Long wavelength spin dynamics of ferromagnetic condensates

(see arXiv:0710.1848)

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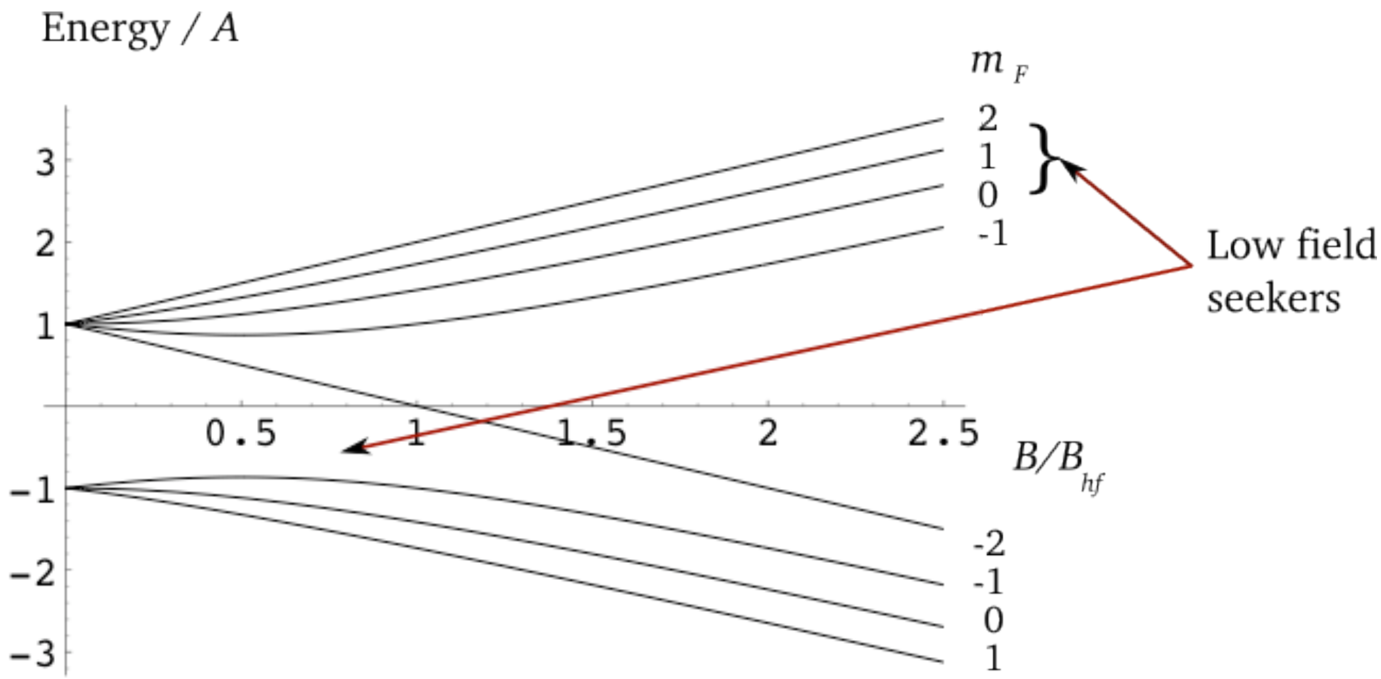
Outline

- Spinor condensates
- Equations of motion in the incompressible limit
- Conservation laws / Topological aspects
- Dipolar interactions
- Precessional instabilities

Hyperfine structure in a typical alkali

J=1/2 I=3/2 ⟶ F=2 or 1



- Magnetic traps - spin state follows field adiabatically
- Optical traps?

Order in spinor condensates

Every condensate has some `magnetic' order!

Single component case $\langle \hat{\varphi}(\mathbf{r}) \rangle = \Phi(\mathbf{r})$

$$\longrightarrow \langle \hat{\varphi}_s(\mathbf{r}) \rangle = \Phi_s(\mathbf{r})$$

ODLRO implies 2 particle order *of some kind*

$$\langle \hat{\varphi} \otimes \hat{\varphi}^\dagger \rangle \neq 0$$

`Spin 1/2'

Pseudospin 1/2 by populating only two states

Any Φ_s is an eigenstate of $\mathbf{n}\cdot\mathbf{S}$ for some $\mathbf{n}$

$$\Phi = e^{i\alpha}\begin{pmatrix} e^{-i\phi/2}\cos\theta/2 \\ e^{i\phi/2}\sin\theta/2 \end{pmatrix}$$

A spin-1/2 condensate is *always* ferromagnetic

Siggia and Ruckenstein, 1980

Spin 1

Decompose density matrix

$$\langle \hat{\phi} \otimes \hat{\phi}^{\dagger} \rangle = \rho + \mathbf{m} \cdot \mathbf{S} + N$$

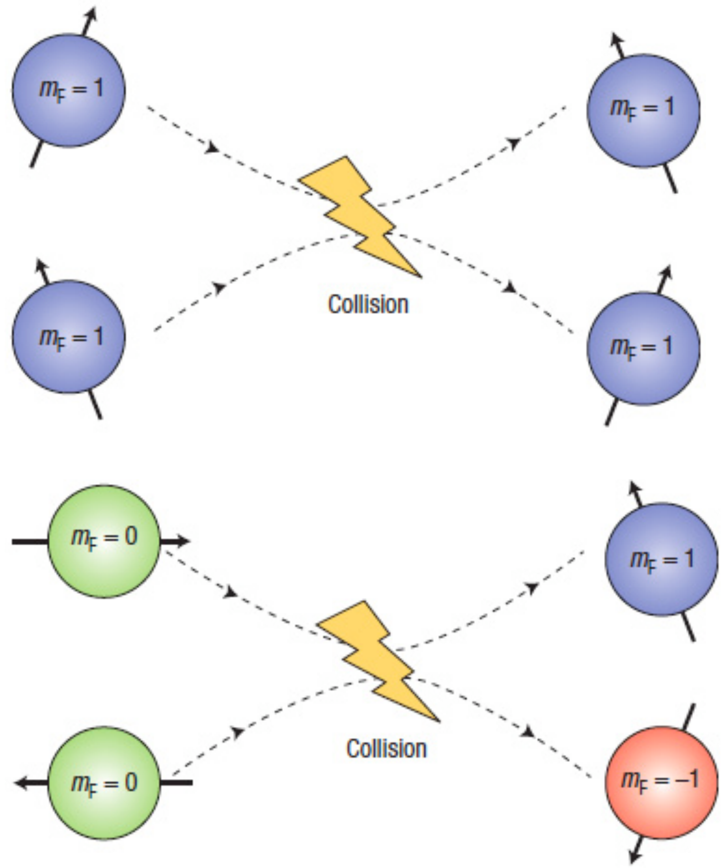
Nematicity *N* is traceless symmetric part

$$N_{\alpha\beta} = \left[\rho_{\alpha\beta} + \rho_{\beta\alpha}\right]/2 - (1/3)\delta_{\alpha\beta}\rho$$

Interactions in a F=1 condensate

Total spin 2

Total spin 0



$$\boxed{H_{\text{int}} = \frac{1}{2} c_0 n^2 + \frac{1}{2} c_2 \mathbf{F} \cdot \mathbf{F}}$$

$$c_0 = (g_0 + 2g_2)/3 \qquad c_2 = g_2 - g_0$$

Possible phases

$c_2<0$ gives ferromagnetism (e.g. ^{87}Rb) $\mathbf{m} \neq 0$

$$\varphi = \sqrt{n} e^{-i\tau} \begin{pmatrix} e^{-i\alpha} \cos^2 \frac{\beta}{2} \\ \sqrt{2} \cos \frac{\beta}{2} \sin \frac{\beta}{2} \\ e^{i\alpha} \sin^2 \frac{\beta}{2} \end{pmatrix}$$

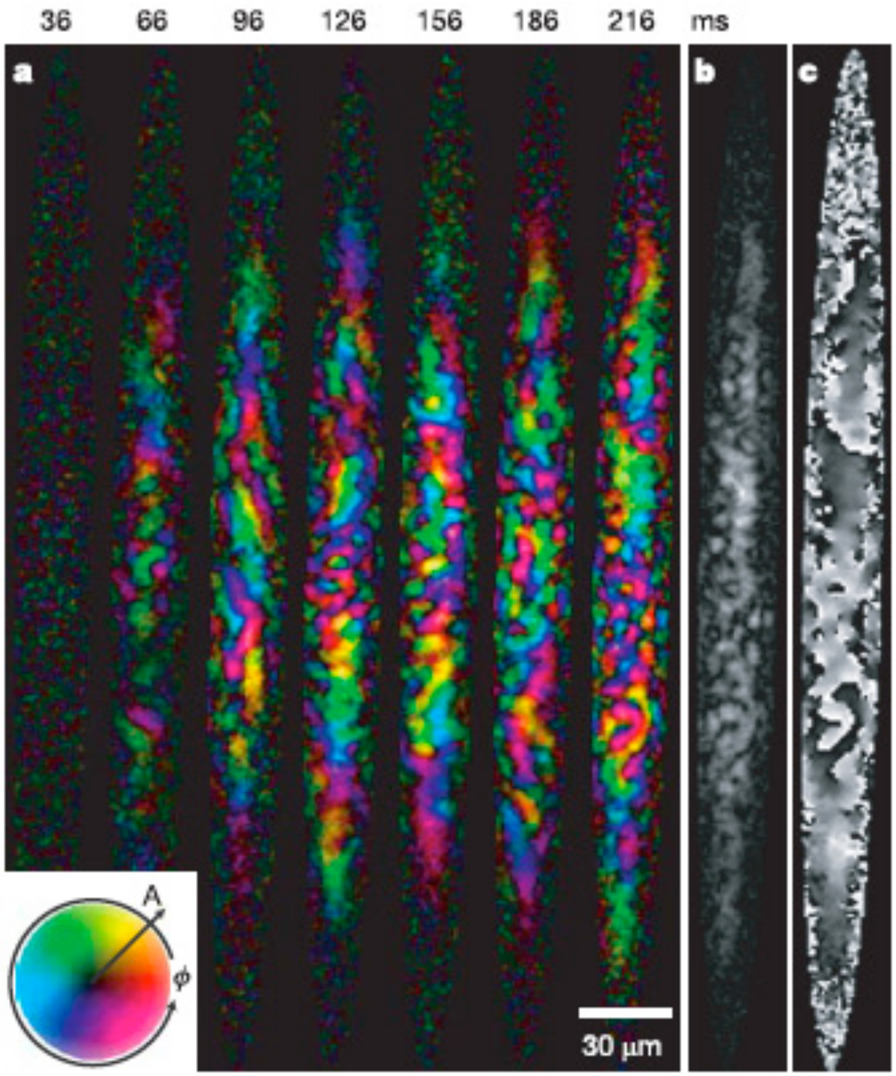
$T_{Ferro} > T_{BEC}$

$c_2>0$ gives polar or nematic state (e.g. ^{23}Na) $N \neq 0$

General case of spin *s* bosons: *s+1* scattering lengths

⟶ Complex! But always have the possibility of ferromagnetism

Berkeley experiment



Sadler et al, Nature 2006

Kibble-Zurek physics?

AL, 2007

M Uhlmann, R Schützhold, UR Fischer, 2006

H Saito, Y Kawaguchi, M Ueda, 2007

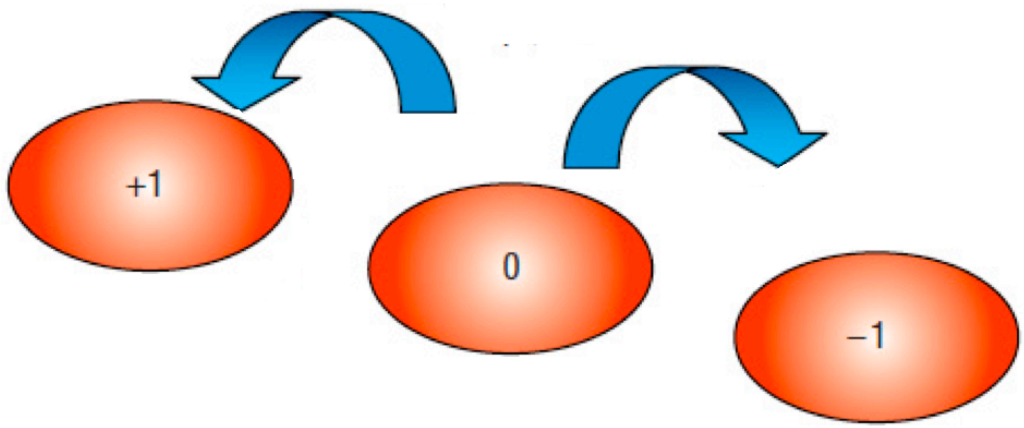
B Damski, WH Zurek, 2007

G Mias, N Cooper, S Girvin 2007

Origin of the instability

$$E_m^{\mathsf{Z}} \equiv -\tilde{p}m + qm^2 \qquad m = +1, 0, -1$$

At large q all atoms in m=0



Beneath some critical q_0 *instability* occurs

Leaves phase difference between $|\uparrow\rangle, |\downarrow\rangle$ undecided

→*XY ordering of transverse magnetization*

Equations of motion - motivation I

Practical question: how to study dynamics?

TDGPE usually sufficient in dilute systems

$$\begin{aligned} \mathcal{L} &= i\Phi^\dagger \partial_t \Phi - \mathcal{H}(\Phi^\dagger, \Phi) \\ \mathcal{H} &= \frac{1}{2}\left[\nabla\Phi^\dagger \nabla\Phi + c_0\left(\Phi^\dagger\Phi\right)^2 + c_2\left(\Phi^\dagger \mathbf{S}\Phi\right)^2\right], \end{aligned}$$

Avoid working with $2s+1$ component spinor?

Would prefer a description just of *spinwaves* and *superfluid flow*, even at high spin

Equations of motion - motivation II

Normal fluids

➔ approximately *incompressible* at low Mach number

$$\partial_t + \mathbf{v} \cdot \nabla \mathbf{v} = -\frac{\nabla P}{\rho}$$

$$(\nabla \cdot \mathbf{v} = 0)$$

"Normal" *superfluids*

➔ $\nabla \cdot \mathbf{v} = \nabla \times \mathbf{v} = 0$

Leaves only possibility of isolated vortex lines

In the spinor case this limit is non-trivial!

The Mermin-Ho relation

The velocity

$$\mathbf{v} = -i\Phi^\dagger \nabla \Phi$$

is not irrotational as in single-component case

$$\nabla \times \mathbf{v} = -i\nabla\Phi^\dagger \times \nabla\Phi$$

On the spin coherent states $\Phi_{\mathbf{n}}$

$$\nabla \times \mathbf{v} = s\nabla\varphi \times \nabla \cos\theta = \frac{s}{2}\varepsilon_{ijk} n_i \nabla n_j \times \nabla n_k$$

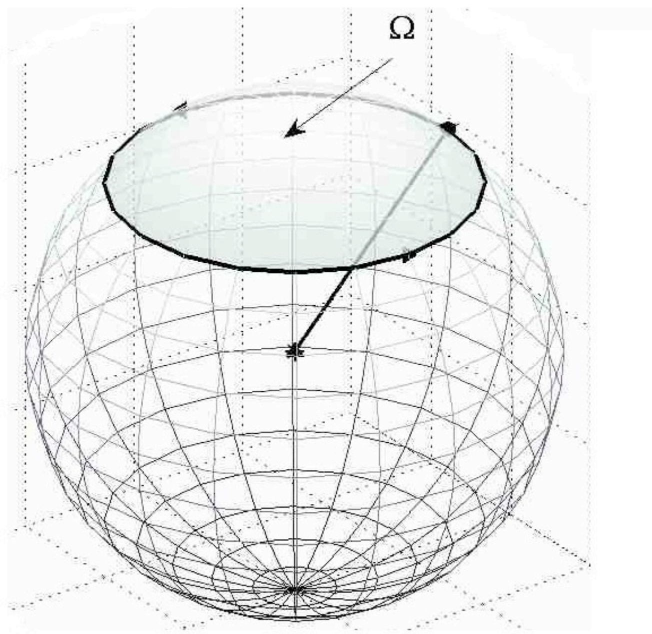
Geometrical interpretation: $\mathbf{n}$ constant on vorticity lines

Vorticity lines fill fluid, not confined to vortices!

Consequences for vortices

$$\nabla \times \mathbf{v} = s\nabla\phi \times \nabla \cos\theta = \frac{s}{2}\varepsilon_{ijk} n_i \nabla n_j \times \nabla n_k$$

$$\oint \mathbf{v} \cdot d\mathbf{l} = 2\pi n + s\Omega$$



Unwind in units of $4\pi s$

Implementing the incompressible limit I

$$
\begin{aligned}
\mathcal{L} &= i\Phi^\dagger\partial_t\Phi - \mathcal{H}(\Phi^\dagger,\Phi) \\
\mathcal{H} &= \frac{1}{2}\left[\nabla\Phi^\dagger\nabla\Phi + c_0\left(\Phi^\dagger\Phi\right)^2 + c_2\left(\Phi^\dagger\mathbf{S}\Phi\right)^2\right],
\end{aligned}
$$

Constrained system: uniform density, *maximal polarization*

Φ an eigenstate of spin or *coherent state*

Implementing the incompressible limit II

Usual prescription: parameterize constraint manifold

$$\mathbf{\Phi} = \mathbf{\Phi}_{\mathbf{n}} e^{i\theta}$$

There is a gauge freedom in the choice of phase

$$\mathbf{v} = \nabla\theta - \mathbf{a}$$

Incompressibility implies $\nabla \cdot \mathbf{v} = \nabla \cdot (\nabla\theta - \mathbf{a}) = 0$

$\mathbf{a} \equiv i\mathbf{\Phi}_{\mathbf{n}}^{\dagger}\nabla\mathbf{\Phi}_{\mathbf{n}}$ is Berry potential

But: Mermin-Ho relation *gauge invariant*

$$\nabla \times \mathbf{v} = -i\nabla\mathbf{\Phi}^{\dagger} \times \nabla\mathbf{\Phi} = \frac{s}{2}\varepsilon_{ijk} n_i \nabla n_j \times \nabla n_k$$

Variation of the action

Lagrangian takes the form

$$\begin{aligned}\mathcal{H} &= \left[\frac{1}{4}s\,(\nabla\mathbf{n})^2 + \frac{1}{2}(\nabla\theta - \mathbf{a})^2\right]\\ \mathcal{L} &= (a_t - \dot{\theta}) - \mathcal{H}\end{aligned}$$

We need the following variation

$$\delta\int d\mathbf{r}dt\, a_\mu j_\mu = \int d\mathbf{r}dt\ \left[s j_\mu \mathbf{n}\times\partial_\mu\mathbf{n}\cdot\delta\mathbf{n} - i\langle\mathbf{n}|\delta\mathbf{n}\rangle\partial_\mu j_\mu\right]$$

$$j_\mu = (1, \mathbf{v})$$

Gauge invariant for conserved current!

Equation of motion

$$j_\mu \partial_\mu \mathbf{n} \times \mathbf{n} - \frac{1}{2}\nabla^2 \mathbf{n} = 0$$

j_μ is just $(1, \mathbf{v})$

$$\frac{D\mathbf{n}}{Dt} - \frac{1}{2}\mathbf{n} \times \nabla^2 \mathbf{n} = 0$$

Modifed Landau-Lifshitz with *advection term* $\partial_t + \mathbf{v} \cdot \nabla$

Continuity equation for spin current

$$\mathbf{J}_\alpha = n_a \mathbf{v} - \frac{1}{2}\epsilon_{\alpha\beta\gamma} n_\beta \nabla n_\gamma$$

Final form of the equations of motion

$$\frac{D\mathbf{n}}{Dt} - \frac{1}{2}\mathbf{n} \times \nabla^2 \mathbf{n} = 0$$
$$\nabla \cdot \mathbf{v} = 0$$
$$\nabla \times \mathbf{v} = \frac{s}{2}\varepsilon_{\alpha\beta\gamma} n_\alpha \nabla n_\beta \times \nabla n_\gamma$$

Supplemented by the condition that $\mathbf{v}_\perp = 0$ at the boundary

(required so that $\delta\theta$ term vanishes

recall $\nabla \cdot (\nabla\theta - \mathbf{a}) = 0$)

Conservation laws

Vorticity obeys

$$\frac{D\omega_i}{Dt} = -\frac{s}{2}\varepsilon_{ijk}\partial_k\partial_l\sigma_{lj}.$$

$$\sigma_{ij} \equiv \frac{1}{2}\delta_{ij}\partial_k\mathbf{n}\cdot\partial_k\mathbf{n} - \partial_i\mathbf{n}\cdot\partial_j\mathbf{n}.$$

Implies conservation of *hydrodynamic impulse*

$$\mathbf{I} \equiv \frac{1}{2}\int \mathbf{r}\times\boldsymbol{\omega}$$

Rate of change given by total external forces on fluid
(with hard walls momentum remains zero)

What is the momentum of a spin wave?

Expand about uniform ferromagnet $n_z(\mathbf{r}) = 1$

$\mathbf{v}$ is 2nd order in deviation, so linear spinwaves unaffected

($\omega = \frac{1}{2}k^2$)

$$\mathbf{v} = is\mathcal{P}_T n_+ \nabla n_-$$

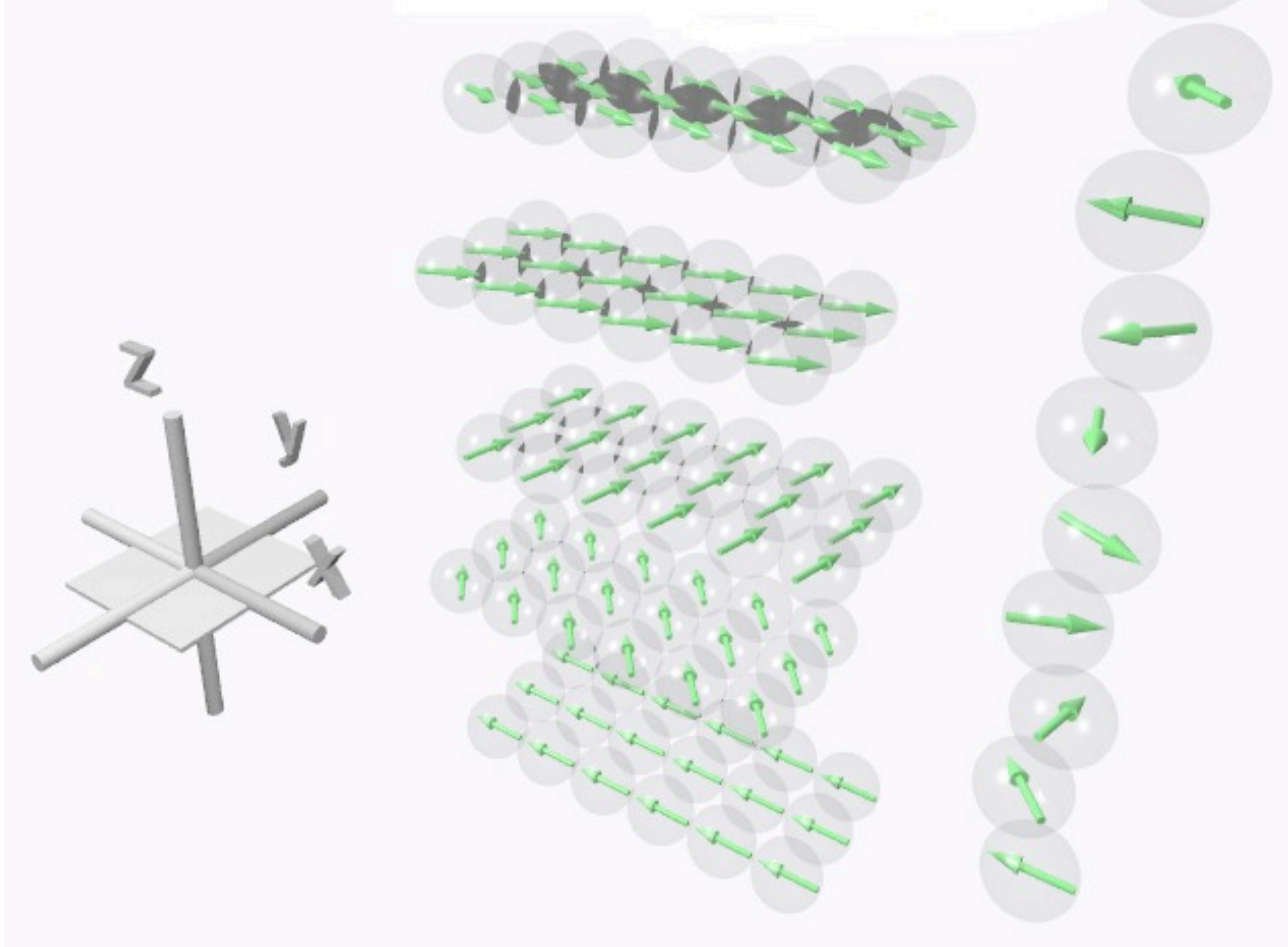
$$\mathcal{P}_T(\mathbf{q}) = 1 - \frac{\mathbf{q}(\mathbf{q}\cdot\ldots)}{q^2}$$

$$n_\pm = n_x \pm i n_y$$

C. Nayak *et al.* Phys. Rev. B 64, 235113(2001)

A simple example: the helix

$$\mathbf{n} = \mathbf{e}_x \cos \kappa z + \mathbf{e}_y \sin \kappa z$$



Instabilities of the helix - LL theory

Within the ordinary LL equation helix is unstable

$$\dot{\mathbf{n}} = \tfrac{1}{2}\mathbf{n} \times \nabla^2 \mathbf{n}$$

Linearized evolution reveals *isotropic* instability

$$\omega^2 = k^2/2(k^2/2 - \kappa^2/2)$$

in region $0 < k < \kappa$

Helix instability in the full theory

Because of the helical background spinwaves…

$$\delta\mathbf{n} = \eta_z(\mathbf{r},t)\mathbf{e}_z + \eta_\perp(\mathbf{r},t)\mathbf{e}_\perp(z)$$

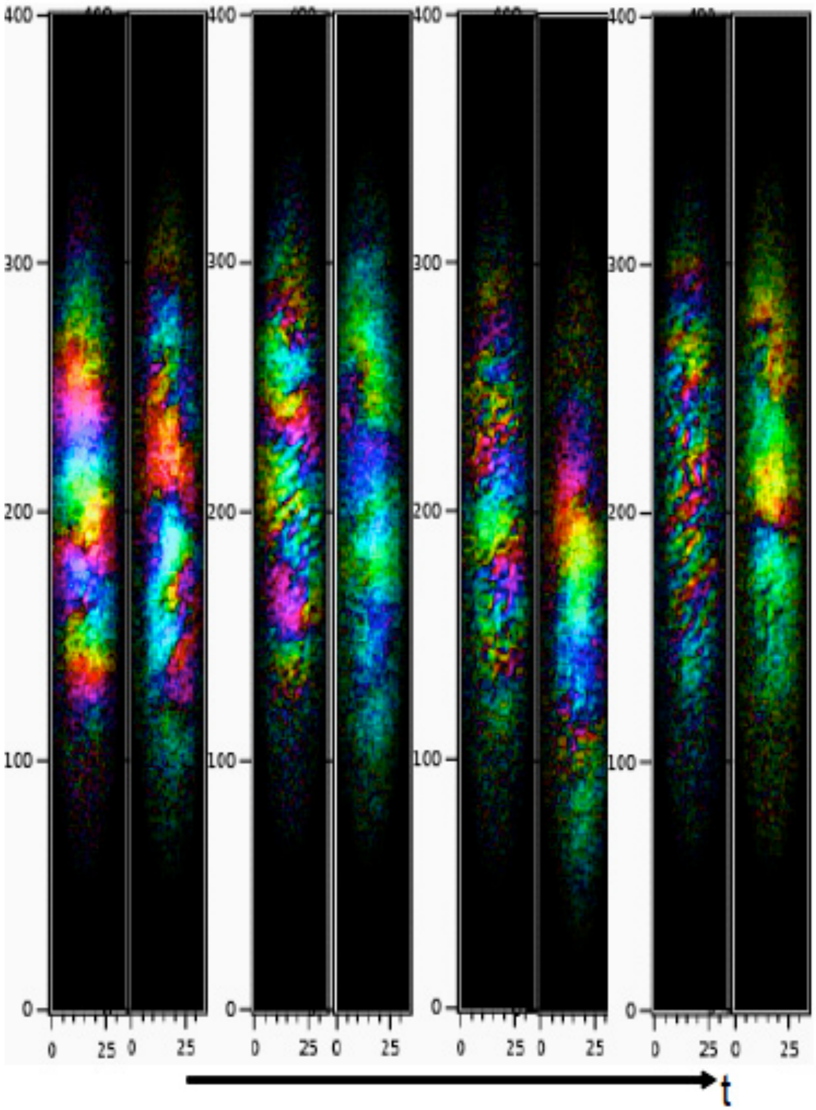
…generate a nonzero MH vorticity

$$\omega_{\mathsf{MH}}(\mathbf{r}) = \kappa s\left(\partial_y\eta_z\mathbf{e}_x - \partial_x\eta_z\mathbf{e}_y\right)$$

Transverse waves advect magnetization up helix

$$\omega^2 = \frac{k^2}{2}\left[\frac{k^2}{2} + \kappa^2\left(s\frac{k_\perp^2}{k^2} - \frac{1}{2}\right)\right]$$

Experiment



Berkeley group studied evolution of helices.

Observe short wavelength instability << pitch

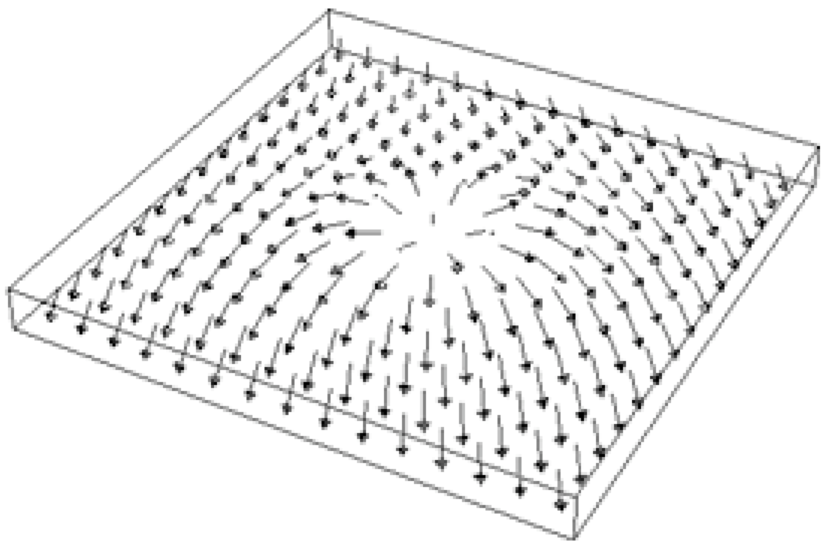
More detailed analysis

arXiv:0710.2499

R.W. Cherng *et al.*

Topological aspects

Skyrmions in 2D O(3) model



associated with nonzero values of integer topological invariant

$$\frac{1}{8\pi}\int d^2\mathbf{r}\epsilon_{ij}\epsilon_{\alpha\beta\gamma}n_\alpha\partial_i n_\beta\partial_j n_\beta$$

(*Degree* of map from S^2 to R^2)

Skyrmions in the ferromagnetic condensate?

$\Phi = \Phi_{\mathbf{n}} e^{i\theta}$ manifold now $SU(2) \sim S^3$

Homotopy group $\pi_i(S^n)$ trivial for $i < n$

$$\pi_n(S^n) = \mathbb{Z}$$

MH relates degree of map $\mathbb{R}^3 \longrightarrow SU(2)$ to helicity invariant of hydrodynamics

$$\int d^3\mathbf{r}\,\mathbf{v}\cdot\nabla\times\mathbf{v} = \frac{s}{2}\int d^3\mathbf{r}\,\mathbf{v}\varepsilon_{\alpha\beta\gamma}n_\alpha\nabla n_\beta\times\nabla n_\gamma$$

Attempts to find stable Skyrmions in 3D have so far failed!

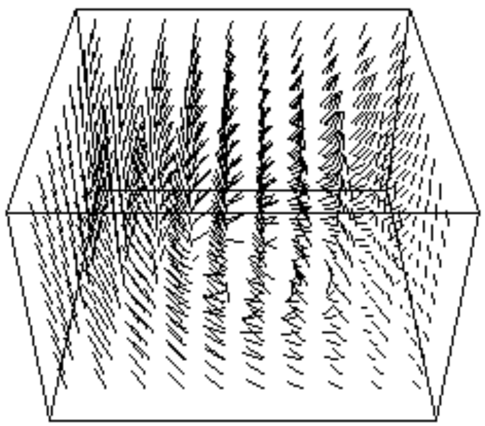
A simple ansatz

$$\Phi = \exp(i\omega(r)\hat{\mathbf{r}}\cdot\mathbf{S})\Phi_{\mathbf{e}_z}$$

(Herbut + Oshikawa 2006)

Produces a static solution for *spin 1/2 only*, if

$$\omega''(r) + \frac{2\omega'(r)}{r} - \frac{2\sin\omega}{r^2} = 0$$

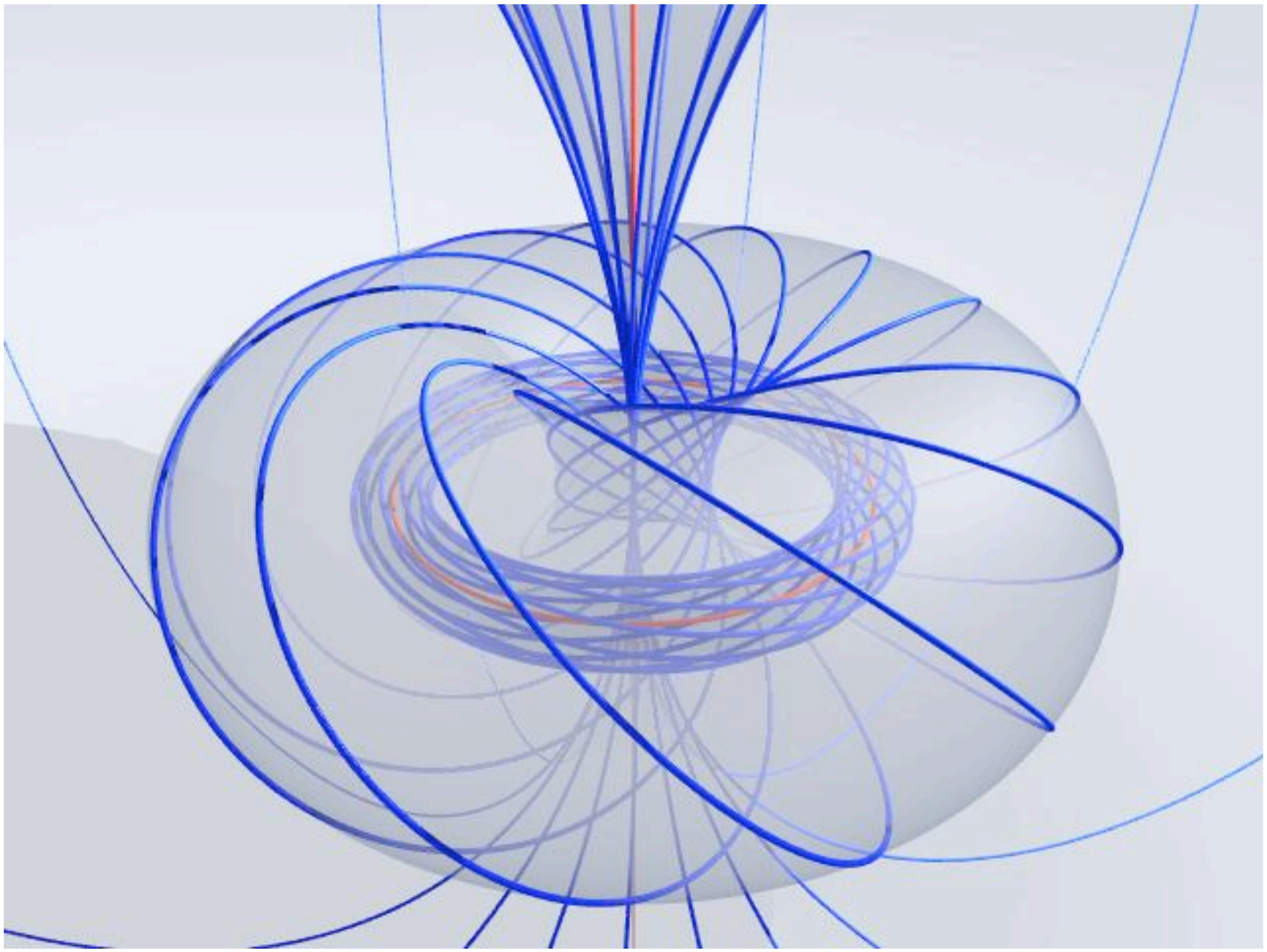


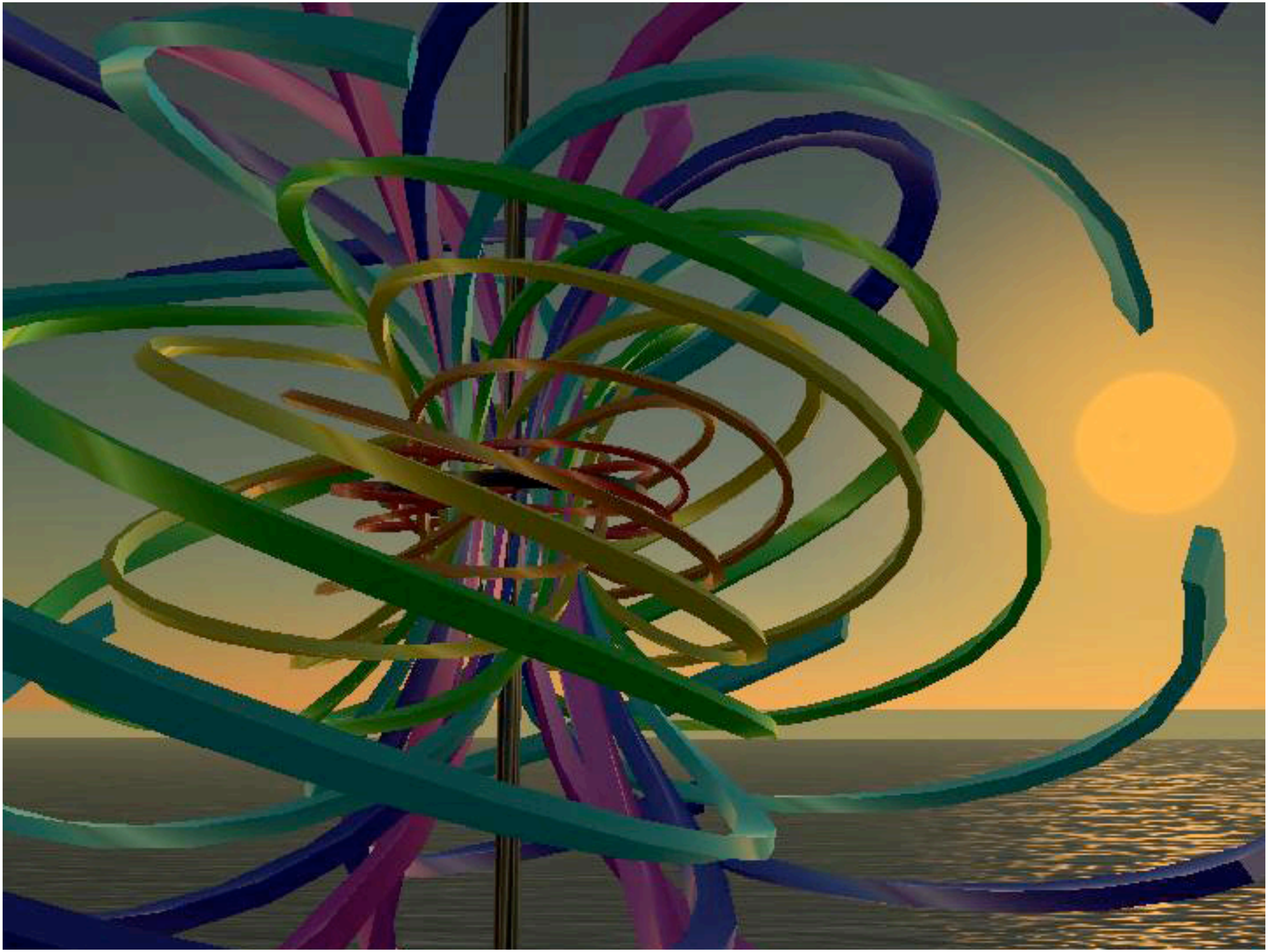
(unfortunately, it's a *meron*)

Ansatz far from symmetric

Not clear why it works (at least to me)

Hopf map





What about dipolar interactions?

$$U_{\mathsf{dip}} = \frac{\mu_0 (g\mu_B)^2}{4\pi r^3} \left[\mathbf{S}_1 \cdot \mathbf{S}_2 - 3 \left(\mathbf{S}_1 \cdot \hat{\mathbf{r}} \right) \left(\mathbf{S}_2 \cdot \hat{\mathbf{r}} \right) \right]$$

Interesting many body physics?

Problem - energy scales are tiny (nK / 10 Hz); easily swamped by magnetic fields, etc.

Recent excitement:

- ^{52}Cr condensate (spin 3, but J=3, U_{dip} 36x bigger)
- Polar molecules (permanent electric dipoles)

Easy plane anisotropy

Consider an infinite plane. Mag. energy density is

$$e_{\mathsf{mag}}(\mathbf{M}) = \mu_0 d \left[\frac{1}{2} M_z^2 - \mathbf{H} \cdot \mathbf{M}\right]$$

For a gas of density 10^{-14} cm^{-3}, B_{demag} is of order 10^{-5} Gauss.

For $k \gg d$ can use this as a *local* energy functional

Deviations have the non-analytic form $\sim k_\alpha k_\beta / |k|$

Precession with easy plane anisotropy

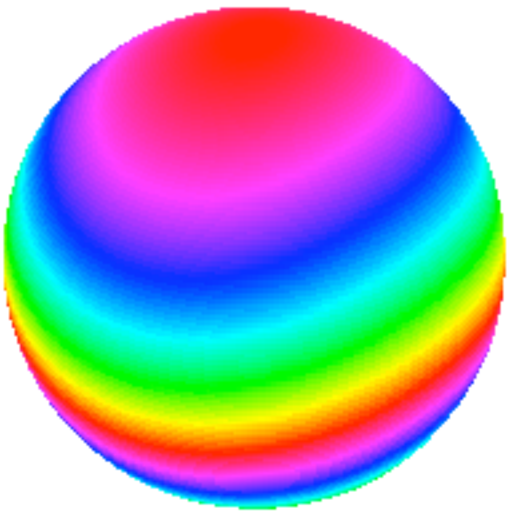
$$\dot{\mathbf{n}} = \mathbf{n} \times \frac{\partial e_{\mathsf{mag}}}{\partial \mathbf{n}}$$

$$e_{\mathsf{mag}}(\mathbf{n}) = \frac{1}{2}\omega_{\perp} n_x^2 + \omega_L n_z$$

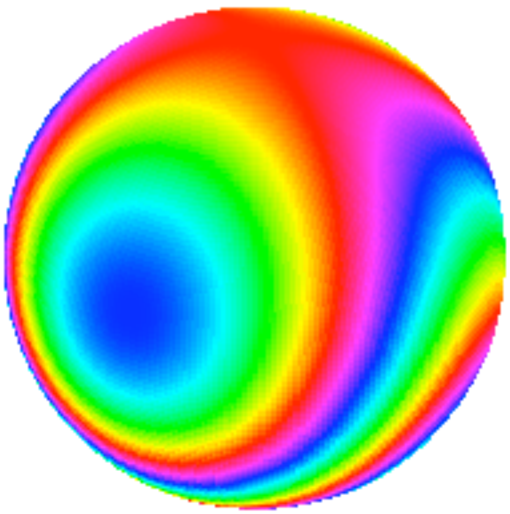
$$\omega_{\mathsf{res}} = \sqrt{\omega_L(\omega_L + \omega_{\perp})}$$

Kittel, 1948

$\omega_{\perp} < \omega_L$



$\omega_{\perp} > \omega_L$



Precessional instabilities I

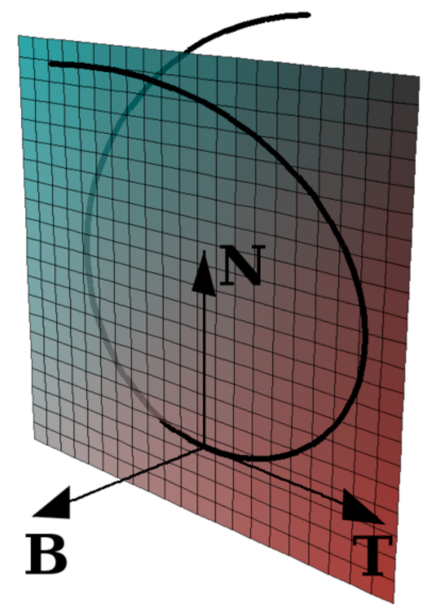
Linearized spinwaves in precessing frame on top of $\mathbf{n}_0(t)$

$$\epsilon_{ab}\partial_t\eta_b = \frac{1}{2}\nabla^2\eta_a + \omega_\perp\left(\eta_a\cos^2\phi(t) - \mathbf{e}_a^x(t)\mathbf{e}_b^x(t)\eta_b\right)$$

Kashuba, 2006

$\{\mathbf{n}_0, \mathbf{e}_1, \mathbf{e}_2\}$ form orthonormal Frenet-Serret frame

$$\partial_t\begin{pmatrix}\mathbf{n}_0\\ \mathbf{e}_1\\ \mathbf{e}_2\end{pmatrix} = \begin{pmatrix}0 & \kappa & 0\\ -\kappa & 0 & \tau\\ 0 & -\tau & 0\end{pmatrix}\begin{pmatrix}\mathbf{n}_0\\ \mathbf{e}_1\\ \mathbf{e}_2\end{pmatrix}$$



Precessional instabilities II

Linearized spinwaves in precessing frame

$$\epsilon_{ab}\partial_t\eta_b = \frac{1}{2}\nabla^2\eta_a + \omega_\perp\left(\eta_a\cos^2\phi(t) - \mathbf{e}_a^x(t)\mathbf{e}_b^x(t)\eta_b\right)$$

Kashuba, 2006

Dipolar interactions give possibility of *parametric resonance*

for spinwaves with $\omega_k = n\omega_L$

Lowest order resonance

$$\omega = \omega_k - \omega_L \pm \sqrt{(\omega_k - \omega_L)^2 - \frac{\omega_\perp^2}{4}\left(1 + n_z\right)^4}$$

Smoking gun would be variation of wavelength with fields due to resonance condition

Summary

- Equations of motion in the incompressible limit
- Conservation laws (how to include vortices?)
- Precessional instabilities

• Atomic gases may be a good place to look for *highly non-linear* magnetic configurations (topological defects, large angle precession)

• Dynamics may be more relevant for experiment than equilibrium physics

• Looking for postdocs! See faculty.virginia.edu/austen/