Formal Proofs for Quantifiers in F

Computability and Logic

Quantifier Rules in *F*

- There are 4 quantifier rules in *F*:
  - Universal Introduction and Elimination
  - Existential Introduction and Elimination
- As we saw last time, Universal Introduction and Existential Elimination have restrictions in that the rules cannot be applied relative to just any individual constant. The system *F* deals with those restrictions through the use of subproofs. We'll see later how that works.
- Fortunately, Universal Elimination and Existential Introduction do not have any restrictions, so we'll start with those.

Notation

- In describing the rules, the following notation is useful:
  - φ(x) is a wff with zero or more instances of x as the only free variable.
  - φ(a/x) is the statement that results when substituting 'a' for all occurrences of 'x' that are free in φ(x).
  - If it is clear which variable we are substituting, we will simply write φ(a).

∀ Elim

- Universal Elimination (∀ Elim) allows one to conclude that any thing has a certain property if everything has that property:

$$\forall x \; \varphi(x)$$

$$\vdots$$

$$\varphi(a)$$

Good and Bad Uses of ∀ Elim

Good
$$\forall x\ SameSize(x,x)$$
$$\vdots$$
$$SameSize(a,a)$$

Bad
$$\forall x\ SameSize(x,x)$$
$$\vdots$$
$$SameSize(a,b)$$
→ The *same* individual constant should be used!

Bad
$$\forall x\ SameSize(x,x)$$
$$\vdots$$
$$SameSize(x,a)$$
→ *All* free occurrences of x should be replaced!

Bad
$$\forall x\ (Tet(x) \rightarrow \forall x\ Large(x))$$
$$\vdots$$
$$Tet(a) \rightarrow \forall x\ Large(a)$$
→ Only *free* occurrences of x should be replaced!

∃ Intro

- Existential Introduction (∃ Intro) allows one to conclude that something has a certain property if some thing has that property:

$$\varphi(a)$$

$$\vdots$$

$$\exists x \, \varphi(x)$$

Good and Bad Uses of ∃ Intro

Good
$$\begin{array}{|l}
\text{SameSize(a,a)} \\
\vdots \\
\exists x\ \text{SameSize(x,x)}
\end{array}$$

Good
$$\begin{array}{|l}
\text{SameSize(a,a)} \\
\vdots \\
\exists x\ \text{SameSize(a,x)}
\end{array}$$

→ Not all occurrences of a have to be replaced!

Bad
$$\begin{array}{|l}
\text{SameSize(a,b)} \\
\vdots \\
\exists x\ \text{SameSize(x,x)}
\end{array}$$

→ The *same* individual constant should be used!

Bad
$$\begin{array}{|l}
\exists x\ \text{SameSize(a,x)} \\
\vdots \\
\exists x\ \exists x\ \text{SameSize(x,x)}
\end{array}$$

→ Doesn't follow the rule (no free x's in ∃x SameSize(x,x))

∀ Intro

- Universal Introduction (∀ Elim) allows one to conclude that everything has a certain property if anything has that property:

$$
\begin{array}{|l}
\rhd a \\
\\
\quad \vdots \\
\\
\varphi(a) \\
\hline
\forall x\ \varphi(x)
\end{array}
$$

a may not occur before the subproof, unless all subproofs in which it occurs have been closed. a may not occur in $\varphi(x)$ either.

Good and Bad Uses of ∀ Intro

Good

| ▷a
| ⋮
| SameSize(a,a)
∀x SameSize(x,x)

Still
Good

| ▷a
| ⋮
| Tet(a)
∀x Tet(x)

→ a occurs outside subproof, but only in a subproof that has been closed.

Bad

Tet(a)
| ▷a
| ⋮
| SameSize(a,a)
∀x SameSize(x,x)

→ a occurs before subproof!

Bad

| ▷a
| ⋮
| SameSize(a,a)
∀x SameSize(a,x)

→ a occurs in SameSize(a,x)!

∃ Elim

- Existential Elimination (∃ Elim) allows one to conclude anything that follows from some thing having a certain property, given that something has that property.

$\exists x\ \varphi(x)$

 ▷a $\varphi(a)$

  ⋮

 Q

Q

a may not occur before the subproof, unless all subproofs in which it occurs have been closed. a may not occur in Q either.

Good and Bad Uses of ∃ Elim

Good

$\exists x \, SameSize(x,x)$

  ▷a $SameSize(a,a)$
  ⋮
  $\exists x \, Cube(x)$

Still
Good

$\exists x \, Cube(x)$

  ▷a $Cube(a)$
  ⋮
  $Tet(b)$

$Tet(b)$

→ a occurs before subproof,
but only in a subproof
which has been closed.

Bad

$Tet(a)$
⋮
$\exists x \, SameSize(x,x)$

  ▷a $SameSize(a,a)$
  ⋮
  $\forall x \, Large(x)$

$\forall x \, Large(x)$

→ a occurs before subproof!

Bad

  ▷a $SameSize(a,a)$
  ⋮
  $Large(a)$

$Large(a)$

→ a occurs in $Large(a)$!

General Conditional Proof

- Most universal claims are proven by the application of $\forall$ Intro. Also, most universal claims are of the form $\forall x\, (\varphi(x) \rightarrow \psi(x))$. Thus, most proofs of universal claims would look like this:

$$\triangleright a$$
$$\varphi(a)$$
$$\vdots$$
$$\psi(a)$$
$$\varphi(a) \rightarrow \psi(a)$$
$$\forall x\, (\varphi(x) \rightarrow \psi(x))$$

General Conditional Proof (Continued)

- Because this is such a common pattern, the rule of General Conditional Proof allows us to take a little short cut:

$$\triangleright a \; \varphi(a)$$

$$\vdots$$

$$\psi(a)$$

a may not occur before the subproof, unless all subproofs in which it occurs have been closed. a may not occur in $\varphi(x) \rightarrow \psi(x)$ either.

$$\forall x \, (\psi(x) \rightarrow \varphi(x))$$