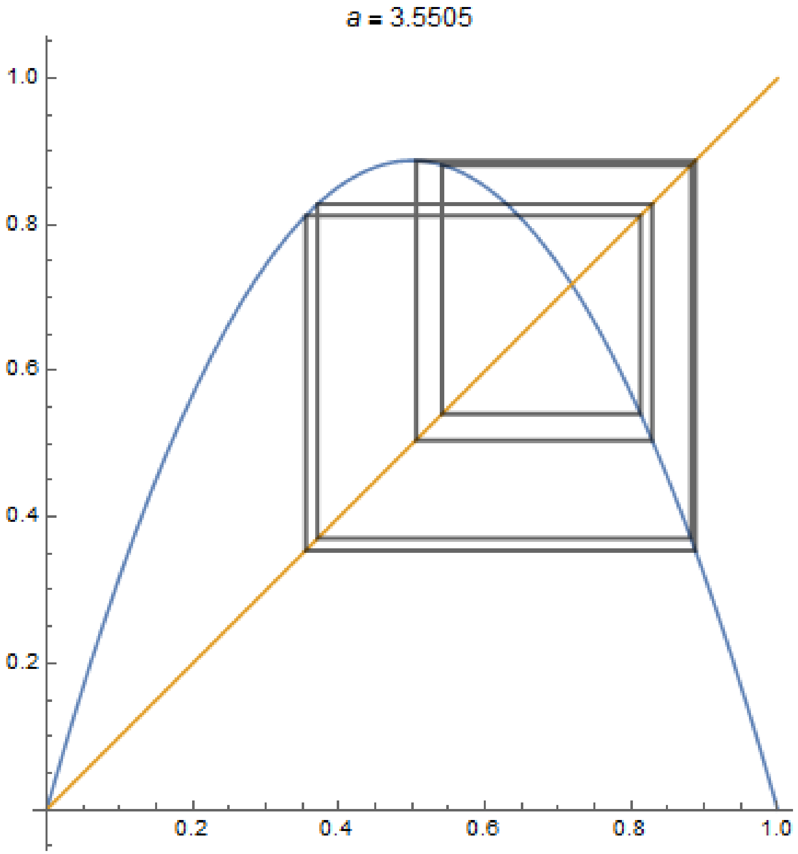
Orbital Dynamics: Period 4



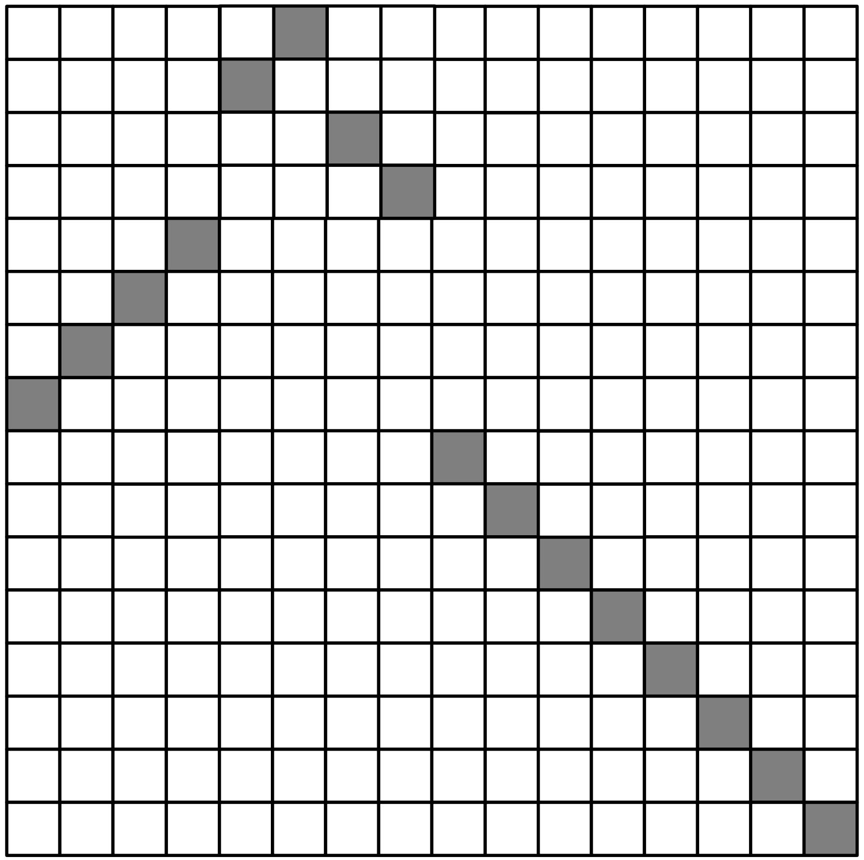


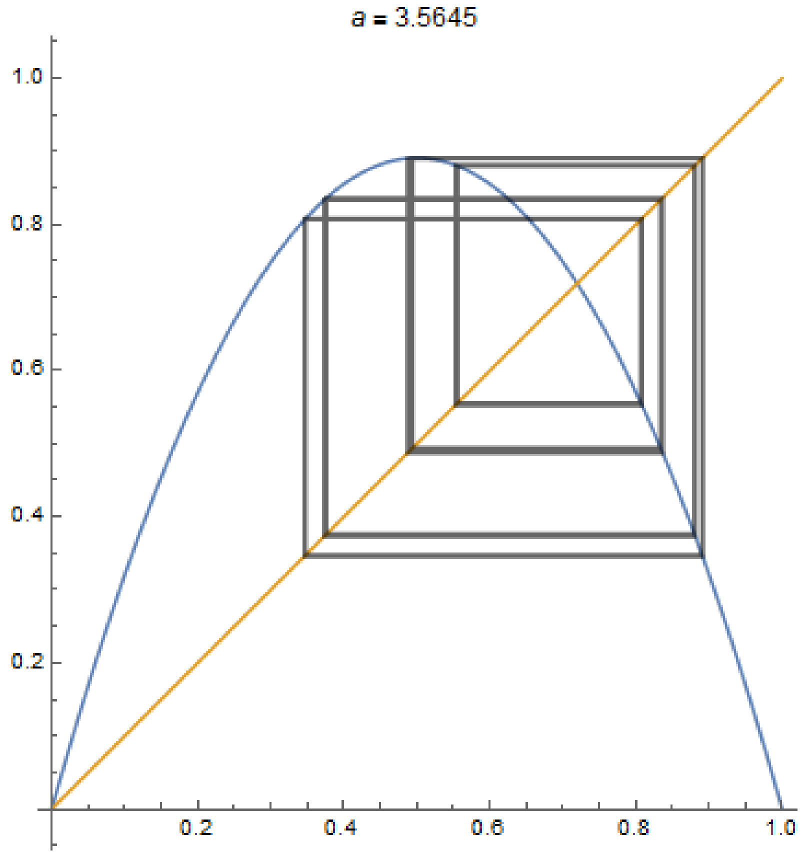
Orbital Dynamics: Period 8



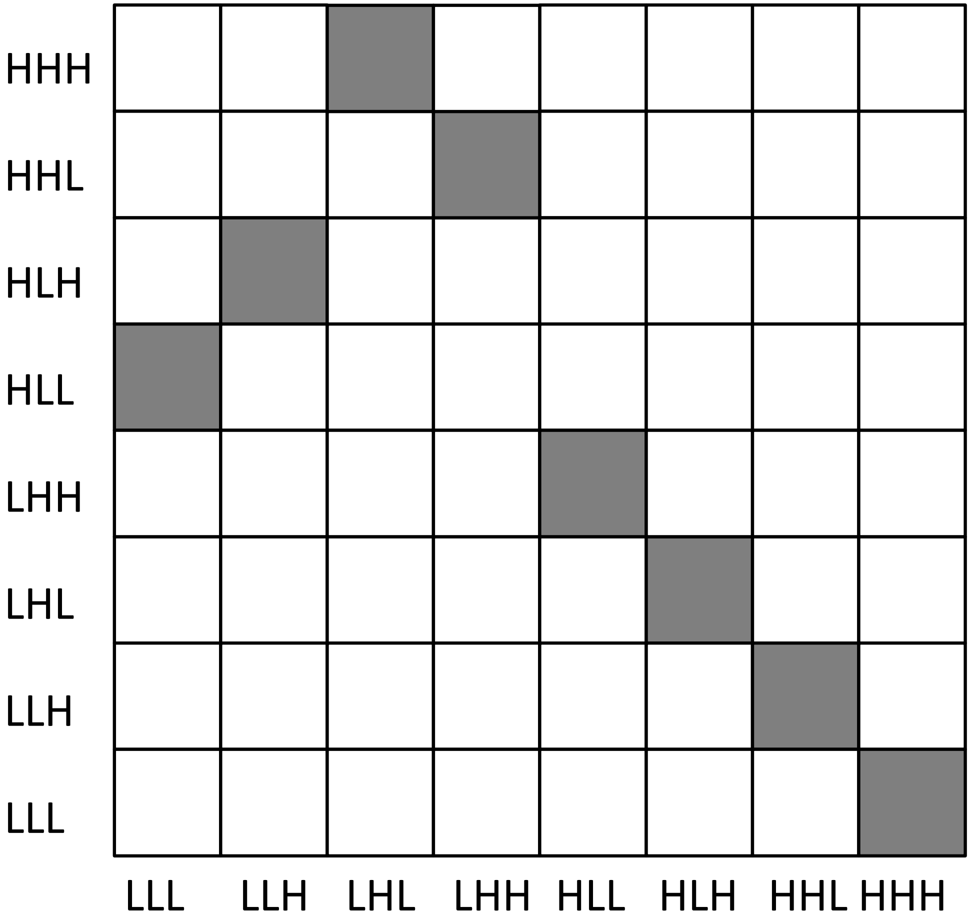


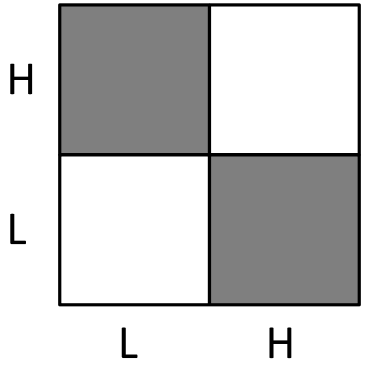
Orbital Dynamics: Period 16



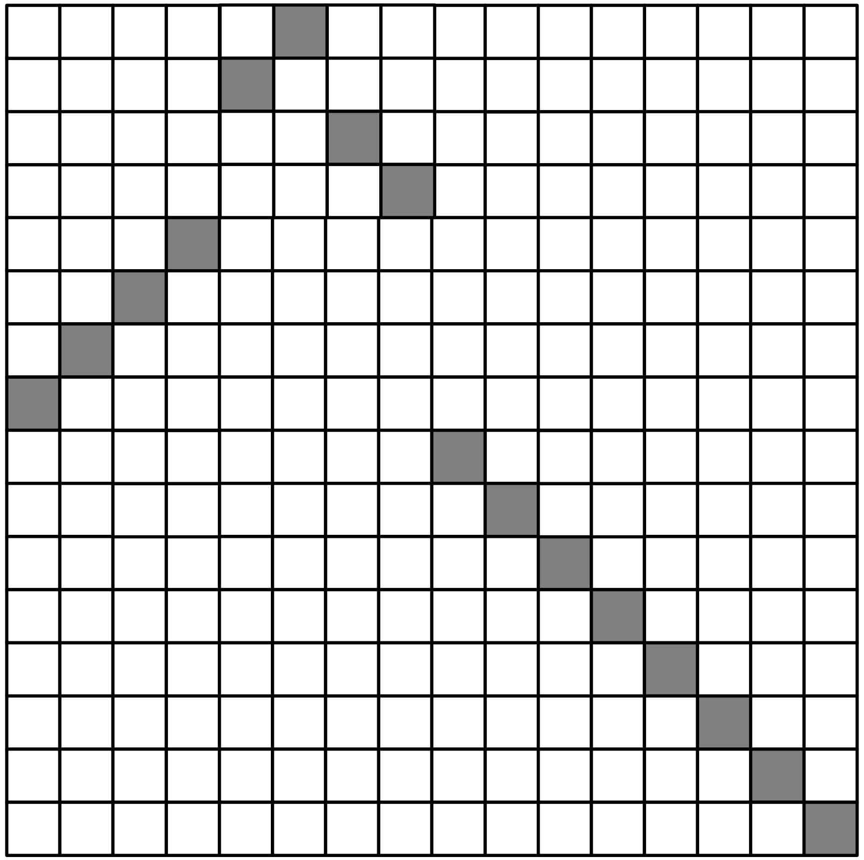


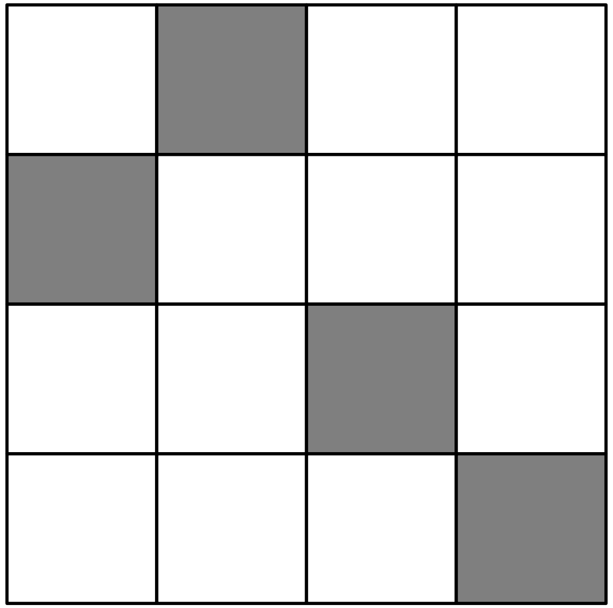
Orbital Dynamics: Period 8 and 2





Orbital Dynamics: Period 16 and 4





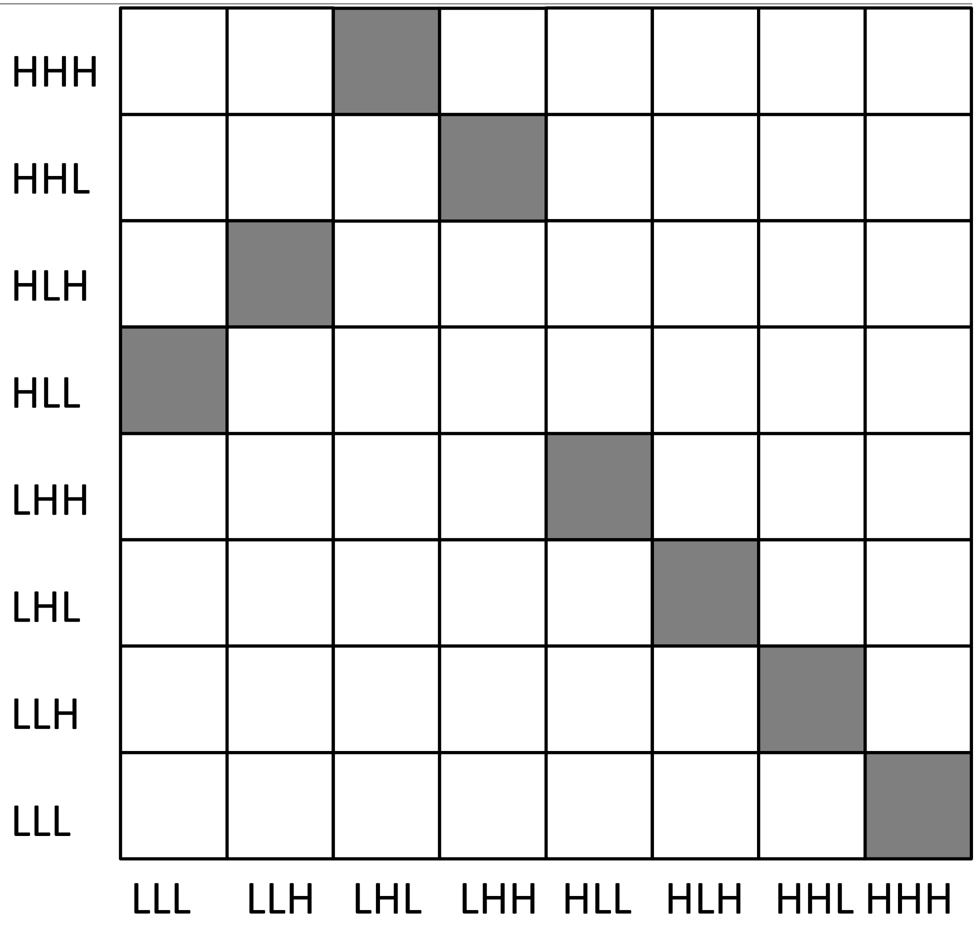
Orbital Dynamics as a Function

$$f_\infty : A_\infty \mapsto A_\infty$$

1. $f_\infty(HX_2X_3X_4...) = LX_2^TX_3^TX_4^T...$

2. $f_\infty(LLX_3X_4X_5...) = HLX_3X_4X_5...$

3. $f_\infty(LHX_3X_4X_5...) = HHf_\infty(X_3X_4X_5...)$



Transformation of an Address String

Step	Address	Rule
0	LLLL...	2
1	HLLL...	1
2	LHHH...	3 and 1
3	HHLL...	1
4	LLHH...	2
5	HLHH...	1
6	LHLL...	3 and 2
7	HHHL...	1
8	LLLH...	2
9	HLLH...	1
10	LHHL...	3 and 1
11	HHLH...	1
12	LLHL...	2
13	HLHL...	1
14	LHLH...	3 (twice)
15	HHHH...	1
16	LLLL...	2

1. $f_\infty(HX_2X_3X_4...) = LX_2^TX_3^TX_4^T...$
2. $f_\infty(LLX_3X_4X_5...) = HLX_3X_4X_5...$
3. $f_\infty(LHX_3X_4X_5...) = HHf_\infty(X_3X_4X_5...)$

Transformation of an Address String

It is true that after 2^n iterations:

- The first n letters will go through a cycle of all possible permutations of H and L of length n
- Thus the first n letters will repeat themselves
- However the letters following will be different→ The orbit is not periodic
- In fact, the orbit gets arbitrarily close to any point in the address space

Orbital Dynamics: At the Feigenbaum Point

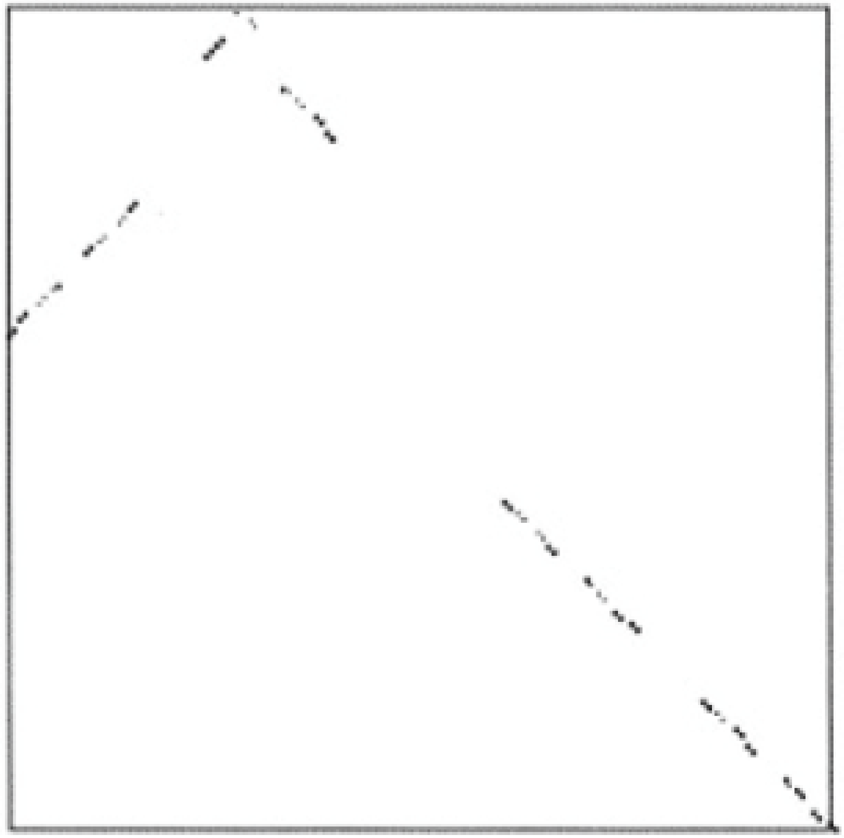
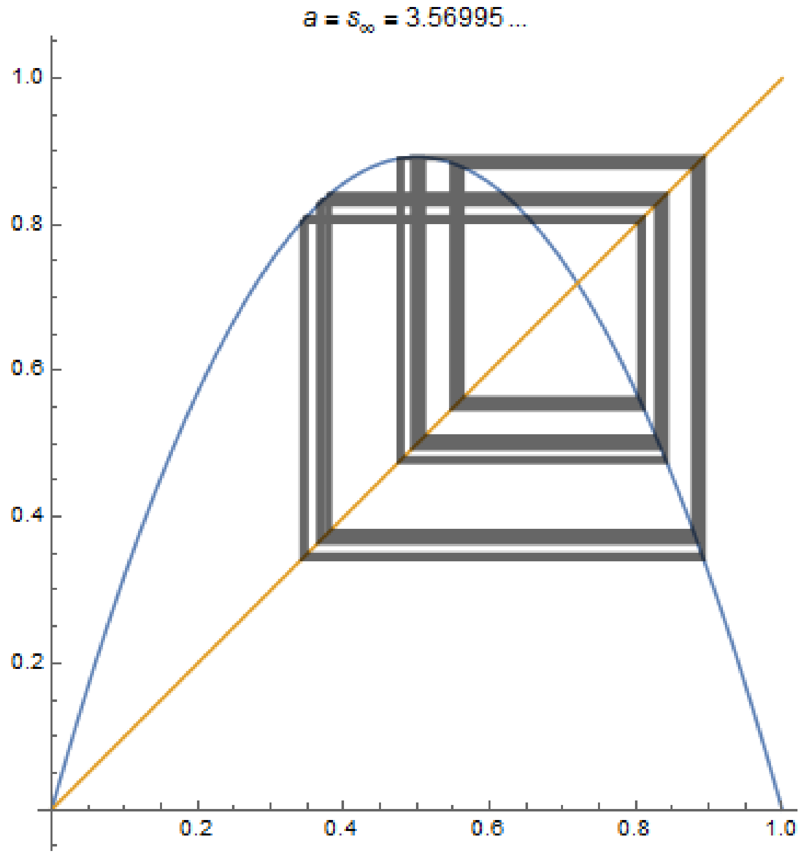


Image: Heinz-Otto Peitgen, Hartmut Jürgens, and Dietmar Saupe. *Chaos and fractals: new frontiers of science.* Springer Science & Business Media, 2004.



A Summary