

Extending classical logic for reasoning about quantum systems

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Abstract

A brief presentation of the (essence of) EQPL (\dagger),
a **decidable** (\ddagger) **conservative extension of propositional logic**
for reasoning about the **state** of a **quantum system**,

including:

- key design options (starting from the QM postulates);
- syntax, semantics, (complete) axiomatization;
- decidability;
- practical impact so far.

(\dagger) Joint work with **Paulo Mateus**.

(\ddagger) Joint work also with **Rohit Chadha** and **Cristina Sernadas**.

Work environment

Transdisciplinary team at IST since 2004

- Physicists;
- Mathematical logicians;
- Computer scientists;
- Information theoreticians (recent addition).

PhD Program in Physics and Mathematics of Information

Just starting with FCT support for 10 new scholarships per year.

From classical to quantum reasoning

Birkhoff and von Neumann (1936)

Algebraic semantics

CPL → QPL

Boolean algebra → complete lattice of closed subspaces
of the Hilbert space at hand
with orthogonal complement

Exogenous approach (2006)

Valuation semantics

CPL → EQPL

valuation → superposition of valuations

Key design options

A matter of cats

Is the cat awake or asleep?

Is the cat indoors or outdoors?

NO CAT WAS HARMED IN THE PREPARATION OF THIS
TALK OR WILL BE HARMED DURING THE TALK.

The classical cat

Either awake or asleep and either indoors or outdoors

Four classical valuations:

$$|00\rangle = \begin{cases} \text{awake} \mapsto 0 \\ \text{indoors} \mapsto 0 \end{cases}$$

$$|10\rangle = \begin{cases} \text{awake} \mapsto 1 \\ \text{indoors} \mapsto 0 \end{cases}$$

$$|01\rangle = \begin{cases} \text{awake} \mapsto 0 \\ \text{indoors} \mapsto 1 \end{cases}$$

$$|11\rangle = \begin{cases} \text{awake} \mapsto 1 \\ \text{indoors} \mapsto 1 \end{cases}$$

Classical valuation space:

$$\mathcal{V} = \{|00\rangle, |01\rangle, |10\rangle, |11\rangle\}$$

The quantum cat

Somewhere between asleep outdoors and awake indoors...

Each quantum valuation is a unit **superposition** of classical valuations:

$$|\psi\rangle = z_{00}|00\rangle + z_{01}|01\rangle + z_{10}|10\rangle + z_{11}|11\rangle$$

with **amplitudes** (the z 's) in \mathbb{C} and such that $\|\psi\| = 1$

Quantum valuation space:

$\mathcal{H}_1^{\mathbb{C}}(\mathcal{V})$ = the surface of the unit sphere of

$\mathcal{H}^{\mathbb{C}}(\mathcal{V})$ = the Hilbert space over \mathbb{C} freely generated from \mathcal{V}
 = $\text{span}^{\mathbb{C}}(\mathcal{V})$ pt \mathcal{V} is finite as in the case at hand.

RECALL

QM Postulate 1

The *state space* of an isolated quantum system is the surface of the unit sphere of a *Hilbert space* over \mathbb{C} .

QM Postulate 2

The Hilbert space of a quantum system composed of a finite number of *independent components* is the *tensor product* of the component Hilbert spaces.

If awokeness and whereabouts are independent

not all superposition are possible

$$\mathcal{H}^{\mathbb{C}}(\mathcal{V}) = \mathcal{H}^{\mathbb{C}}(\mathcal{V}_{[\text{awake}]}) \otimes \mathcal{H}^{\mathbb{C}}(\mathcal{V}_{[\text{indoors}]})$$

$$|\psi\rangle = |\psi'\rangle \otimes |\psi''\rangle$$

$$|\psi'\rangle \in \mathcal{H}_1^{\mathbb{C}}(\mathcal{V}_{[\text{awake}]}) =$$

$$\mathcal{H}_1^{\mathbb{C}}(\{|00\rangle|_{\text{awake}}, |01\rangle|_{\text{awake}}, |10\rangle|_{\text{awake}}, |11\rangle|_{\text{awake}}\}) =$$

$$\mathcal{H}_1^{\mathbb{C}}(\{\mathbf{awake} \mapsto 0, \mathbf{awake} \mapsto 1\})$$

$$|\psi''\rangle \in \mathcal{H}_1^{\mathbb{C}}(\mathcal{V}_{[\text{indoors}]}) =$$

$$\mathcal{H}_1^{\mathbb{C}}(\{\mathbf{indoors} \mapsto 0, \mathbf{indoors} \mapsto 1\})$$

If awakesness and whereabouts are not independent

the two qubits may be entangled

$$\mathcal{H}^{\mathbb{C}}(\mathcal{V}) \neq \mathcal{H}^{\mathbb{C}}(\mathcal{V}_{[\text{awake}]}) \otimes \mathcal{H}^{\mathbb{C}}(\mathcal{V}_{[\text{indoors}]})$$

$|\psi\rangle \neq |\psi'\rangle \otimes |\psi''\rangle$ is possible

for instance

$$|\psi\rangle = \sqrt{\frac{1}{2}} |00\rangle + \sqrt{\frac{1}{2}} |11\rangle = \sqrt{\frac{1}{2}} (|00\rangle + |11\rangle)$$

(a state with maximum entanglement — a Bell state)

The probabilistic result of observing a quantum cat

From amplitudes to probabilities

If the quantum cat is in state

$$|\psi\rangle = \sqrt{\frac{1}{3}} (|00\rangle + i|01\rangle + |11\rangle)$$

the distribution of the random outcome of the logical projective **observation** of its **awakeness** is as follows:

$$\mu_{|\psi\rangle}^{\text{awake}} = \begin{cases} (\text{awake} \mapsto 0) \mapsto \frac{2}{3} \\ (\text{awake} \mapsto 1) \mapsto \frac{1}{3} \end{cases}$$

RECALL

QM Postulate 3

Every *measurable physical quantity* of an isolated quantum system is described by an *observable* acting on its Hilbert space.

FURTHERMORE

QM Postulate 4 (discrete case)

The only possible *outcomes* of the measurement of a physical quantity are the *eigenvalues* of its observable.

When the physical quantity is measured on a system at state $|\psi\rangle$ using *observable* A with countable spectrum E^A , the resulting outcomes are random with distribution

$$\mu_{|\psi\rangle}^A = e \mapsto \|Proj_e|\psi\rangle\|^2 : E^A \rightarrow [0, 1].$$

When outcome e is observed, the state of the system becomes the unit vector obtained by normalizing $Proj_e|\psi\rangle$.

EN PASSANT

Statistical ensembles

Hence, when measurements are made along the way, for dealing with the **evolution of the system** one needs to work with **mixed states** (probabilistic mixtures of superpositions).

Such mixed states are better represented by **density operators**.

Changes to the system correspond to **superoperators** that subsume both unitary transformations and measurements in the realm of density operators.

NOT TODAY :-)

Observing some of a finite number of qubits

Logical projective measurement of $F \subseteq Q$

$$A^F = \left(\sum_{v \in \mathcal{V}_{[F]}} |v\rangle\langle v| \right) \otimes \text{Id}_{[F]^c}$$

$$E^F = \mathcal{V}_{[F]}$$

$$\mathcal{E}_v^F = v \otimes \text{span}^{\mathbb{C}}(\mathcal{V}_{[F]^c})$$

where

$\mathcal{V}_{[F]}$ = set of Q -valuations restricted to F

$\mathcal{V}_{[F]^c}$ = set of Q -valuations restricted to $Q \setminus F$

By the way...

Non-deterministic cat

Non-empty set of valuations.

A quantum cat induces a non-deterministic cat:

$$V = \text{support of } |\psi\rangle$$

Probabilistic cat

Random valuation.

Measuring qubits of a quantum cat results in a probabilistic cat.

EQPL

EQPL overview

Signature

Q = finite set of qubit symbols

Semantics

Each **quantum valuation** is a unit superposition of classical valuations over Q :

$$|\psi\rangle \in \mathcal{H}_1^{\mathbb{C}}(2^Q)$$

In fact, we need a bit more of structure...

Language

Rich enough to express:

- classical constraints;
- independence of some qubits from the other qubits;
- probabilities of measurement outcomes;
- amplitudes of the quantum state at hand.

Only

for reasoning about the [state](#) of a [quantum system](#).

Extensions required

for reasoning about the [evolution](#) of a [quantum system](#)

taking also into account [QM Postulate 5](#).

EQPL syntax

Classical formulas

$$\alpha := \text{ff} \mid q \mid (\alpha \Rightarrow \alpha)$$

Real and complex terms ($F \subseteq G \subseteq Q$)

$$t := x \mid 0 \mid 1 \mid (t + t) \mid (t t) \mid \text{Re}(u) \mid \text{Im}(u) \mid |u| \mid (\int \alpha)$$

$$u := z \mid \langle F|G \rangle \mid (t + it) \mid \bar{u} \mid (u + u) \mid (u u) \mid (\alpha \triangleright u; u)$$

Quantum formulas ($G \subseteq Q$)

$$\gamma := \alpha \mid (t \leq t) \mid [G] \mid \mathbb{F} \mid (\gamma \sqsupset \gamma)$$

Other connectives as abbreviations

Classical

Quantum

 \neg \boxplus \wedge \boxcap \vee \boxcup \Leftrightarrow \equiv

For instance,

 $(\neg \alpha)$ for $(\alpha \Rightarrow \mathbf{ff})$ $(\boxplus \gamma)$ for $(\gamma \boxsup \mathbf{FF})$

Modalities as abbreviations of probabilistic assertions

$(\diamond \alpha)$ for $(0 < (\int \alpha))$

$(\square \alpha)$ for $(1 = (\int \alpha))$

Other useful abbreviations in applications ($F \subseteq G \subseteq Q$)

$$\left(\bigwedge_G F \right) \quad \text{for} \quad \left(\left(\bigwedge_{q \in F} q \right) \wedge \left(\bigwedge_{q \in G \setminus F} (\neg q) \right) \right)$$

$$\langle F \supset \alpha | G \rangle \quad \text{for} \quad \left(\left(\left(\bigwedge_G F \right) \Rightarrow \alpha \right) \triangleright \langle F | G \rangle; 0 \right)$$

$$\left(\diamond_{[G]}^u \alpha \right) \quad \text{for} \quad \left([G] \sqcap (0 < |u|) \sqcap \left(\bigsqcup_{F \subseteq G} (\langle F \supset \alpha | G \rangle = u) \right) \right)$$

in the superposition of independent component G
there is a valuation making α true with amplitude u

Examples assuming $\mathbf{awake, indoors, running} \in Q$

1. $(\mathbf{running} \Rightarrow \mathbf{awake})$
2. $[\mathbf{awake, indoors, running}]$
3. $(\boxplus [\mathbf{awake}])$
4. $((\diamond \mathbf{awake}) \sqcap (\diamond(\neg \mathbf{awake})))$
5. $((\int \mathbf{awake}) = \frac{1}{9})$
6. $\left(\diamond_{[\mathbf{awake, running}] }^{i\frac{1}{3}} \mathbf{running} \right)$

EQPL (relaxed) semantics

Quantum structure over Q

$$w = (\mathcal{K}, V, \mathcal{S}, |\psi\rangle, \zeta)$$

replacing \mathbb{C} by \mathcal{K}^\bullet

(the algebraic closure of the ordered real closed field \mathcal{K}).

Quantum structure over Q : $w = (\mathcal{K}, V, \mathcal{S}, |\psi\rangle, \zeta)$

\mathcal{K} ordered real closed field

$\emptyset \neq V \subseteq 2^Q$ (admissible valuations)

\mathcal{S} partition of Q (independent components)

$|\psi\rangle = \{|\psi\rangle_S\}_{S \in \mathcal{S}}$ with each $|\psi\rangle_S \in \mathcal{H}_1^{\mathcal{K}^\bullet}(2^S)$

$$|\psi\rangle_\emptyset = 1$$

$$|\psi\rangle_{S_1 \cup \dots \cup S_n} = |\psi\rangle_{S_1} \otimes \dots \otimes |\psi\rangle_{S_n}$$

such that $\langle v | \psi \rangle_Q = 0$ whenever $v \notin V$

$\zeta = \{\zeta_G^F\}_{F \subseteq G \subseteq Q}$ with each $\zeta_G^F \in \mathcal{K}^\bullet$

such that $\zeta_G^F = \langle v_G^F | \psi \rangle_G$ whenever $G \in \text{Alg}(\mathcal{S})$

$$\text{where } v_G^F(q) = \begin{cases} 1 & \text{if } q \in F \\ 0 & \text{if } q \in G \setminus F \end{cases}$$

Denotation of terms by w and ρ

$$\llbracket x \rrbracket_{w\rho} = \rho(x)$$

$$\llbracket t_1 + it_2 \rrbracket_{w\rho} = \llbracket t_1 \rrbracket_{w\rho} + i\kappa \cdot \llbracket t_2 \rrbracket_{w\rho}$$

$$\llbracket (f \alpha) \rrbracket_{w\rho} = \sum_{v \Vdash^c \alpha} \|\langle v | \psi \rangle\|^2$$

$$\llbracket z \rrbracket_{w\rho} = \rho(z)$$

$$\llbracket \langle F | G \rangle \rrbracket_{w\rho} = \zeta_G^F$$

$$\llbracket (\alpha \triangleright u_1; u_2) \rrbracket_{w\rho} = \begin{cases} \llbracket u_1 \rrbracket_{w\rho} & \text{if } V \Vdash^c \alpha \\ \llbracket u_2 \rrbracket_{w\rho} & \text{otherwise} \end{cases}$$

...

Satisfaction of quantum formulas by w and ρ

$$w\rho \Vdash \alpha \quad \text{iff} \quad \mathcal{V} \Vdash^c \alpha$$

$$w\rho \Vdash (t_1 \leq t_2) \quad \text{iff} \quad \llbracket t_1 \rrbracket_{w\rho} \leq_{\mathcal{K}\bullet} \llbracket t_2 \rrbracket_{w\rho}$$

$$w\rho \Vdash [G] \quad \text{iff} \quad G \in \text{Alg}(\mathcal{S})$$

$$w\rho \not\Vdash \mathbb{F}$$

$$w\rho \Vdash (\gamma_1 \supset \gamma_2) \quad \text{iff} \quad w\rho \not\Vdash \gamma_1 \text{ or } w\rho \Vdash \gamma_2$$

...

Quantum connectives do not collapse into classical connectives

For instance:

$$(\neg \alpha) \quad \vDash \quad (\boxplus \alpha)$$

$$(\boxplus \alpha) \quad \not\vDash \quad (\neg \alpha)$$

$$\not\vDash \quad ((\boxplus \alpha) \sqsupset (\neg \alpha))$$

$$(\alpha_1 \Rightarrow \alpha_2) \quad \vDash \quad (\alpha_1 \sqsupset \alpha_2)$$

$$(\alpha_1 \sqsupset \alpha_2) \quad \not\vDash \quad (\alpha_1 \Rightarrow \alpha_2)$$

But quantum tautologies are still valid. For example:

$$\vDash \quad (\gamma \sqsupset \gamma)$$

$$\vDash \quad (\gamma \vee (\boxplus \gamma))$$

EQPL axiomatization

Inference rules

CMP $\alpha_1, (\alpha_1 \Rightarrow \alpha_2) \vdash \alpha_2$

QMP $\gamma_1, (\gamma_1 \sqsupset \gamma_2) \vdash \gamma_2$

Axioms

CTaut $\vdash \alpha$ for each classical tautology α

QTaut $\vdash \gamma$ for each quantum tautology γ

ACORCF $\vdash \kappa_{\vec{t}, \vec{u}}^{\vec{x}, \vec{z}}$ for each instance $\kappa_{\vec{t}, \vec{u}}^{\vec{x}, \vec{z}}$
of a theorem κ of the ACORCF theory

$$\kappa := (a \leq a) \mid \mathbb{F} \mid (\kappa \sqsupset \kappa)$$

$$a := x \mid 0 \mid 1 \mid (a + a) \mid (a a) \mid \text{Re}(b) \mid \text{Im}(b) \mid |b|$$

$$b := z \mid (b + ib) \mid \bar{b} \mid (b + b) \mid (b b)$$

Axioms (cont)

$$\text{Lift } \Rightarrow \quad \vdash ((\alpha_1 \Rightarrow \alpha_2) \sqsupset (\alpha_1 \sqsupset \alpha_2))$$

$$\text{Clps } \mathbb{F} \quad \vdash (\mathbb{f} \equiv \mathbb{F})$$

$$\text{Ref } \sqcap \quad \vdash ((\alpha_1 \sqcap \alpha_2) \sqsupset (\alpha_1 \wedge \alpha_2))$$

Axioms (cont)

$$\text{Ind } \emptyset \quad \vdash [\emptyset]$$

$$\text{Ind } \cup \quad \vdash ([G_1] \supset ([G_2] \supset [G_1 \cup G_2]))$$

$$\text{Ind } \setminus \quad \vdash ([G] \equiv [Q \setminus G])$$

Axioms (cont)

$$\text{Empty} \quad \vdash (\langle \emptyset | \emptyset \rangle = 1)$$

$$\text{NAdm} \quad \vdash \left(\left(\neg \left(\bigwedge_G F \right) \right) \supset (\langle F | G \rangle = 0) \right)$$

$$\text{Unit} \quad \vdash \left([G] \supset \left(\left(\sum_{F \subseteq G} |\langle F | G \rangle|^2 \right) = 1 \right) \right)$$

$$\text{Prod} \quad \vdash \left(([G_1] \sqcap [G_2]) \supset \right. \\ \left. (\langle F_1 \cup F_2 | G_1 \cup G_2 \rangle = (\langle F_1 | G_1 \rangle \langle F_2 | G_2 \rangle)) \right) \\ \text{provided that } G_1 \cap G_2 = \emptyset$$

Axioms (conc)

$$\text{If } \# \quad \vdash (\alpha \sqsupset ((\alpha \triangleright u_1; u_2) = u_1))$$

$$\text{If } \# \# \quad \vdash ((\exists \alpha) \sqsupset ((\alpha \triangleright u_1; u_2) = u_2))$$

$$\text{Prob} \quad \vdash \left(\left(\int \alpha \right) = \sum_{F \subseteq Q} |\langle F \triangleright \alpha | Q \rangle|^2 \right)$$

Metatheorems

- MTD:

$$\Gamma, \gamma_1 \vdash \gamma_2 \text{ iff } \Gamma \vdash (\gamma_1 \supset \gamma_2).$$

- MTRA:

$$\text{If } \Gamma, \gamma \vdash \mathbb{F} \text{ then } \Gamma \vdash (\boxplus \gamma).$$

- PSQEF:

$$\vdash ((\gamma_1 \equiv \gamma_2) \supset (\gamma \equiv \gamma'))$$

provided that γ' is obtained by replacing zero or more **q-occurrences** of γ_1 by γ_2 in γ .

- PSCEF:

$$\vdash ((\alpha_1 \Leftrightarrow \alpha_2) \supset (\gamma \equiv \gamma'))$$

provided that γ' is obtained by replacing zero or more occurrences of α_1 by α_2 in γ .

Main results

Main results

Conservativeness of the extension $\text{CPL} \hookrightarrow \text{EQPL}$

$$\vDash^c \alpha \text{ iff } \vDash \alpha$$

Soundness and completeness of EQPL

$$\vDash \gamma \text{ iff } \vdash \gamma$$

Decidability of EQPL

\emptyset^+ is decidable

Strong versions of these results also hold.

Outline of the proof of weak completeness and decidability

Fagin–Halpern–Megiddo technique (1990)

Originally proposed for a probabilistic logic
(simpler than the probabilistic fragment of EQPL):

- Key idea:
Reduce any formula to a disjunction of systems of linear inequations over the real numbers where each variable represents the probability of a classical molecular formula.
- Model exists iff at least one of the systems has a solution.
- Decidability results from the algorithmic nature of the model construction.

What else was needed for EQPL

Significant revamp of the FHM technique
in order to cope with the novel aspects of EQPL:

- non-deterministic semantics of quantum connectives;
- classical formulae mixed with arithmetic (in)equations and non-entanglement constraints;
- amplitude terms besides probability terms;
- quantum structures instead of probability spaces.

Maximal extension technique used thrice:

- for removing alternative terms;
- for constructing the set V of admissible valuations;
- for building the partition \mathcal{S} of the set Q of qubits.

Further developments and practical impact

Extensions and probabilistic ramifications

Dynamics of quantum systems

- Linear-time temporal EQPL.
- Branching-time temporal EQPL.
- Hoare calculus over EQPL.

QM Postulate 5 (discrete case)

Excluding measurements, each state transition of an isolated quantum system is described by a unitary transformation.

Dynamics of probabilistic systems

Unexpected significant contribution to the field of **verification of probabilistic imperative programs** (Hoare calculus).

Applications

Model checking of quantum protocols

- Warwick quantum model checker.
- Linear-time temporal extension of EQPL.
- Subspace of stabilizer states in $\mathcal{H}^{K^\bullet}(2^Q)$.

S. J. Gay, R. Nagarajan, and N. Papanikolaou,

QMC: A model checker for quantum systems.

Proceedings of the 20th International Conference on Computer Aided Verification (**CAV'08**), Princeton, USA. A. Gupta and S. Malik, editors. Vol. 5123 LNCS, Springer, 2008, pp. 543–547.

Outlook

Ongoing and future work on quantum logic

- Axiomatization of stabilizer semantics.
- Quantum Hoare calculus over density operators.
- More general extensions for reasoning about the dynamics of quantum systems.
- Linear algebra techniques in classical, non-deterministic, probabilistic and quantum program verification.
- Planning under (probabilistic or quantum) uncertainty.
- Knowledge representation with (probabilistic or quantum variants of) description logic.
- FOL theory of quantum systems.

Ongoing and future work (not directly related but maybe still worth mentioning here)

- Kolmogorov complexity of computable functions.
- Quantum Turing machines with classical control.
- Robust notions of:
 - quantum universal function;
 - quantum Kolmogorov complexity.
- Applications to distinguishability of quantum states.
- Quantum Kolmogorov complexity of computable unitary transformations.
- Quantum Turing reducibility.

Sources

For more details on EQPL

- P. Mateus and A. Sernadas, *Weakly complete axiomatization of exogenous quantum propositional logic*, **Information and Computation**, vol. 204 (2006), no. 5, pp. 771–794.
- R. Chadha, P. Mateus, A. Sernadas, and C. Sernadas, *Extending classical logic for reasoning about quantum systems*, **Handbook of Quantum Logic and Quantum Structures: Quantum Logic** (D. Gabbay K. Engesser and D. Lehmann, editors), Elsevier, 2009, pp. 325–372.