Nilpotent orbits of a reductive group over a local field

(Algebraic Groups and Invariant Theory
Ascona)

George McNinch

Tufts University

September 4, 2009

Reductive groups: some conditions

We'll consider a connected reductive group G over a field K.
I want to suppose G is "D-standard."

- if G is semisimple, D-standard just means the characteristic is "very good" for G.
- Any K-form of GL_n is D-standard; a form of SL_n is D-standard $\iff$ n is invertible in K.
- $\mathrm{Sp}(V)$ is D-standard just when $p \neq 2$.
- a Levi factor of a parabolic subgroup of a D-standard group is again D-standard.

Nilpotent orbits, geometrically

- Let G be D-standard with Lie algebra $\mathfrak{g}$
- Recall G-orbits in the nilpotent variety $\mathcal{N} \subset \mathfrak{g}$ are classified *geometrically* by "Bala-Carter data"
- ...in particular, one can label the *geometric* nilpotent orbits using data derived just from the root datum of G.
- More complicated in general: study of $G(K)$-orbits in $\mathcal{N}(K)$.

Nilpotent centralizers

- if char. K is 0, $\mathfrak{sl}_2$-triples containing X are a useful tool; unavailable (and less useful) in general.
- For a general D-standard group, one replaces the ss elt H of a triple by a suitable cocharacter $\phi : \mathbf{G}_m \to G$ "associated with X".
- following Premet, one knows such a cocharacter to exist by using geometric invariant theory result of Kempf-Rousseau (since nilpotent elements are precisely the unstable vectors in the adjoint representation)
- If ϕ is a cocharacter associated with X, one knows that $M = C \cap C_G(\operatorname{im}\phi)$ is a Levi factor of C (over K).

Optimal SL_2's

- Assume $X^{[p]} = 0$ (or $p = 0$). If the cocharacter ϕ is associated to X, this data determs a so-called "optimal" homomorphism $\psi : \mathrm{SL}_2 \to G$ for which X is in the image of $d\psi$, and $\psi \begin{pmatrix} t & 0 \\ 0 & t^{-1} \end{pmatrix} = \phi(t)$.
- the adjoint representation of the image of ψ on $\mathfrak{g}$ is a direct sum of indecomposable *tilting modules* each of highest weight $0 \leq \lambda \leq 2p - 2$.
- a tilting module V is one for which both V and $V^\vee$ have a filtration by standard modules $H^0(\mu)$

Structure of a nilpotent centralizer

Theorem (M - Nagoya Math. J. 2008)

Assume G is D-stndrd, and let $X \in \mathfrak{g}$ nilpotent. Then the (geometric) root datum of a Levi factor of the centralizer $C = C_G(X)$ is independent of p.

Method of proof: may suppose $K = K_{\mathrm{alg}}$.

- let $\mathcal{A}$ be a DVR with residues K and fractions of char 0.
- let $\mathcal{G}_{/\mathcal{A}}$ be split reductive with root datum of G
- find a nilpotent section $X_1 \in \mathfrak{g}(\mathcal{A})$ specializing to X for which the $\mathcal{A}$ group scheme $C_{\mathcal{G}}(X_1)$ is *smooth* over $\mathcal{A}$.
- find the Levi factor "over $\mathcal{A}$".

$\square$

Local K

The object of this talk is related to the method of proof of the previous result. I want to consider reductive groups over a local field.

Notations for local fields

$\mathcal{A}$ complete DVR, with

- fractions K,
- residues $k = \mathcal{A}/\pi\mathcal{A}$ – assume k *perfect*
- valuation $v : K^\times \to \mathbf{Z}$

- e.g. $\mathcal{A} = k[[t]]$, $K = k((t))$
- or $\mathcal{A} = \mathbf{Z}_p$, $K = \mathbf{Q}_p$, $k = \mathbf{Z}/p\mathbf{Z}$.
- From now on, G is a connected reductive group over K (additional assumptions as we proceed).

Results for local K

- DeBacker (Annals of Math, 2002) described $G(K)$-orbits in $\mathcal{N}(K)$, provided the residue char. is sufficiently large.
- his result relates nilpotent $G(K)$-orbits with nilpotent orbits for the reductive quotients of special fibers of corresponding parahoric group schemes.
- "labelling" is achieved using the *Bruhat-Tits building* of G.
- *my goal* (in this talk): give a description/construction of DeBacker's mapping, under milder assumptions on G
- apology in advance: I'm going to ignore a number of issues (in particular, I won't describe DeBacker's labelling).

Goal, a bit more precisely

- let $\mathcal{P}$ be a parahoric group scheme attached to G (more about parahorics in a bit...)
- and let X_0 be a nilpotent element in $\mathrm{Lie}(\mathcal{P}_{/k,\mathrm{red}})$ (I'll need to assume $X_0^{[p]} = 0$).
- idea is to produce a corresponding nilpotent orbit in $\mathfrak{g}(K)$ (with reasonable properties).
- it suffices to carry out the construction when the residue field k is algebraically closed; *we assume this from now on*
- (explanation: since k is anyhow perfect, there is always an *étale extension* K' of K for which k' is an alg. closure of k. And our constructions will descend for étale base change.)

Parahoric group schemes

- since k alg. closed, thm of Steinberg $\implies$ G *quasisplit* $/K$.
- fix a max'l K-split torus S and K-Borel subgroup $S \subset B$.
- the centralizer T of S is a max K-torus of G, and $\exists$ smooth $\mathcal{T}$ over $\mathcal{A}$ with $T = \mathcal{T}_{/K}$, containing the "canonical" $\mathcal{A}$-torus $\mathscr{S}$ with $\mathscr{S}_{/K} = S$
- $\Phi \subset X^*(S)$: K-roots of G; $\alpha \in \Phi$ determines $U_\alpha \subset G$
- Bruhat-Tits: $\exists$ "valuation of the root datum of G" hence filtration of $U_\alpha(K)$ compatible with the valuation on K and corresp. $\mathcal{A}$-group schemes $\mathfrak{U}_{f,\alpha}$ with generic fiber U_α
- for suitable choices of $\mathfrak{U}_{f,\alpha}$, the data $(\mathcal{T}, (\mathfrak{U}_{f,\alpha})_{\alpha\in\Phi})$ determ's smooth affine $\mathcal{A}$-gp scheme $\mathcal{P}$ with $\mathcal{P}_{/K} = G$.

Parahoric example

Consider $G = \mathrm{Sp}_6 = \mathrm{Sp}(V)$:

- fix "hyperbolic" basis $\{e_i, , f_i \mid 1 \leq i \leq 3\}$ of V.
- $\{X_\alpha\}$ corresp. Chev. basis of $\mathfrak{g}$.
- $S = T$ is 3 dim'l split torus
- B is stab. of isotropic flag
  $Ke_1 \subset Ke_1 + Ke_2 \subset Ke_1 + Ke_2 + Ke_3$.

Parahoric example, continued

Recall

$G = \mathrm{Sp}_6$

- Consider the vertex $\varpi = \dfrac{\varpi_1}{2}$ of the "affine" fund. alcove (where ϖ_i are fund. dom. coweights)
- define $f : \Phi \to \mathbf{Q}$ by $f(\alpha) = -\langle \alpha, \varpi \rangle$.
- $\exists\, \mathfrak{U}_{f,\alpha}$ with $\mathfrak{U}_{f,\alpha}(\mathcal{A}) = \exp(\pi^{\lceil f(\alpha) \rceil} \mathcal{A} X_\alpha)$.
- some sample roots:
  - $f(\alpha_1) = -1/2$, so $\mathfrak{U}_{f,\alpha_1}(\mathcal{A}) = \exp(\mathcal{A} X_\alpha)$ and $\mathfrak{U}_{f,-\alpha_1}(\mathcal{A}) = \exp(\pi \mathcal{A} X_\alpha)$
  - if $\beta = 2\alpha_1 + \alpha_2 + \alpha_3$, $f(\beta) = 1$ so $\mathfrak{U}_{f,\pm\beta} = \exp(\pi \mathcal{A} X_{\pm\beta})$.

Parahoric example, continued

Recall

$G = \mathrm{Sp}_6, \mathfrak{U}_{f,\alpha}(\mathcal{A}) = \exp(\pi^{\lceil f(\alpha) \rceil} \mathcal{A} X_\alpha)$

- the resulting group scheme $\mathcal{P}$ is the stabilizer of lattice flag $\langle \pi^{-1} e_1, e_2, e_3, f_2, f_3, f_1 \rangle \subset \langle e_1, e_2, e_3, f_2, f_3, \pi f_1 \rangle$
- $\mathcal{P}_{/k}$ has reductive quotient $\mathrm{Sp}_2 \times \mathrm{Sp}_4$, hence an 8 dim'l unip. radical

Parahoric groups schemes: Levi factor of special fiber

Theorem

The special fiber $\mathcal{P}_{/k}$ of a parahoric $\mathcal{P}$ has a unique Levi factor containing $\mathscr{S}_{/k}$.

- result known to Bruhat-Tits – Tits formulated this result in his "Corvallis" notes. But I'm unaware of a published proof.

Note:

In general, a linear group need have no Levi factor.

e.g. $G = \mathrm{SL}_2(W_2)$ has none, where W_2 is ring of "length 2 Witt vectors"

Levi factors: possible proof

- I have an argument that essentially reduces the problem to a verification in case G has K-rank 1 or 2.
- If G is quasisplit with index ${}^3D_4$ or ${}^6D_4$, there is a parahoric group scheme whose reductive quotient has type A_2. (*right now*) I don't see to give an easy argument for existence of Levi factor. (Which is not to say I doubt the result...)
- the argument I have in mind at least covers the case where G is split, or even $G = R_{L/K}H$ for H split and L a finite separable extension.

Hope 1: adjoint representation of a parahoric

Let $\mathcal{P}$ be a parahoric group scheme attached to G. Write M for a Levi factor of the special fiber $\mathcal{P}_{/k}$.

- Assume that M and G are both D-standard.

Hope 1

The representation of M on $\mathrm{Lie}(\mathcal{P}_{/k})$ is a tilting module.

some examples:

- $\mathcal{P}_{/K} = \mathrm{Sp}_6$ and $\mathcal{P}_{/k,\mathrm{red}} = \mathrm{Sp}_2 \times \mathrm{Sp}_4$.
  $\mathrm{Lie}(\mathcal{P}_{/k}) \simeq \mathrm{Lie}(M) \oplus H^0(\varpi_1; \varpi_1)$ as M-representation.
- $\mathcal{P}_{/K}$ split of type E_7 and $\mathcal{P}_{/k,\mathrm{red}}$ of type $A_2 \times A_5$.
  $\mathrm{Lie}(\mathcal{P}_{/k}) \simeq \mathrm{Lie}(M) \oplus H^0(\varpi_1; \varpi_2) \oplus H^0(\varpi_2; \varpi_3)$.

Hope 1 continued

- more generally, **Hope 1** is true whenever G is split over K.
- to understand **Hope 1** in the non-split case, should consider various “non-split” *échelonnages* found in [BT 1].
- e.g. there is a parahoric $\mathcal{P}$ for which $G = \mathcal{P}_{/K}$ has K-root system C_n and $\mathcal{P}_{/k,\text{red}}$ has k-root system of type D_n. (échelonnage named “B-C_n”).

Hope 2: good filtration for optimal SL_2's

Let F (alg. closed) field, let M be a D-standard reductive group over F.

- Fix nilpotent $X \in \mathrm{Lie}(M)$ with $X^{[p]} = 0$,
- and fix cochar. associated with X.
- these choices determine an *optimal* mapping $\mathrm{SL}_2 \to G$; write J for its image.

Hope 2

J is a good filtration subgroup of M.

Meaning: as J-module, each standard M-module $H^0_M(\lambda)$ has a filtration by modules of form $H^0_J(n)$ for various $n \geq 0$.

Hope 2: continued

Evidence for **Hope 2**:

- $\mathrm{Lie}(M)$ has good filtration as J-module.
- always true when the almost-simple components of M are classical (types A,B,C,D) or of type G_2

Remark

Chuck Hague and I are investigating together this question via Frob. splitting.

Application:

If M is Levi factor of $\mathcal{P}_{/k}$ and $S \subset M$ is an optimal SL_2, then the validity of **Hopes 1 and 2** would mean that $\mathrm{Lie}(\mathcal{P}_{/k})$ is a tilting module for J.

Nilpotent sections with smooth centralizers

- Fix a parahoric $\mathcal{P}$, Levi factor M of $\mathcal{P}_{/k}$, and nilpotent $X \in \operatorname{Lie}(M)(k)$ with $X^{[p]} = 0$; assume that both G and M are D-standard
- choose co-character ϕ of M assoc. to X
- after replacing X by an $M(k)$-conjugate, we may assume that ϕ is a cocharacter of $\mathscr{S}_{/k}$
- thus ϕ "is" also an $\mathcal{A}$-map $\phi : \mathbf{G}_{m/\mathcal{A}} \to \mathscr{S}$.
- *choose* $\tilde{X} \in \mathfrak{p}(\phi; 2) = \mathfrak{p}(\phi; 2)(\mathcal{A})$ with $X = \tilde{X} + \pi\mathfrak{p}$.

...smooth centralizers

View $\tilde{X}$ as a nilpotent element of $\mathfrak{g} = \mathrm{Lie}(\mathcal{P}_{/K})$. Recall that M and G are assumed to be D-standard.

Proposition

Assume that **Hope 1** *and* **Hope 2** *are valid, and that all weights of ϕ on $\mathrm{Lie}(\mathcal{P}_{/k})$ are $\leq 2p - 2$.*

(a) $\dim C_G(\tilde{X}) = \dim C_{\mathcal{P}_{/k}}(X)$.

(b) *the group scheme $C_{\mathcal{P}}(\tilde{X})$ is smooth over $\mathcal{A}$.*

(c) *the cocharacter $\phi \in X_*(S) = X_*(\mathscr{S}_{/K})$ is associated with $\tilde{X} \in \mathfrak{g}(K)$.*

...smooth centralizers

Corollary

There is a natural mapping

$$H^1(k, C_{\mathcal{P}_{/k}}(X)) \to H^1(K, C_G(\tilde{X})),$$

- Since k is *perfect*, $H^1(k, C_{\mathcal{P}_{/k}}(X))$ may be identified with $H^1(k, C_M(X)_{/red})$
- This natural mapping is in some sense *realized* by DeBacker's mapping mentioned earlier

Corollary

If X is distinguished in $\mathrm{Lie}(M)$, then $\tilde{X}$ is K-distinguished (a maximal split torus in $C_G(X)$ is central in G).

Description of mapping

- Write $\mathcal{P}^+$ for pre-image of $(R_u\mathcal{P}_{/k})(k)$ in $\mathcal{P}(\mathcal{A})$.
- And write $\mathfrak{p}^+$ for the pre-image of $\mathrm{Lie}(R_u\mathcal{P}_{/k})$ under the mapping $\mathfrak{p} \to \mathfrak{p}_{/k} = \mathrm{Lie}(\mathcal{P}_{/k})$.
- Let $\tilde{X} \in \mathfrak{p} = \mathfrak{p}(\mathcal{A})$ be a lift of $X \in \mathrm{Lie}(M)(k)$ as before.

Proposition (following DeBacker (following Waldspurger))

The $G(K)$-orbit of $\tilde{X}$ is the nilpotent orbit of minimal dimension having non-empty intersection with $\tilde{X} + \mathfrak{p}^+$.

Remark

The prop. characterizes the G-orbit $\tilde{X}$ among all “lifts” of $X \in \mathrm{Lie}(\mathcal{P}_{/k,\mathrm{red}})$.

description, continued

Using **Hopes 1 & 2**, construct $\mathcal{A}$-submodule $C \subset \mathfrak{p}$ which is an $\mathcal{A}$-direct summand, stable under the image of the cocharacter ϕ, and for which

(a) $\mathfrak{g} = C_{/K} \oplus [\tilde{X}, \mathfrak{g}]$ and

(b) $\mathrm{Lie}(\mathcal{P}_{/k}) = C_{/k} \oplus [X, \mathrm{Lie}(\mathcal{P}_{/k})]$.

Proof of previous proposition uses:

Proposition

$\tilde{X} + \mathfrak{p}^+ = \mathrm{Ad}(\mathcal{P}^+)(\tilde{X} + C \cap \mathfrak{p}^+)$.

Sketch of idea.

It suffices to prove the result holds mod π^n for all n. There is a unipotent k-group U whose k-points coincide with the image of $\mathcal{P}^+$ in $\mathcal{P}(\mathcal{A}/\pi^n\mathcal{A})$. One uses that U-orbits are closed to facilitate the proof. □