Extrapolation Techniques for Computing Accurate Solutions of Elliptic Problems with Singular Solutions

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Outline

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- Zenger Correction
- Multigrid and Extrapolation
- Numerical Results
  - Test cases
  - Bioelectric field problem
  - Optical Flow
- Conclusions

Problem Description

Discretization of elliptic PDEs with

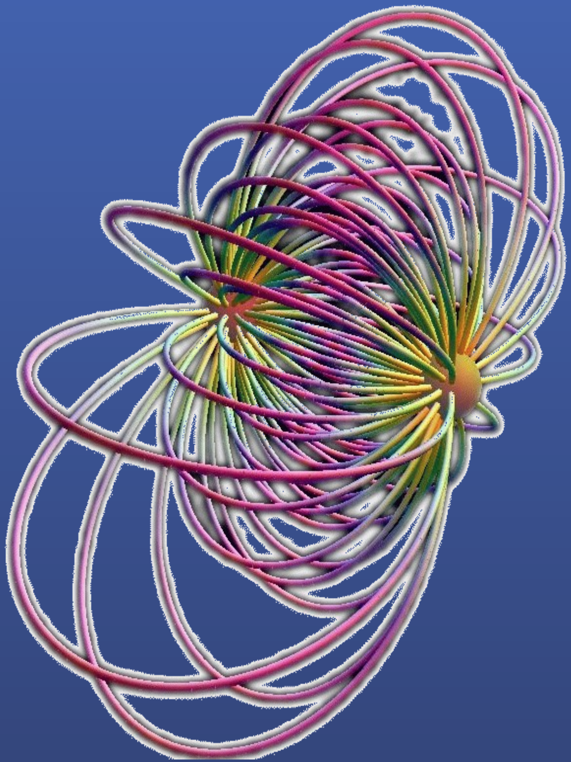
$$\text{singularities} \left\{ \begin{array}{l} \text{point load} \\ \text{dipole} \\ \text{quadrupole} \end{array} \right\} \text{in the source terms}$$

$$\begin{array}{l} \text{standard discretization} \\ \text{relies on smoothness} \end{array} \Rightarrow \left\{ \begin{array}{l} \bullet \ \text{accuracy deteriorates} \\ \bullet \ \text{standard analysis fails} \end{array} \right.$$

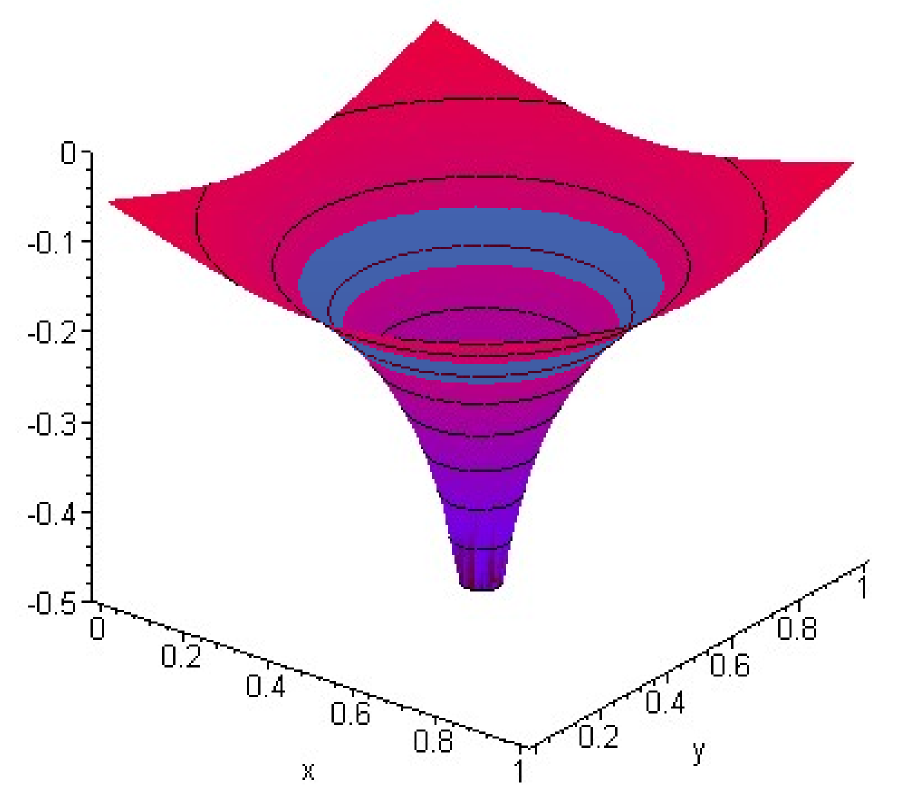
Applications

Fields of application in which problems with singularities of this kind arise are manifold, e.g.

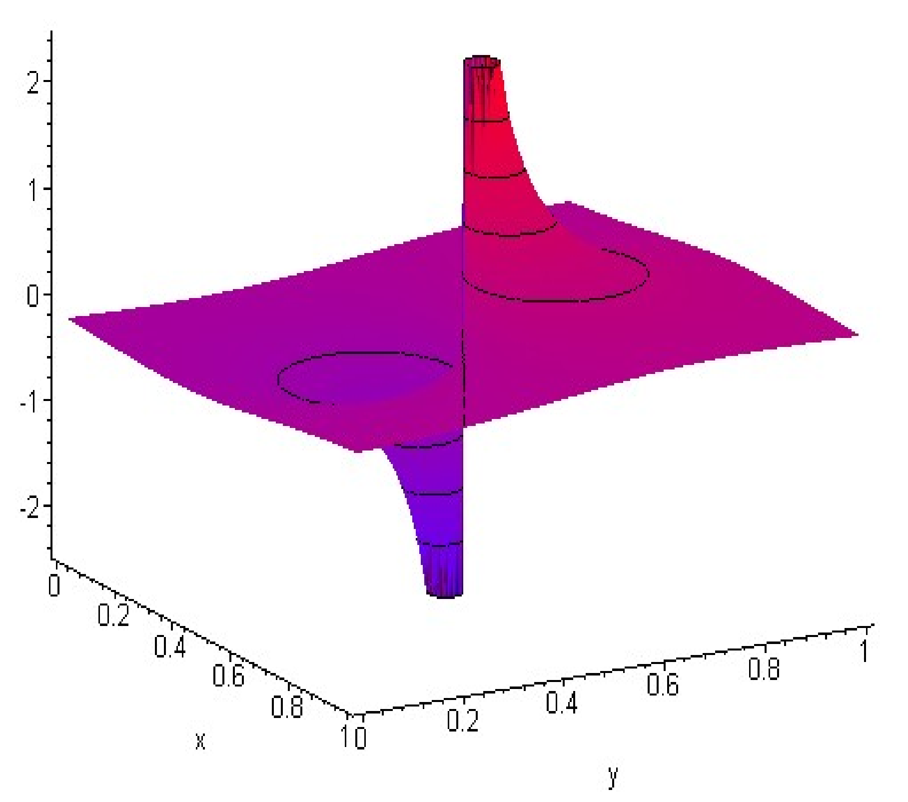
- point sources and sinks in porous media flow that are described by Dirac delta functions
- point loads and dipoles as source terms inducing electrostatic potentials



Electrostatic Potentials



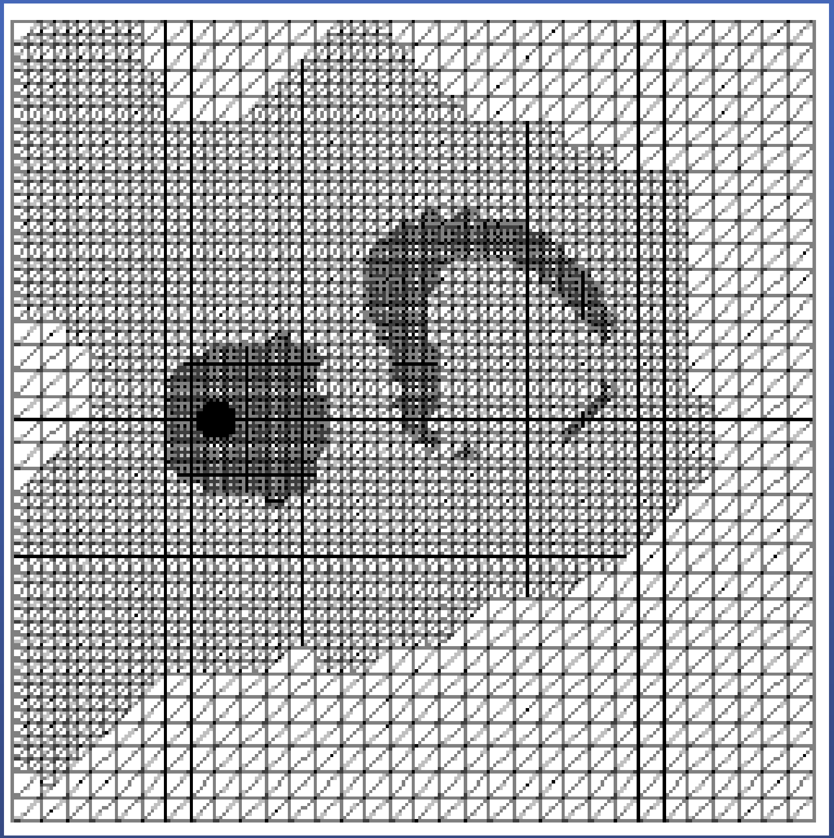
Point load



Dipole

Different approaches

- local mesh refinement
- use knowledge about the asymptotic solution to eliminate the singularity
- construction of special techniques based on weighted norms (Rannacher, Blum)
- Zenger Correction



Zenger Correction

Idea: Replace the original generalized function by its numerical equivalent on a uniform grid.

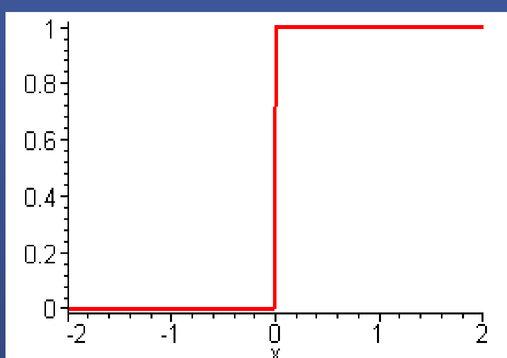
- no local refinement
- construction of numerical representation
  - analytical integration of delta function yields smooth function
  - numerical differentiation yields h-dependent discrete equivalent of delta function

Generalized Functions

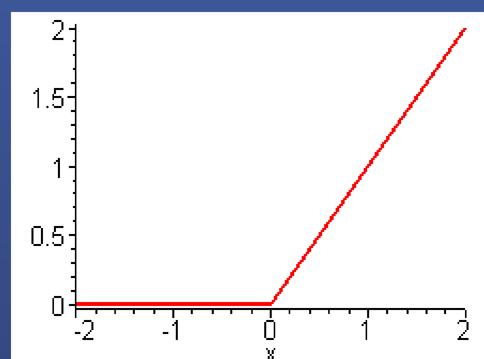
Definition: The family of functions H_i: $\mathbb{R} \to \mathbb{R}$, $i \in \mathbb{Z}$ is defined by

$$H_0(x) := \begin{cases} 0 & : \quad x \leq 0 \\ 1 & : \quad x > 0 \end{cases}, \qquad H_i(x) := \begin{cases} H'_{i-1}(x) & : \quad i > 0 \\ \int\limits_{-\infty}^{x} H_{i+1}(\xi)d\xi & : \quad i < 0 \end{cases}$$

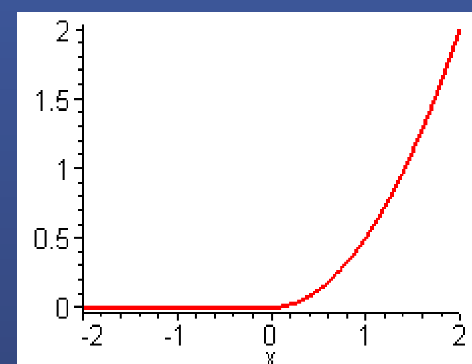
H_0 is the Heaviside step function and H_1 the Dirac-δ-function.



H_0



H_{-1}

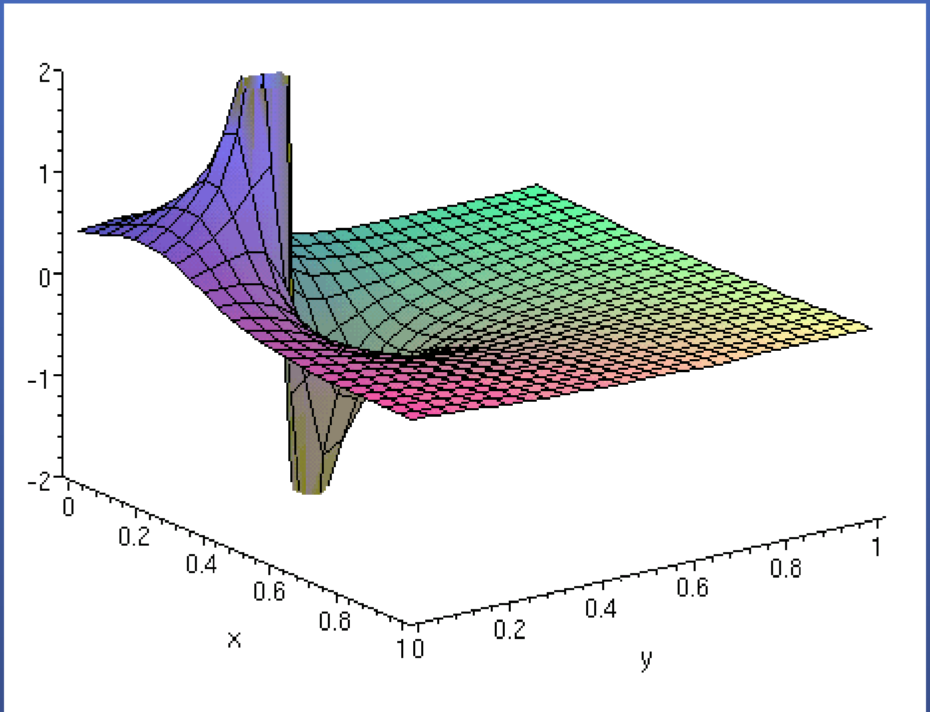


H_{-2}

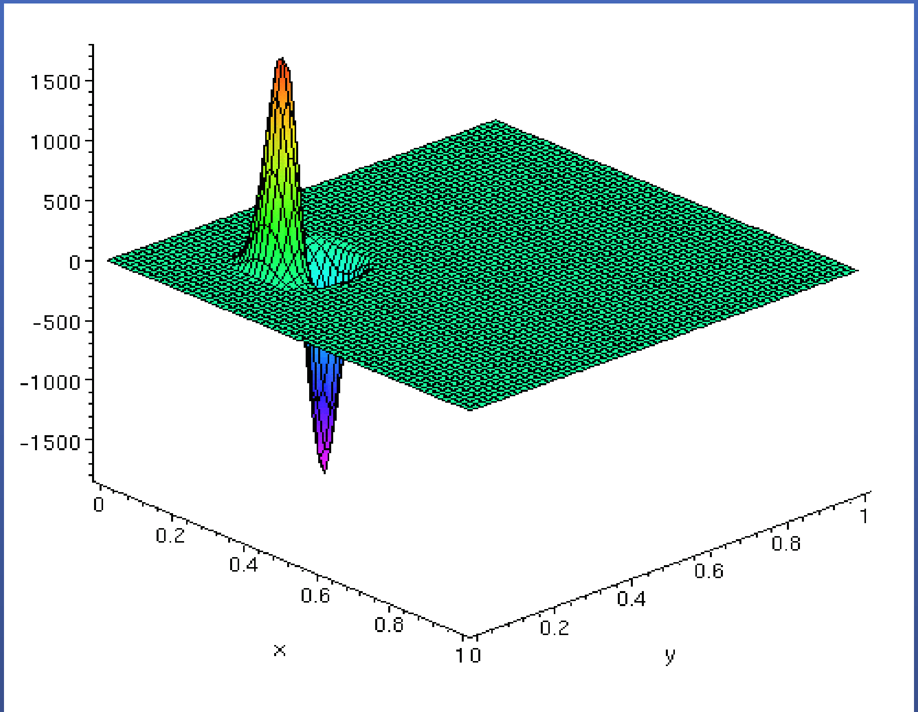
Generalized Functions II

- model problem: Poisson equation $\Delta u = f$
- f dipole $f = \vec{v} \cdot \nabla \mathcal{H}_{\vec{x}_0}^{(1,1)}(\vec{x}) = \vec{v} \cdot \begin{pmatrix} H_2(x - x_0) H_1(y - y_0) \\ H_1(x - x_0) H_2(y - y_0) \end{pmatrix}$
- correction $f_h = \vec{v} \cdot \begin{pmatrix} \delta^{(4,4)} \mathcal{H}_{\vec{x}_0}^{(-2,-3)}(\vec{x}) \\ \delta^{(4,4)} \mathcal{H}_{\vec{x}_0}^{(-3,-2)}(\vec{x}) \end{pmatrix}$

Zenger Correction for a Dipole

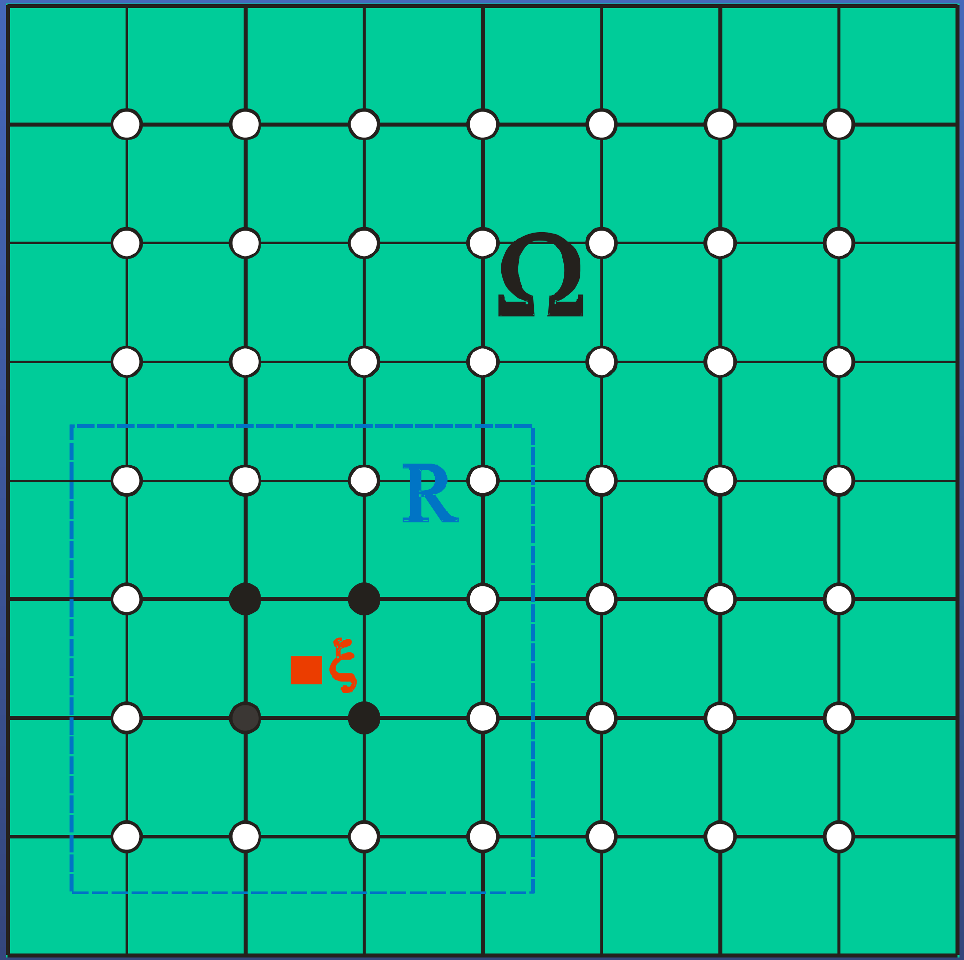


Analytic Solution



Right hand side with Zenger correction

Zenger Correction: Mathematical dipole



Zenger Correction: Summary

Advantages:

+ analytical solution is not needed for correction
+ no changes to grid or solver
+ correction is been applied to a fixed number of points ($\Rightarrow$ independent of meshsize)
+ the pollution effect is eliminated

Disadvantages:

- the accuracy breaks down near the singularity

Richardson Extrapolation

- based on the asymptotic error expansion

  $u_h - u^* = h^2 e_2 + h^4 e_4 + \ldots + \mathcal{O}(h^{2k})$

- we use a fine grid with meshsize h and a coarse grid with $H = 2h$

$$\tilde{u}_H = \frac{4}{3}\,\hat{I}_h^H\, u_h - \frac{1}{3}\, u_H$$

- on the coarse grid a higher accuracy is achieved through the combined solution

  $\Rightarrow$ so the final accuracy can be improved from $\mathcal{O}(h^2)$ to $\mathcal{O}(h^4)$

- u_h and u_H are naturally computed with full multigrid

Multigrid: VCycle

- Goal: solve $A^h u^h = f^h$ using a hierarchy of grids

Relax on $A^h u^h = f^h$

Compute $r^h = f^h - A^h u^h$

Restrict $r^H = I_h^H r^h$

Solve $A^H e^H = r^H$

by recursion

...   ...

Interpolate $e^h = I_H^h e^H$

Correct $u^h \leftarrow u^h + e^h$

Multigrid: τ-Extrapolation

Relax on $A^h u^h = f^h$

Compute $r^h = f^h - \frac{4}{3} A^h u^h + \frac{1}{3} A^H I_h^H u^h$

Restrict $r^H = I_h^H r^h$

Solve $A^H e^H = r^H$

by recursion

...

...

Interpolate $e^h = I_H^h e^H$

Correct $u^h \leftarrow u^h + e^h$

τ-Extrapolation

- τ-Extrapolation is a multigrid specific technique and works for both CS and FAS
- For CS the defects of two different grid levels are combined
- Higher accuracy is achieved by modifying the correction at the finest grid level only

$$\tilde{u}_h^{m+1} = u_h^m + I_H^h A_H^{-1} \left(\frac{4}{3} I_h^H (f_h - A_h u_h^m) - \frac{1}{3} (I_h^H f_h - A_H \hat{I}_h^H u_h^m) \right)$$

- special care has to be taken when choosing the restriction operator and the smoothing procedure in order not to destroy the high accuracy

Numerical Experiments: Test case

- model problem: Poisson equation

$$\begin{aligned} \Delta u &= f \quad \text{in } \Omega = (0,1)^3 \\ u &= g \quad \text{on } \partial\Omega \end{aligned}$$

- finite differences, equidistant grid, meshsize h.
- correction scheme multigrid solver

f is dipole at ξ = (0.26,0.26,0.26) not a grid point

Discretization Errors for a Dipole

h	$(\frac{1}{2},\frac{1}{2},\frac{1}{2})$		L_∞ Ω	L_1 Ω		L_2 Ω		L_1 $\Omega\backslash R$		L_2 $\Omega\backslash R$	
$\frac{1}{16}$	4.26e-02		2.86e+06	8.49e+02		4.93e+04		2.13e-02		2.84e-02	
		2.0			3.1		1.6		2.1		1.9
$\frac{1}{32}$	1.06e-02		2.86e+06	9.62e+01		1.66e+04		5.14e-03		7.64e-03	
		2.0			3.1		1.5		2.0		1.9
$\frac{1}{64}$	2.65e-03		2.86e+06	1.15e+01		5.73e+03		1.26e-03		2.02e-03	
		2.0			3.0		1.5		2.0		2.0
$\frac{1}{128}$	6.63e-04		2.86e+06	1.40e+00		2.00e+03		3.13e-04		5.20e-04	
		2.0			3.0		1.5		2.0		2.0
$\frac{1}{256}$	1.66e-04		2.86e+06	1.74e-01		7.03e+02		7.77e-05		1.32e-04	

Ω = problem domain
R = region around singularity

$$2^\alpha = \frac{|u^* - u_H|}{|u^* - u_h|}$$

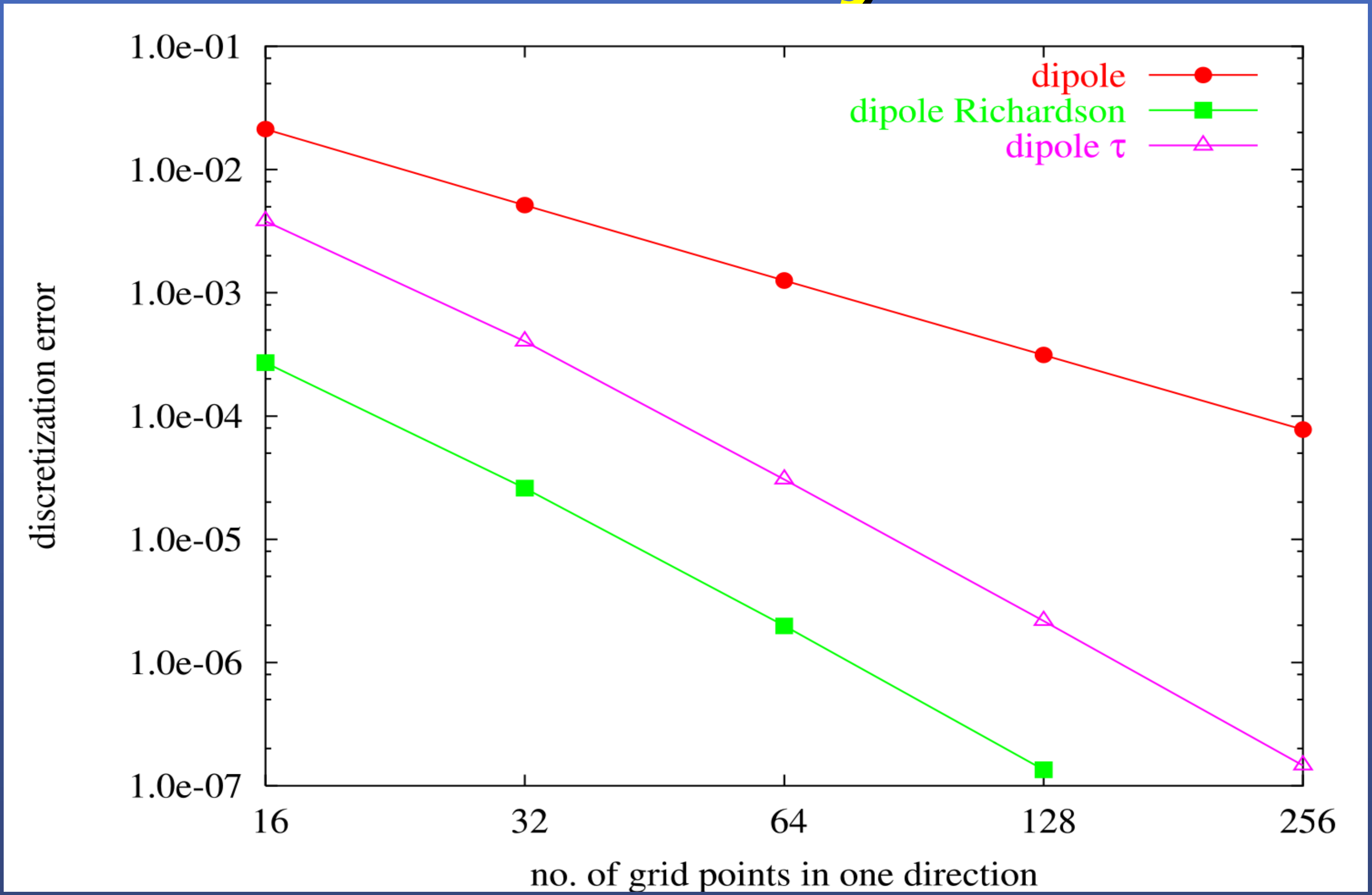
Discretization Errors for a Dipole and Richardson Extrapolation

h	$(\frac{1}{2},\frac{1}{2},\frac{1}{2})$	L_∞ Ω	L_1 Ω	L_2 Ω	L_1 $\Omega\backslash R$	L_2 $\Omega\backslash R$
$\frac{1}{16}$	9.00e-05	2.86e+06	8.49e+02	4.93e+04	2.71e-04	1.29e-03
	3.9		3.1	1.6	3.4	3.2
$\frac{1}{32}$	5.82e-06	2.86e+06	9.62e+01	1.66e+04	2.60e-05	1.36e-04
	4.0		3.1	1.5	3.7	3.5
$\frac{1}{64}$	3.66e-07	2.86e+06	1.15e+01	5.73e+03	1.98e-06	1.17e-05
	4.0		3.0	1.5	3.9	3.8
$\frac{1}{128}$	2.32e-08	2.86e+06	1.40e+00	2.00e+03	1.35e-07	8.45e-07

Discretization Errors for a Dipole and τ-Extrapolation

h	$(\frac{1}{2},\frac{1}{2},\frac{1}{2})$		L_∞ Ω	L_1 Ω		L_2 Ω		L_1 $\Omega\backslash R$		L_2 $\Omega\backslash R$	
$\frac{1}{16}$	6.30e-04	4.5	2.86e+06	8.49e+02	3.1	4.93e+04	1.6	3.81e-03	3.2	1.18e-02	2.4
$\frac{1}{32}$	2.76e-05	6.2	2.86e+06	9.62e+01	3.1	1.66e+04	1.5	4.04e-04	3.7	2.31e-03	3.5
$\frac{1}{64}$	3.85e-07	3.3	2.86e+06	1.15e+01	3.0	5.73e+03	1.5	3.06e-05	3.8	2.03e-04	3.7
$\frac{1}{128}$	4.01e-08	3.4	2.86e+06	1.40e+00	3.0	2.00e+03	1.5	2.17e-06	3.9	1.58e-05	3.8
$\frac{1}{256}$	3.77e-09		2.86e+06	1.73e-01		7.03e+02		1.46e-07		1.13e-06	

Extrapolation for increased Accuracy

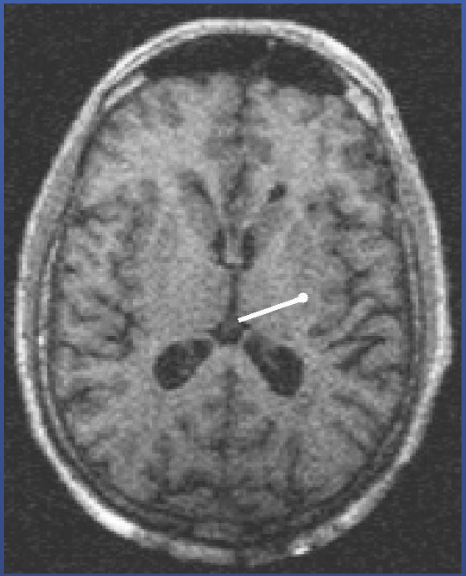


Bioelectric Field Problem I

Reconstruction of electrical behaviour inside the head from measurements (Electroencephalography)

Inverse problem important for Neurology and Neurosurgery

Neurosurgery at Erlangen University Head Center



Localization of an epileptic focus

Bioelectric Field Problem II

Governing equation:

$$\begin{aligned} -\nabla\cdot(\sigma\nabla\Phi) &= I \quad \text{in } \Omega \\ \sigma\frac{\partial\phi}{\partial\vec{n}}\Big|_\Gamma &= 0 \quad \text{on } \partial\Omega \end{aligned}$$

σ: Conductivity tensor
Φ: Potential field (forward problem)
I: Source terms (inverse problem)

Electrically active brain areas are physically modeled as dipoles
⇒ suitable numerical dipole models are needed

Bioelectric Field Problem III

For a mathematical dipole at x_0 and with moment M the source term becomes:

$$I = \nabla \cdot \vec{M}\delta(\vec{x} - \vec{x}_0) = \nabla \cdot \vec{j}^p$$

Another possibility is the Feynman dipole

$$I = \vec{M}(\delta(\vec{x} - \vec{x}_{so}) - \delta(\vec{x} - \vec{x}_{si})) = \vec{j}^p$$

-finite difference discretization $-\nabla_h \cdot (\sigma \nabla_h \Phi_h) = I_h$

-finite element discretization $-\int_\Omega v \nabla \cdot (\sigma \nabla \phi) d\Omega = \int_\Omega v I d\Omega$

Bioelectric Field Problem: Zenger Correction for Finite Elements

We use Greens formula on the source term

$$\int_\Omega v \nabla \cdot \vec{j}^p d\Omega = \int_\Gamma v \frac{\partial \vec{j}^p}{\partial \vec{n}} d\Gamma - \int_\Omega (\nabla v) \vec{j}^p d\Omega$$

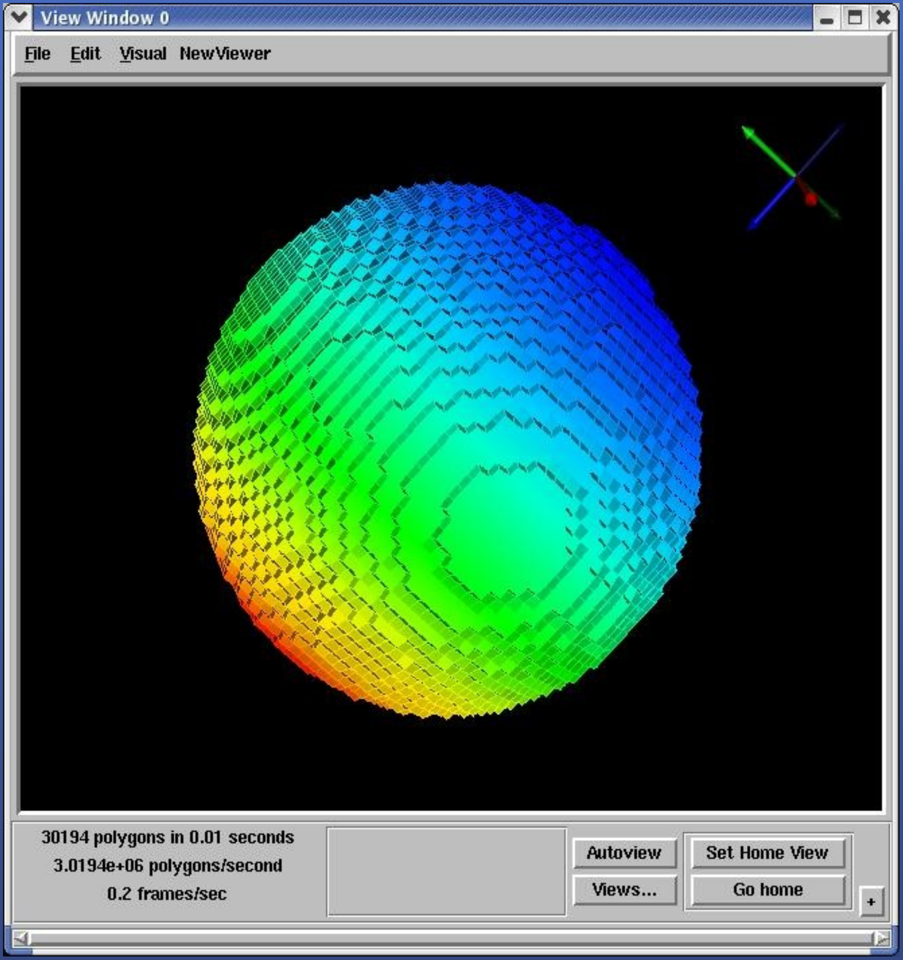
where the surface integral equals 0. Discretization gives for the entry on the RHS for node i in element k

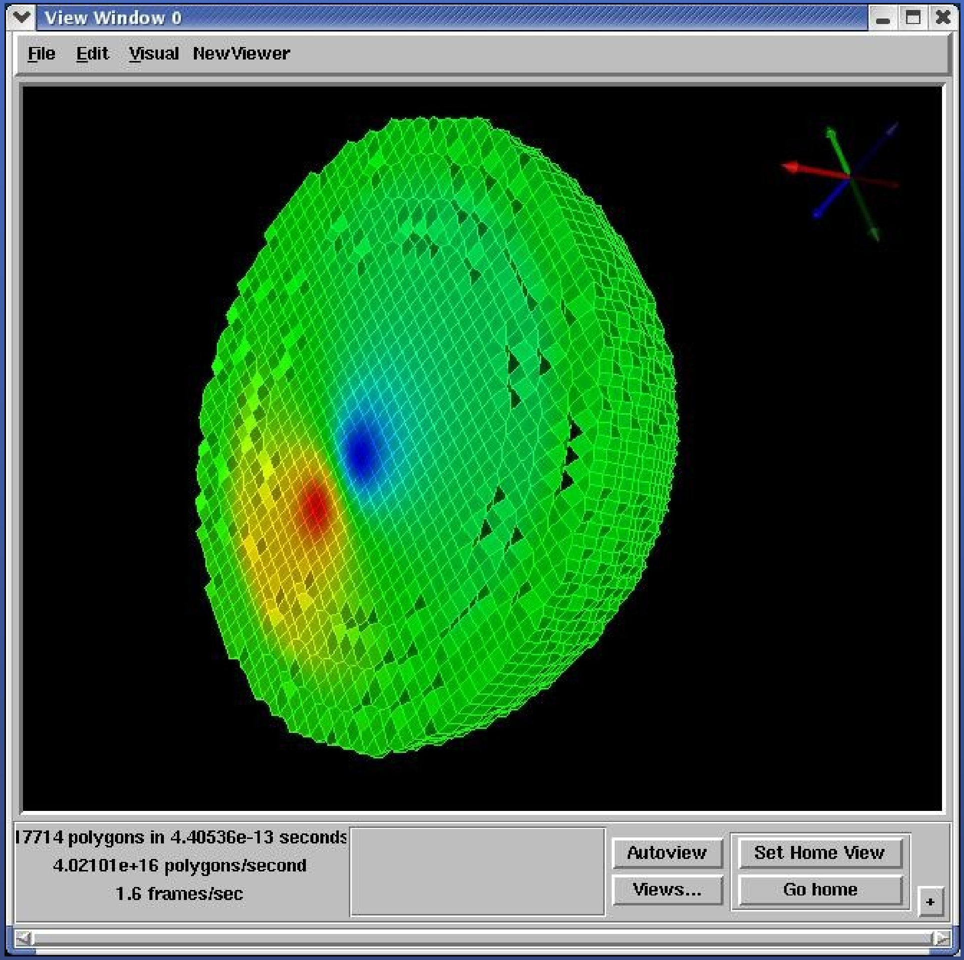
$$b_i^k = \int_{\Omega_k} (\nabla \varphi_i) \vec{M} \delta(\vec{x} - \vec{x}_0) d\Omega = \vec{M} \cdot \nabla \varphi_i(\vec{x}_0)$$

and is evaluated in the reference element with mapping Ψ

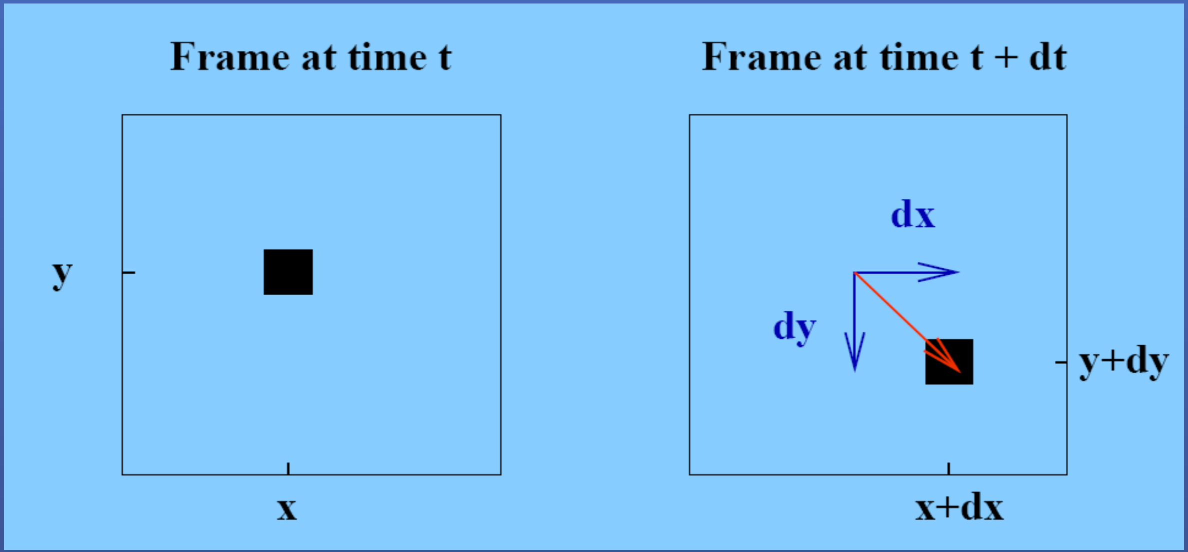
$$b_i^k = \vec{M} \cdot \left[D\Psi(\Psi^{-1}(\vec{x})) \right]^{-T} \nabla \tilde{\varphi}_i(\Psi^{-1}(\vec{x}_0))$$

Bioelectric Field Problem: SCIRun Results





Optical Flow I



- An approximation of the motion
- Optical Flow ≠ Motion

The optical flow at the pixel (x,y) is the 2D-velocity vector

$$(u, v) = \left(\frac{dx}{dt}, \frac{dy}{dt}\right)$$

Optical Flow Constraints

- I(x,y,z,t): The image intensity of the pixel (x,y,z) at time t
- I_x, I_y, I_z, I_t: Spatial and temporal derivatives of I
- Assumption: Objects keep the same intensity over time

$$I(x,y,z,t) = I(x+dx, y+dy, x+dz, t+dt)$$

- Taylor Expansion gives

$$I_x u + I_y v + I_z w + I_t = 0$$

The optical flow constraint equation (OFCE)

Regularization

This leads to the minimization problem

$$\min_{(u,v,w)} \int_x \int_y \int_z E(u,v,w)\, \mathrm{d}x\, \mathrm{d}y\, \mathrm{d}z$$

where $E = E_d + \alpha E_r$.

$$E_d(u,v,w) = (I_x u + I_y v + I_z w + I_t)^2 \quad \text{(optical flow)}$$

$$E_d(\vec{u}) = (T(\vec{x} + \vec{u}(\vec{x})) - R(\vec{x}))^2 \quad \text{(image registration)}$$

E_r is a regularization term and α is a positive scalar for adjustment between E_d and E_r.
The method for solving will depend on the choice of E_r.

Horn-Schunk Algorithm

A standard choice of E_r is the isotropic stabilizer

$$E_r(u,v,w) = \frac{1}{2}\left(|\nabla u|^2 + |\nabla v|^2 + |\nabla w|^2\right)$$

From calculus of variations, we get

$$\alpha \Delta u - I_x(I_x u + I_y v + I_z w + I_t) = 0$$

$$\alpha \Delta v - I_y(I_x u + I_y v + I_z w + I_t) = 0$$

$$\alpha \Delta w - I_z(I_x u + I_y v + I_z w + I_t) = 0$$

We discretize the Laplacian by the standard 7 point stencil.

Optical Flow: Results for moving point

A standard choice of E_r is the isotropic stabilizer

$$E_r(u,v,w) = \frac{1}{2}\left(|\nabla u|^2 + |\nabla v|^2 + |\nabla w|^2\right)$$

From calculus of variations, we get

$$\alpha \Delta u - I_x(I_x u + I_y v + I_z w + I_t) = 0$$

$$\alpha \Delta v - I_y(I_x u + I_y v + I_z w + I_t) = 0$$

$$\alpha \Delta w - I_z(I_x u + I_y v + I_z w + I_t) = 0$$

We discretize the Laplacian by the standard 7 point stencil.

Conclusions & Future Work

- Conclusions
  - Singularities occur in a variety of applications
  - Zenger Correction is an efficient way to compute far field of a singular solution
- Future work
  - evaluate effectiveness of Zenger correction for bioelectric field problem, integrate in NeuroFEM (C. Wolters), compare with other dipole models
  - Do FEM theory for Zenger correction
  - evaluate effectiveness of dipole models for real data (head models)