Voronoi Tessellations for Ocean Modeling: Methods, Modes, and Conservation

Todd Ringler
Theoretical Division
Los Alamos National Laboratory

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Climate, Ocean, and Sea Ice Modeling Project
http://public.lanl.gov/ringler/ringler.html

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Scope

1. Voronoi tessellations and their properties
2. Low (2nd-order) finite-volume methods
3. Structured and unstructured meshes
4. Shallow-water equations / layered OGCM

Outline

1. Definition of a Voronoi Tessellation (VT)
2. Definition of a Centroidal VT -- a special class
3. Discretization -- so many choices
4. Modes and Euler's Formula
5. Mimetic methods -- a route to conservation
6. Structured vs. Unstructured VTs
7. My views on where this all is heading

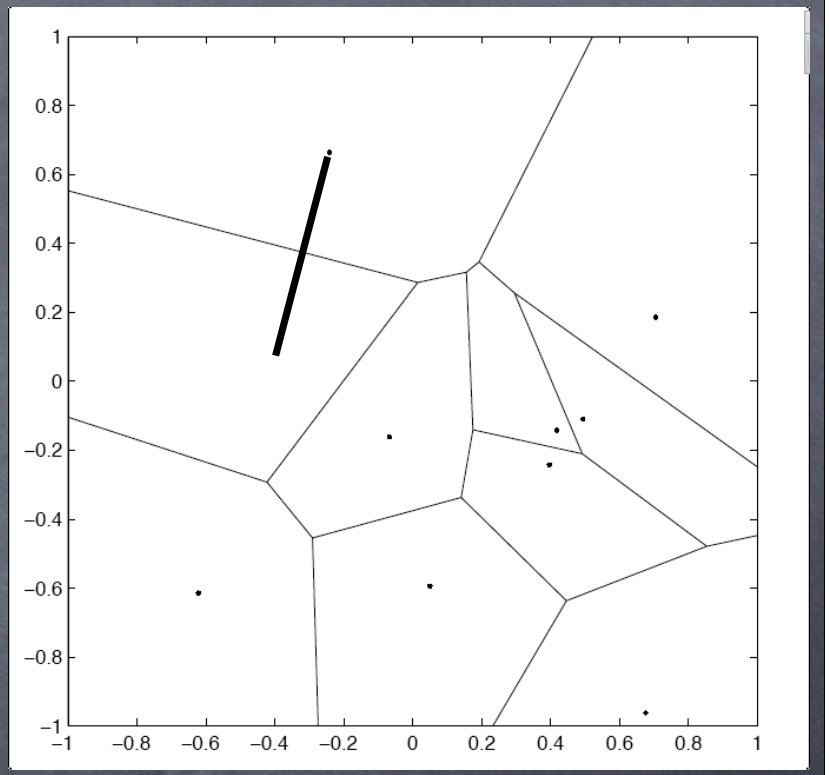
Nearly everything here is applicable to any mesh using FV methods.

Definition of a Voronoi Tessellations

Given a region, S
And a set of generators, z_i ...

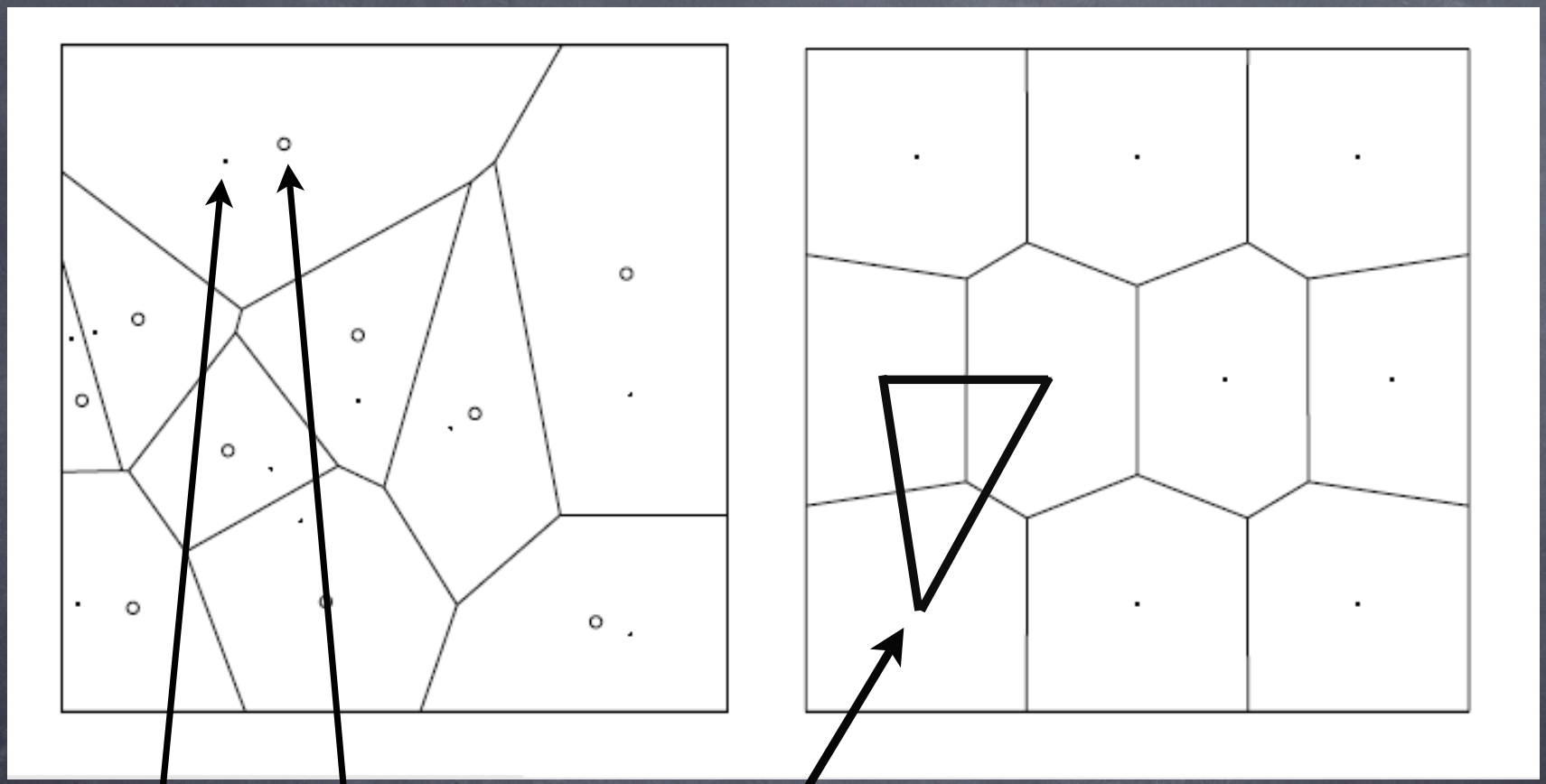
The Voronoi region, V_i, for each z_i is the set of all points closer to z_i than z_j for j not equal to i.

We are guaranteed that the line connecting generators is orthogonal to the shared edge and is bisected by that edge.



But this does not mean that the grid is nice ....

Definition of a Centroidal Voronoi Tessellations



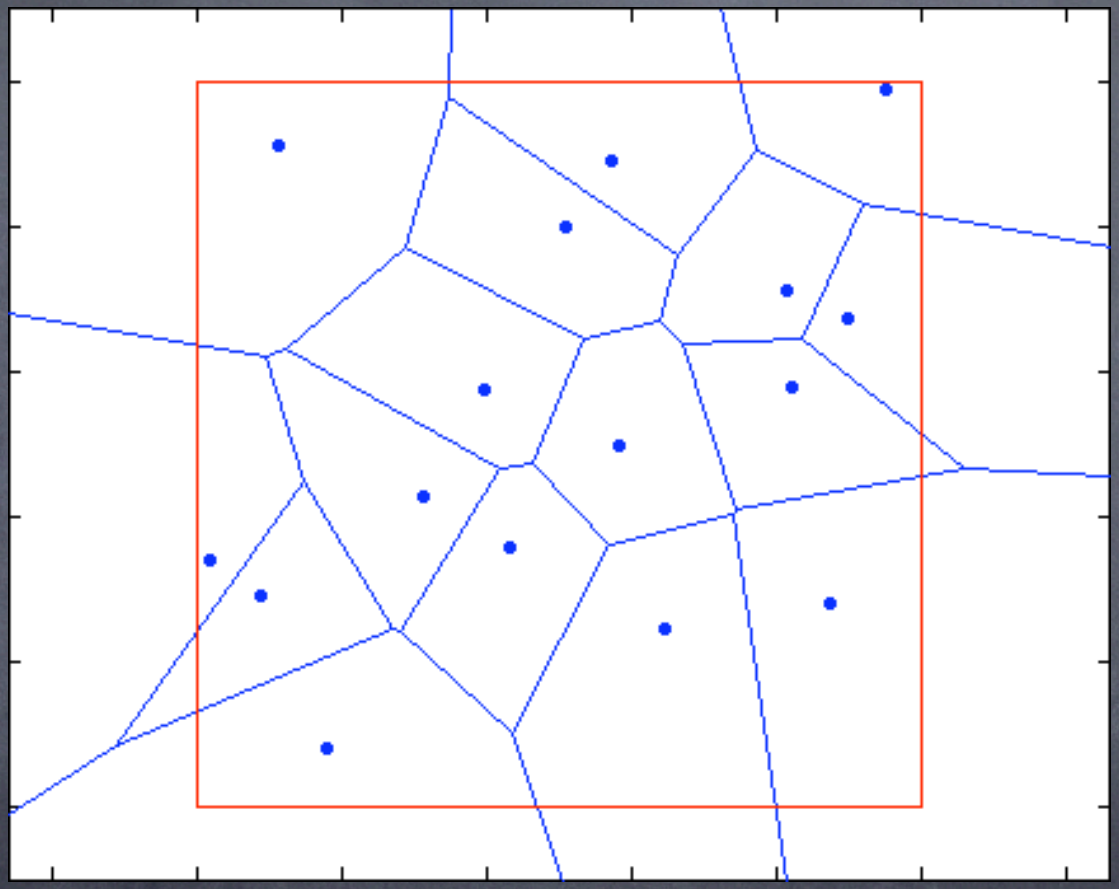
Dual tessellation

z_i

z_i^* = center of mass wrt a user-defined density function

$$z^* = \frac{\int_V w\rho(w)\,dw}{\int_V \rho(w)\,dw}$$

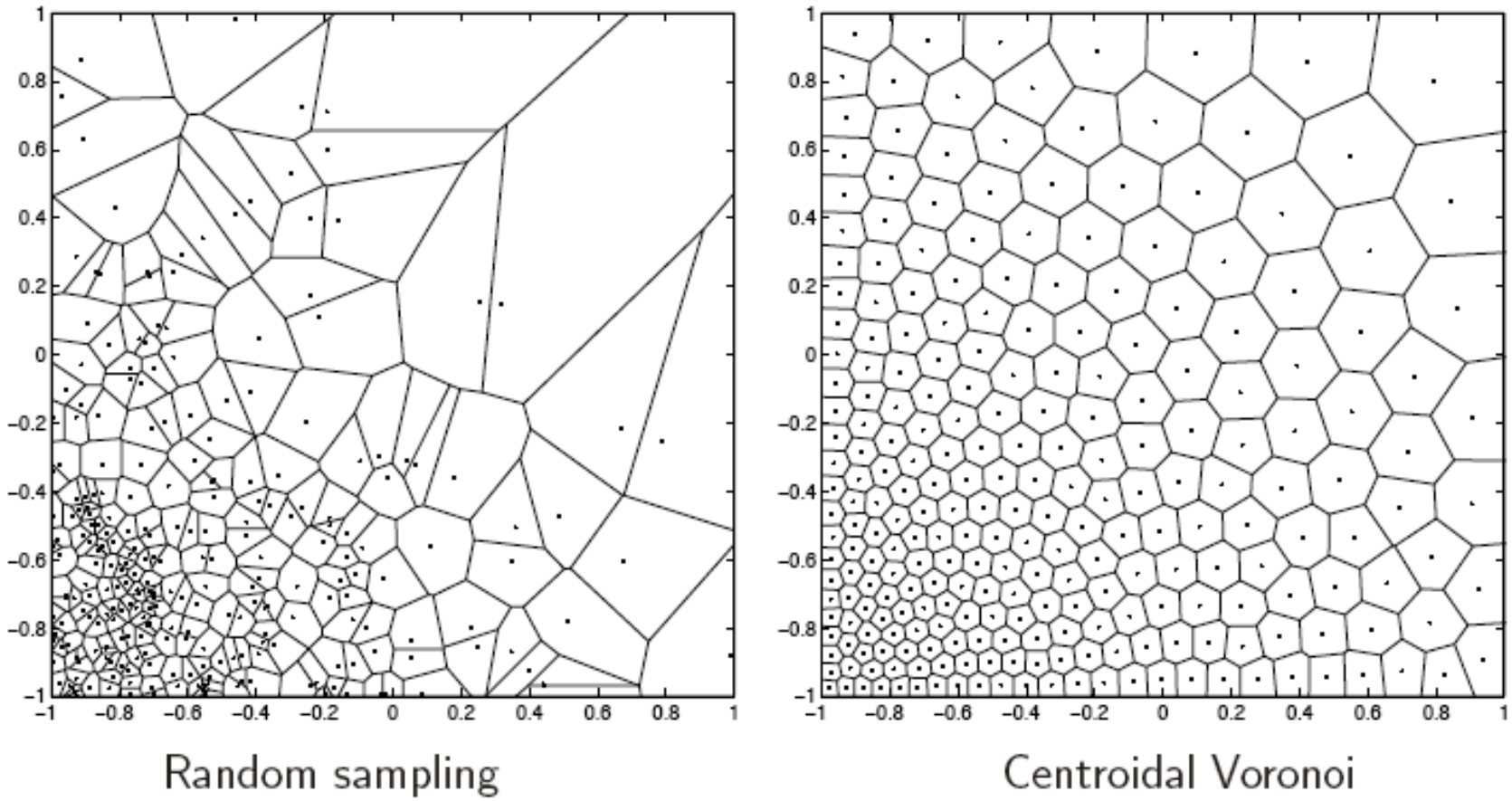
Iterating toward and CVT ....



Non-uniform Centroidal Voronoi Tessellations

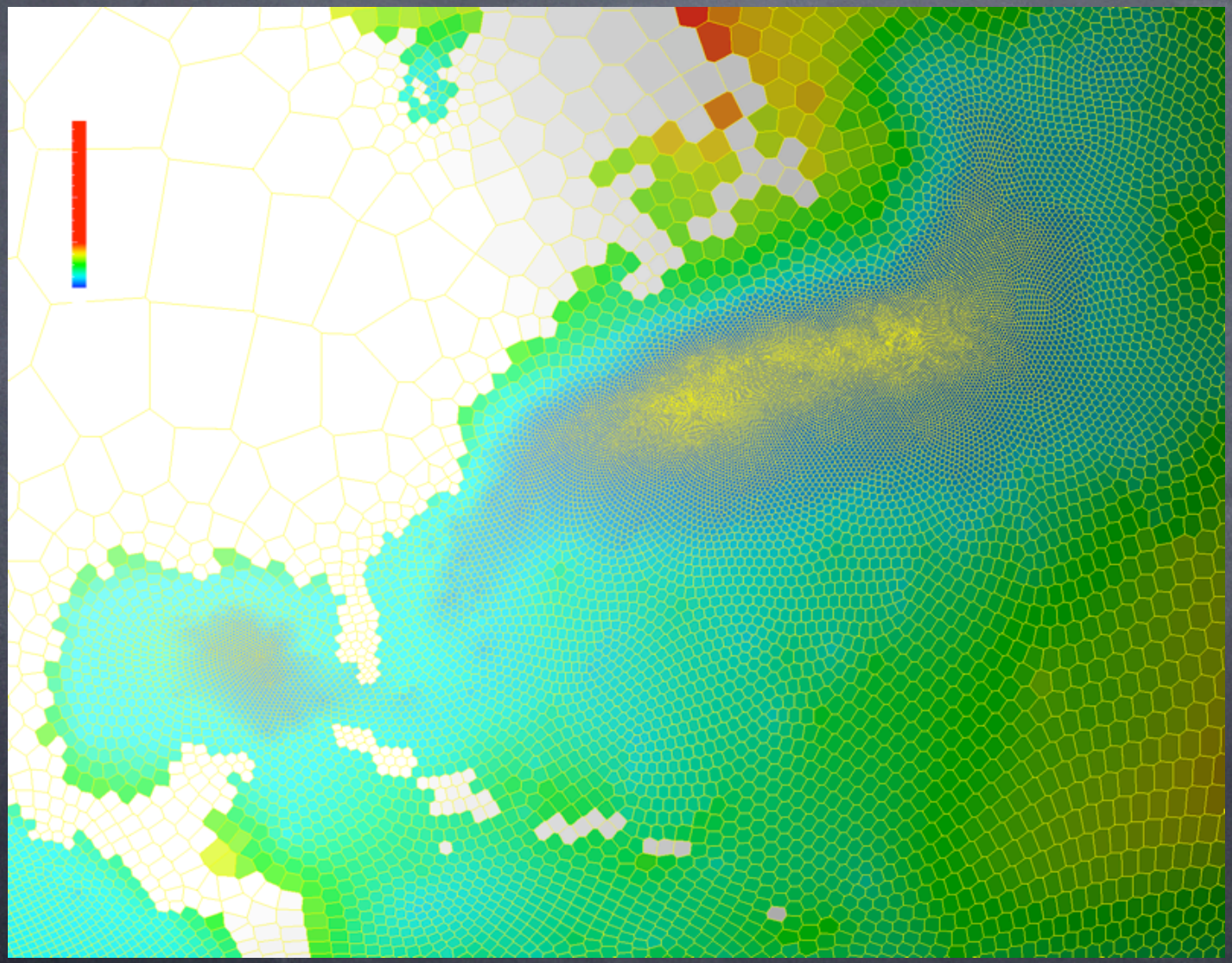
Distribute generators in such a way as to make the grid regular.

Also biases the location of those generators to regions of high density.

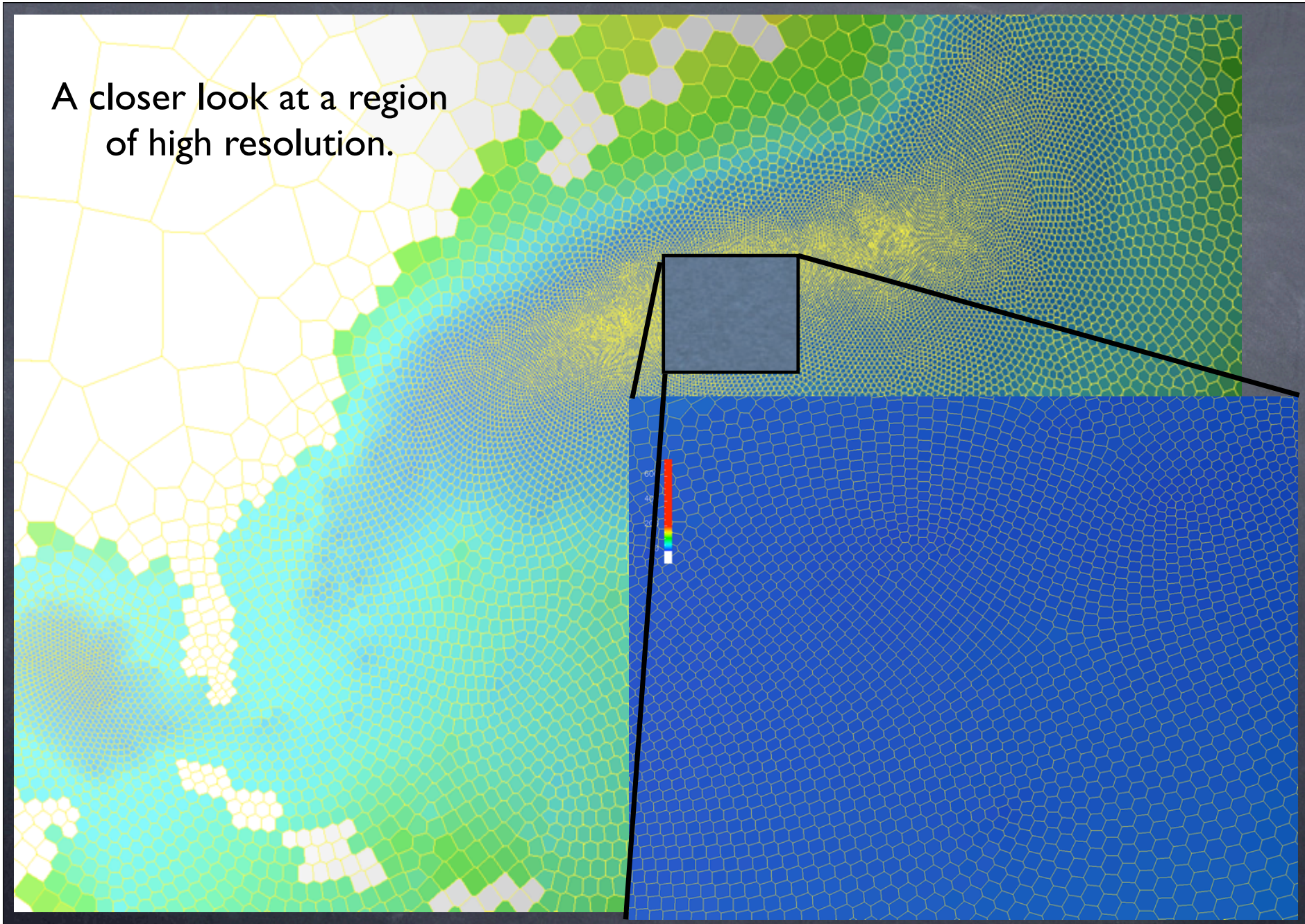


An example of an SCVT

the density proxy use here was TOPEX SSH Variance



A closer look at a region of high resolution.



(S)CVTs have their roots in applied math ...

Gersho conjecture (now proven in 2D): as we added generators, all cells evolve toward perfect hexagons. Meaning that the grid just keeps getting more regular as we add resolution.

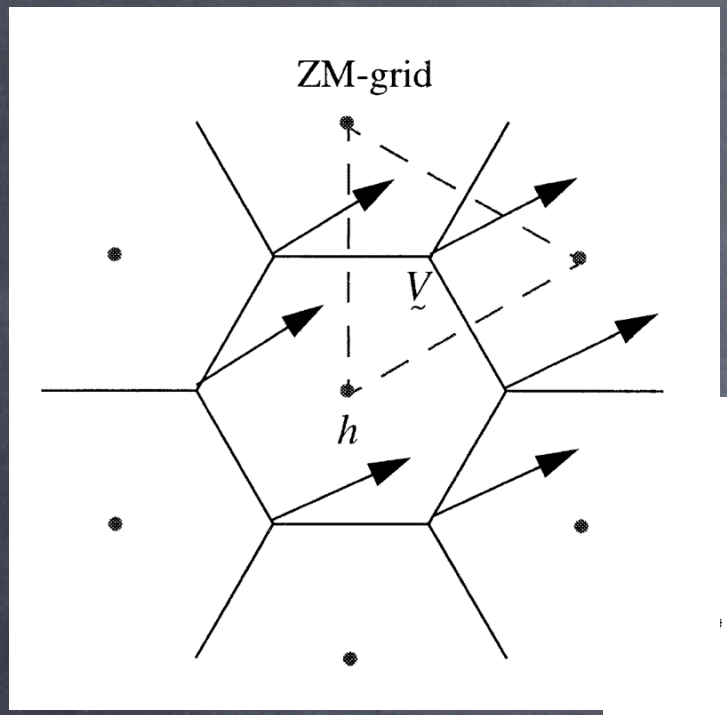
Optimal sampling: given a region, R, and N buckets to measure precipitation in R, the optimal placement of those buckets is a CVT. If a prior distribution, P, of precipitation is known, the CVT takes that information into account with rho=P^1/2.

Guaranteed to have 2nd-order truncation error of Poisson equation.

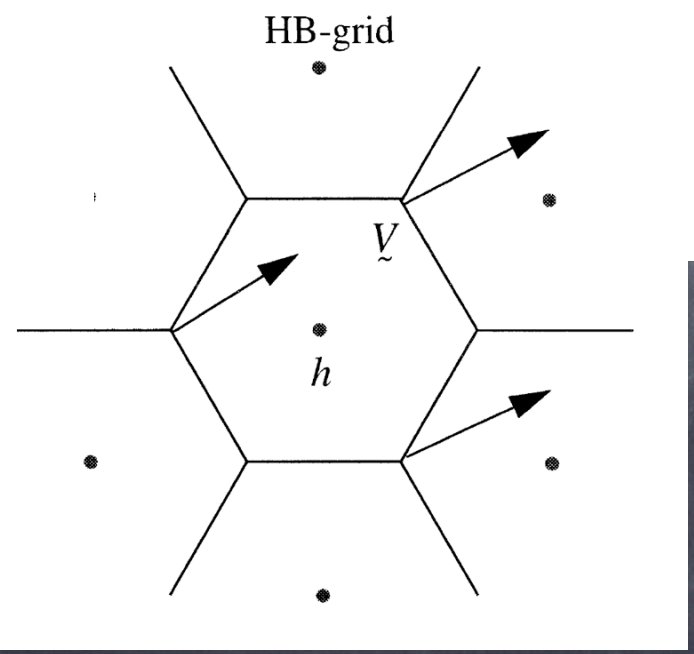
In summary: if Voronoi tessellations are to be used, then there is no good reason not to use Centroidal Voronoi Tessellations.

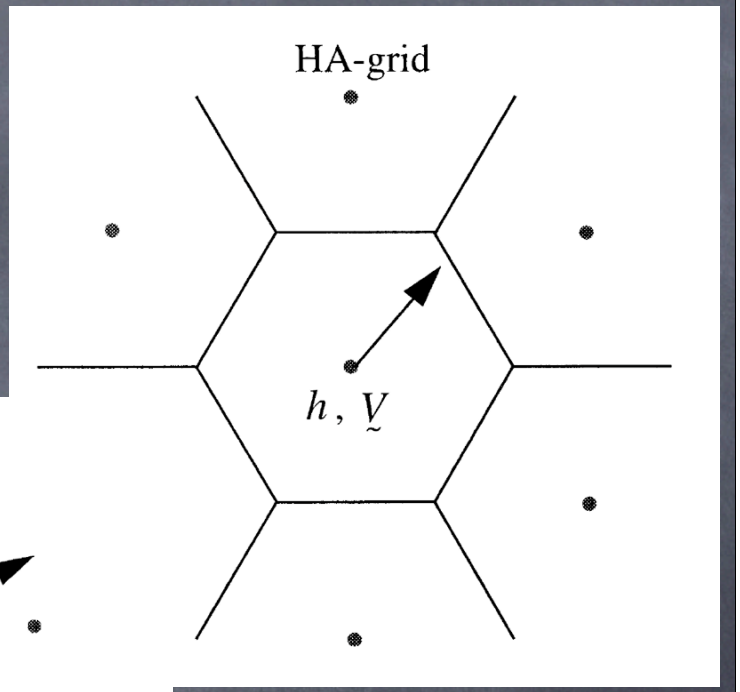
A mesh without robust numerical methods is useless, so what about discretization?

Some grid-staggering options ....



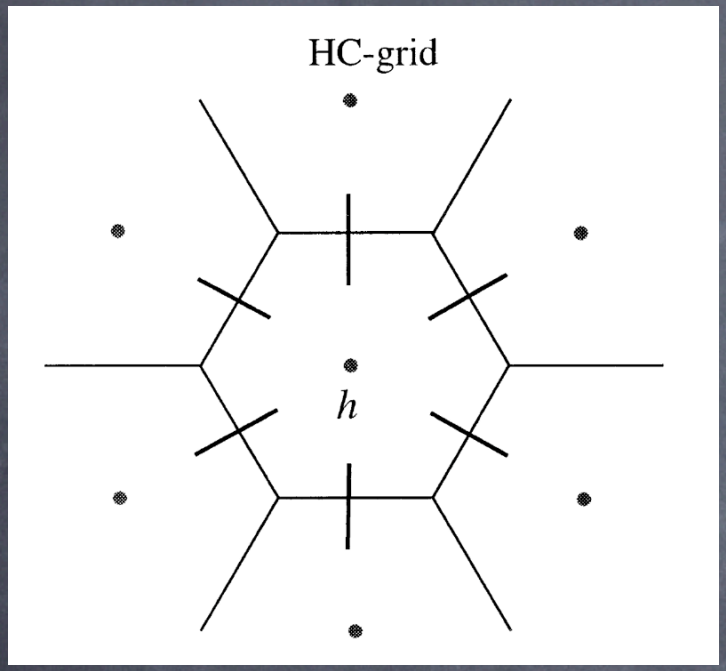
CSU Ocean Model

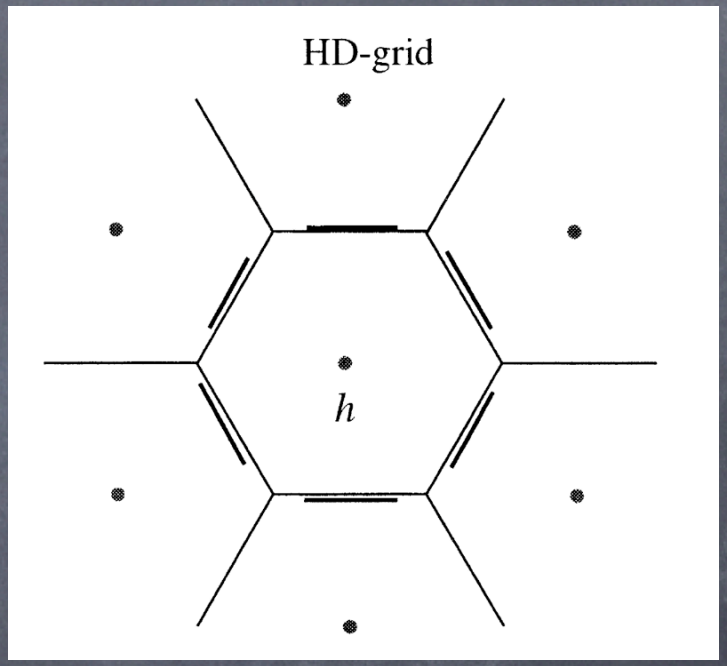




Frontier AGCM

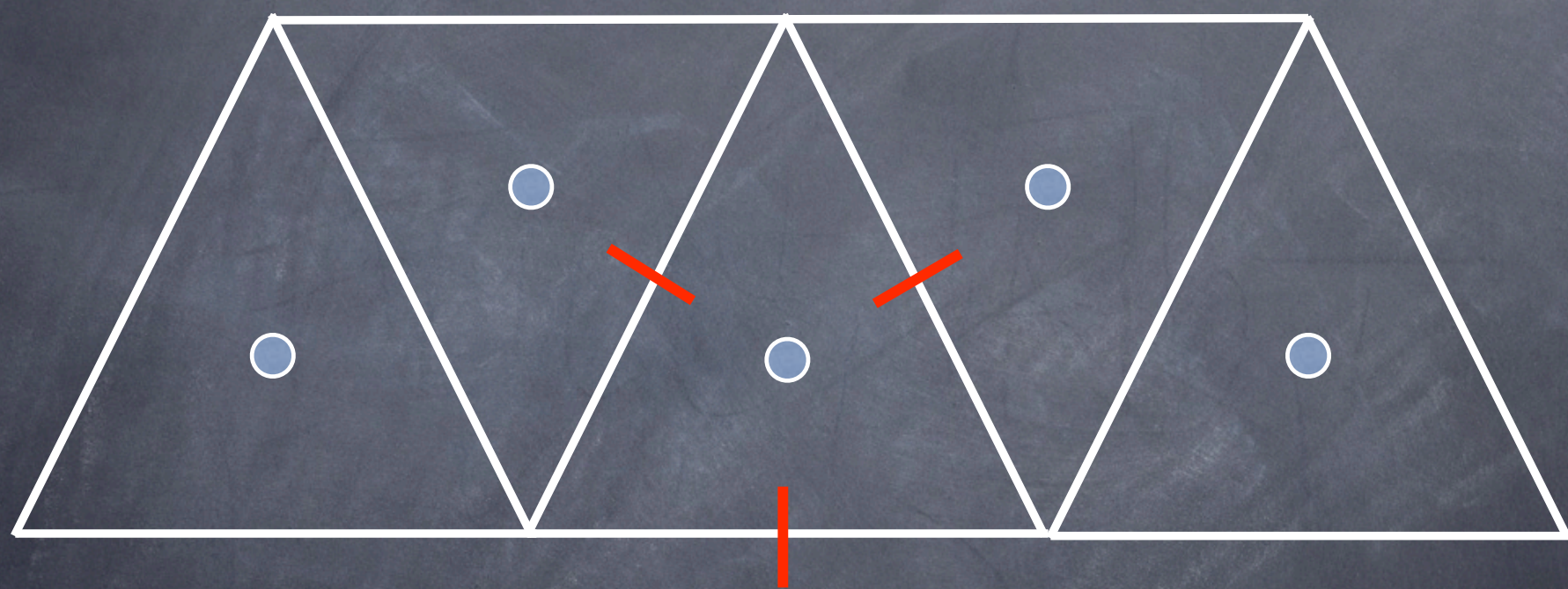
and more options ...





A staggering we are going to consider for our variable resolution grids.

and for those who prefer the dual tessellation of triangles ...



MPI/DWD ICON project

So with so many options, how do we go about choosing the correct grid staggering?

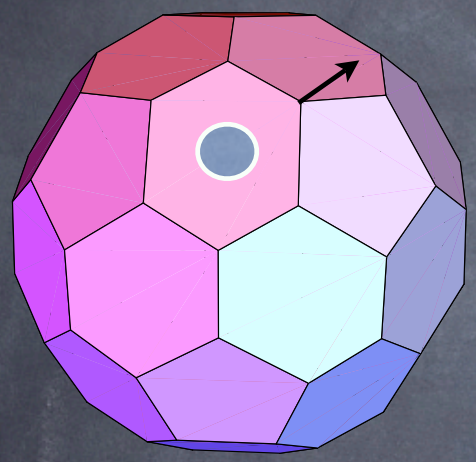
1. Many (by not all) of the known characteristics of quad-staggerings carry over.

2. Detailed look at the linear geostrophic adjustment problem, i.e. gravity waves, geostropic balance, and (most importantly) null spaces.

3. Detailed look at nonlinear properties, such as energy and potential enstrophy (conservation or boundedness).

Euler's Formula and Free Modes

Faces + Vertices - Edges = 2



Faces = 42 ⟶ mass

Vertices = 80 ⟶ velocity

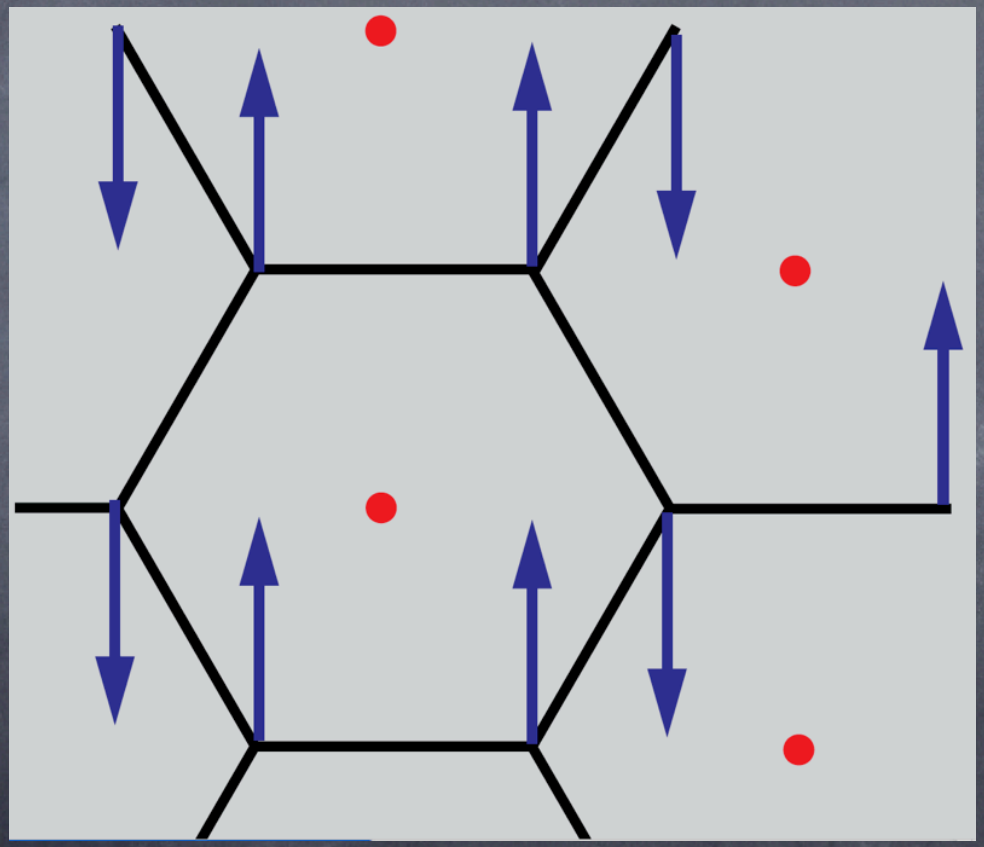
Edges = 120

The continuous shallow-water equations have one full (2d) vector associated with each mass.

80 velocity modes - 42 mass modes = 38 free velocity modes.

Susceptible to grid-scale noise in velocity field.

In this system, the extra velocity modes create patterns with zero div and zero curl.

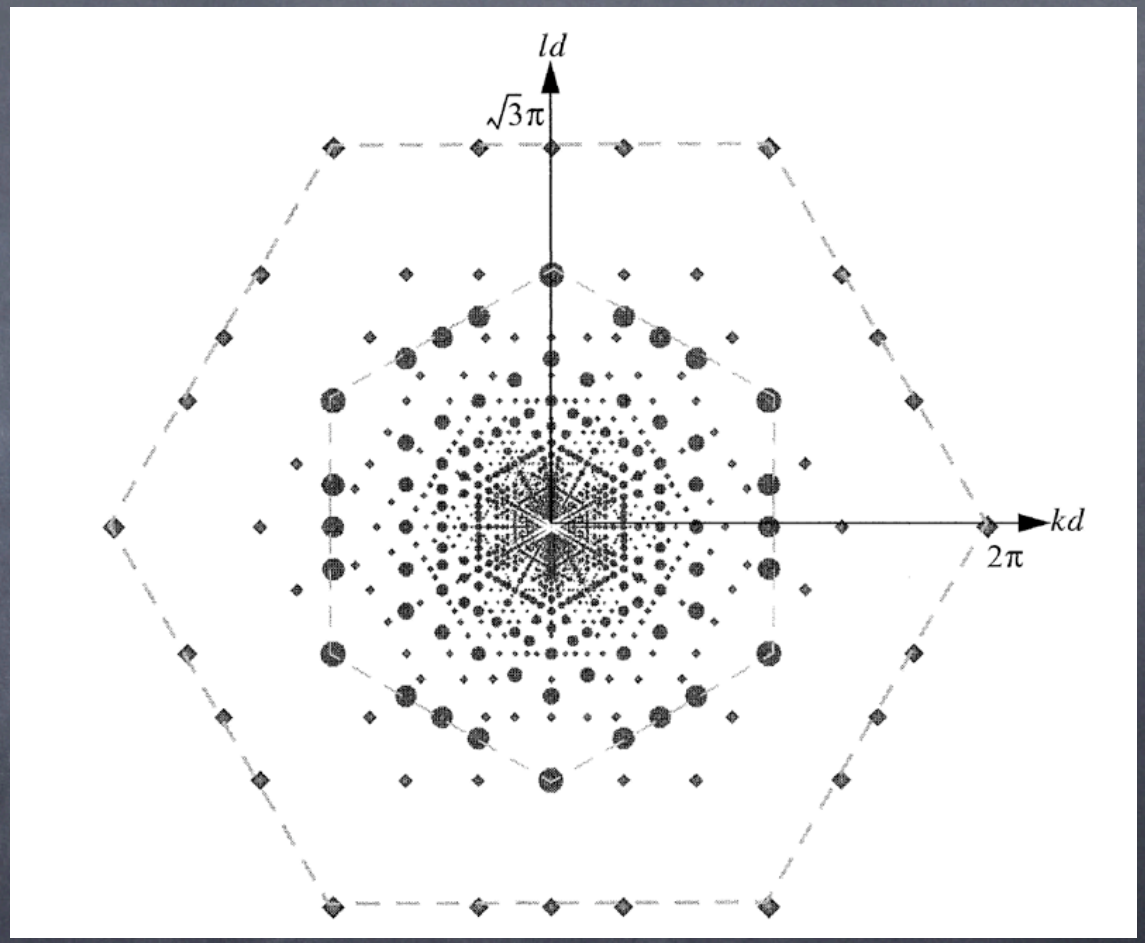


A full analysis of the discrete modes

Circles depict mass modes.

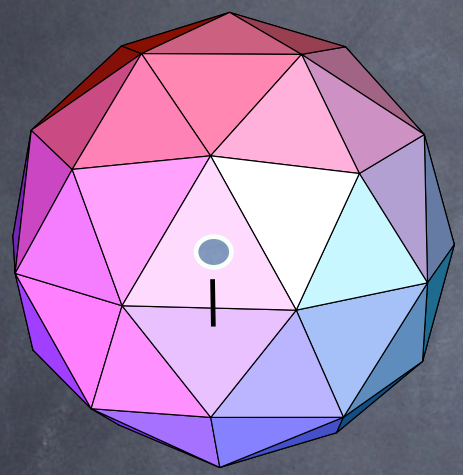
Diamonds depict velocity modes.

In terms of vorticity and divergence, the region between the hexagons is aliased into the inner hexagon.



What about the triangular C-grid?

Faces + Vertices - Edges = 2



Faces = 80 ⟶ mass

Vertices = 42

Edges = 120 ⟶ velocity (1/2)

80 mass modes - 60 velocity modes = 20 free mass modes.

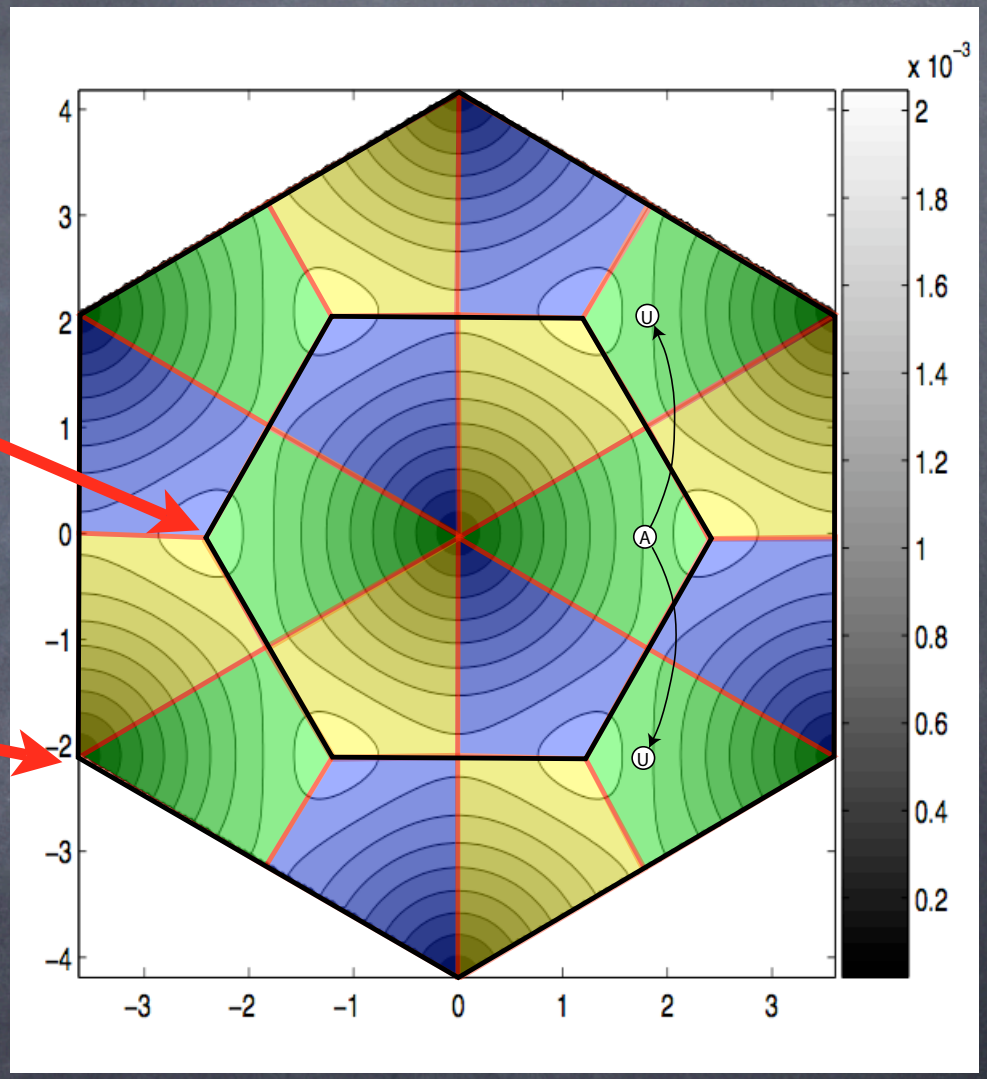
Susceptible to grid-scale noise in mass field.

Dispersion relation on triangular C-grid

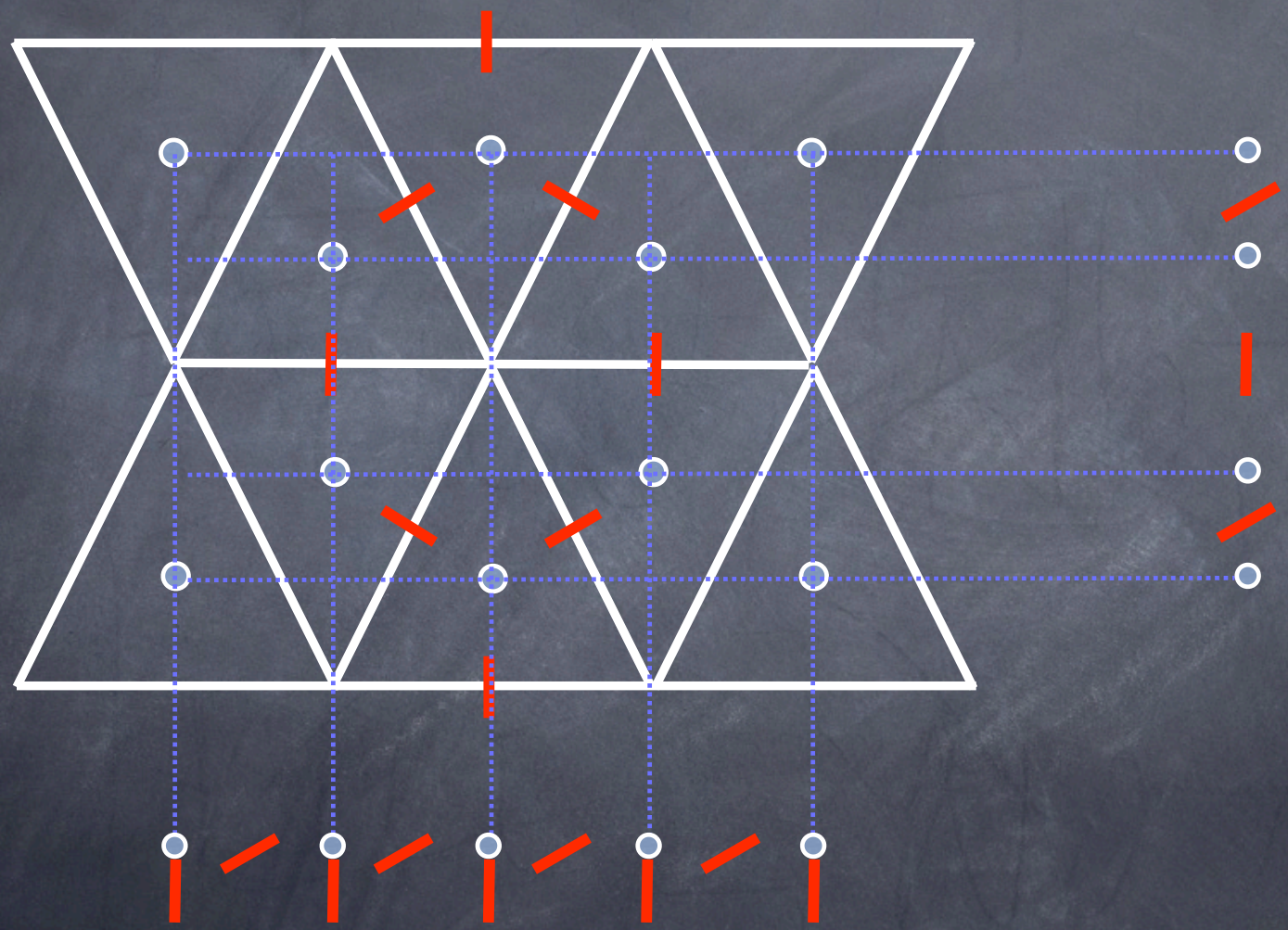
Asymmetry in relation

zero group velocity locations

zero phase velocity locations



Collapse grid in each direction: grid works different in x and y



Unconstrained modes are always going to be an issue on these grids. The solution is to identify them early, determine their severity, and develop stencils to suppress/filter these modes.

Mimetic Methods:

The idea probably has merit, it has been "invented" in at least three different lines of work.

Developing discrete analogs to the weak-form definitions of div, grad, and curl such that certain vector identities hold exactly.

These vector identities are a necessary prerequisite for energy conservation. At a minimum, these vector identities lead to a coherent formulation.

Robust, extensible method applicable to any FV grid.

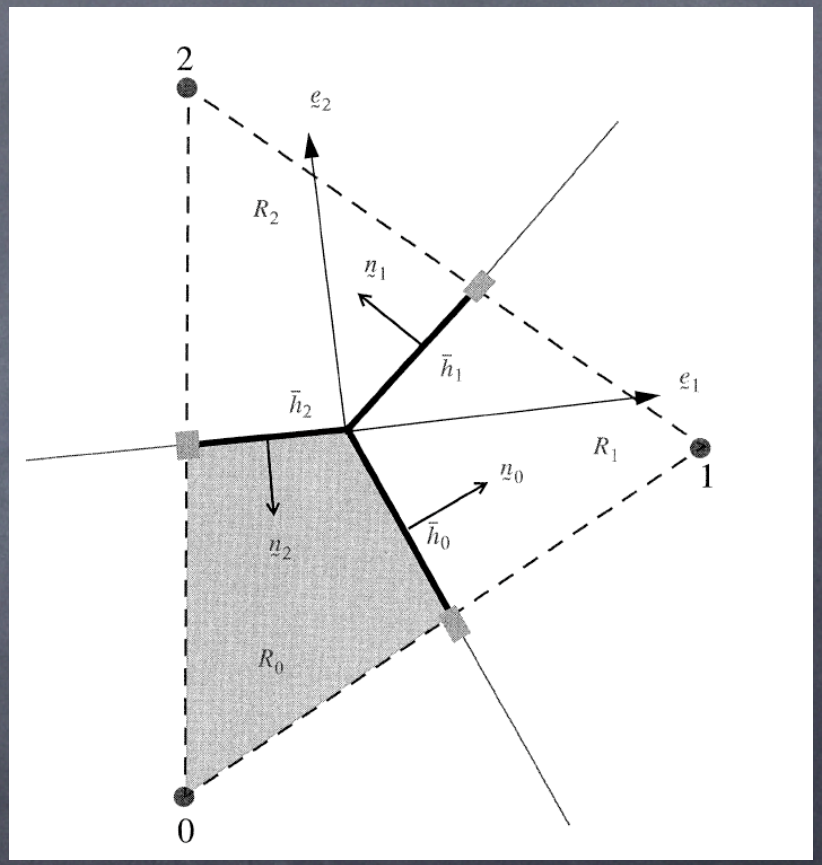
Building operators from line integrals

$$grad(h) = \nabla h = \lim_{A \to 0} \frac{1}{A} \int_c h \tilde{n} \, dl$$

$$div(\tilde{V}) = \nabla \cdot \tilde{V} = \lim_{A \to 0} \frac{1}{A} \int_c \tilde{V} \cdot \tilde{n} \, dl$$

$$\nabla \cdot \left(h \tilde{V} \right) = h \nabla \cdot \tilde{V} + \tilde{V} \cdot \nabla h$$

$$\tilde{\sigma} : \nabla \tilde{V} + \tilde{V} \cdot \left(\nabla \cdot \tilde{\sigma} \right) = 0$$



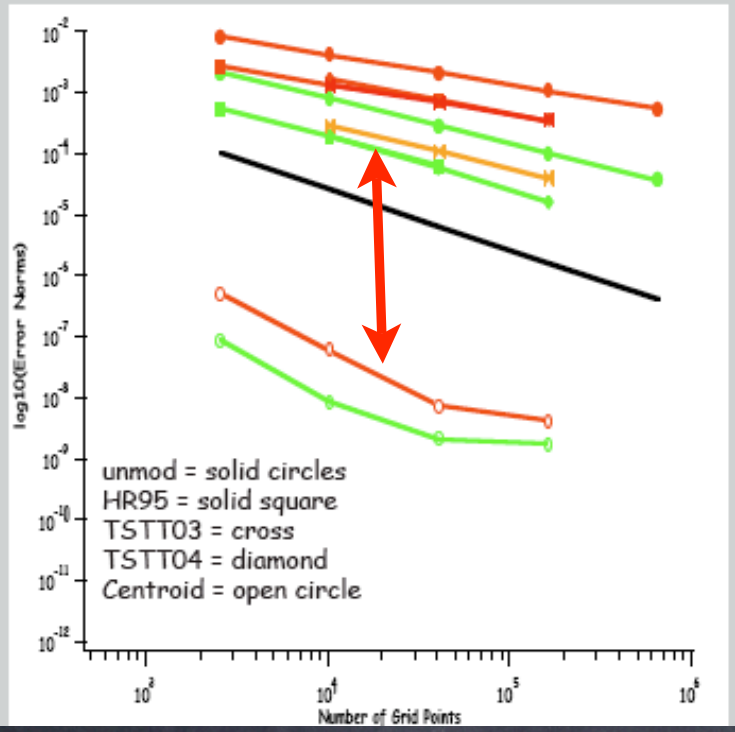
heating in internal energy
(positive definite)

dissipation in momentum
(negative definite)

The accuracy of these operators ... looking at the Laplacian

Solution#1: $u = \sin\phi$

Truncation Error
L2 Norm in Green, Linf Norm in Red
Black Line indicates -2 convergence



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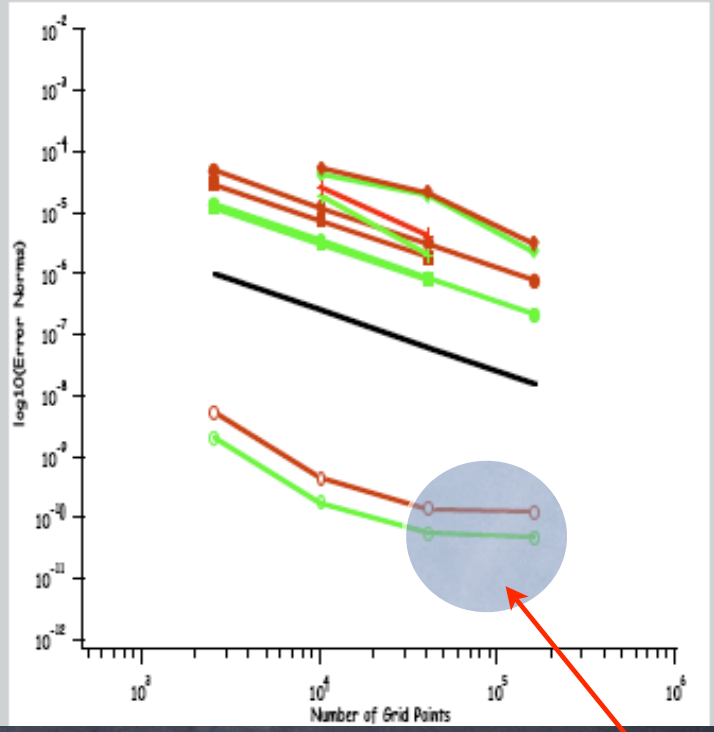


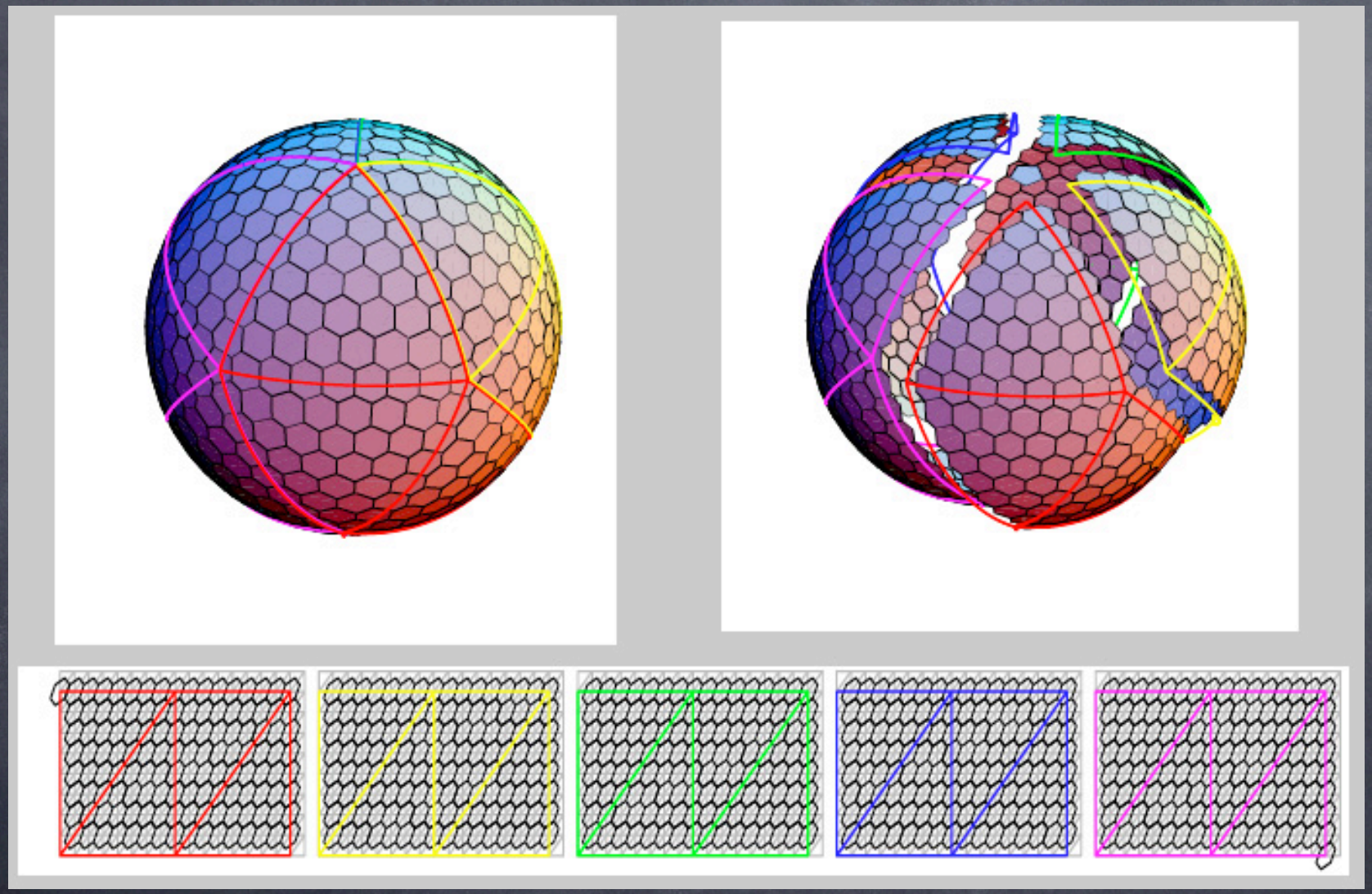
spread is due to the use of a spherical harmonic as the test function

round-off

The mimetic approach provides robust analogs to discrete vector identities with second-order accuracy in the solution error.

Making a model that scales ...
the trade-off between structured
and unstructured meshes.

Voronoi Tessellations: Structured Meshes



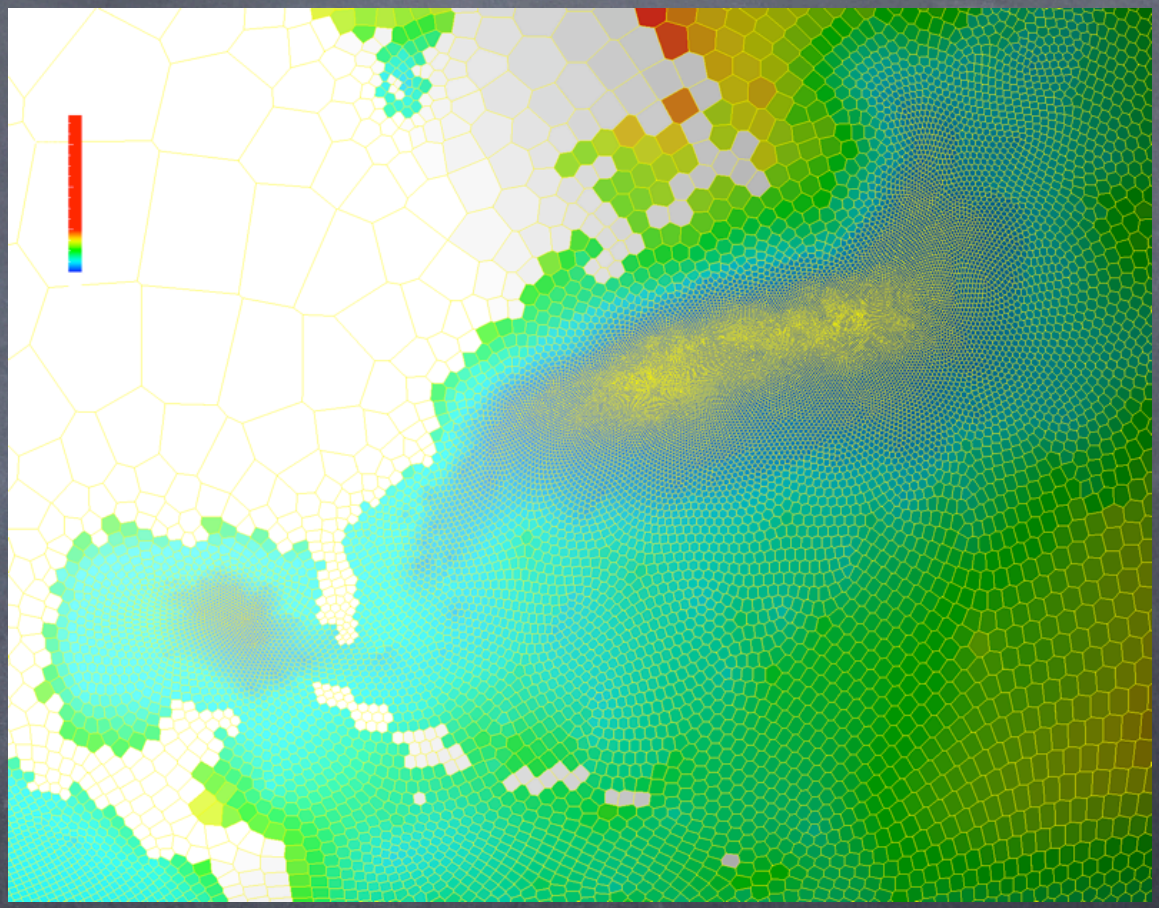
structured topology breaks down for even mildly varying resolutions.

Voronoi Tessellations: Blocks for Domain Decomposition for structured meshes

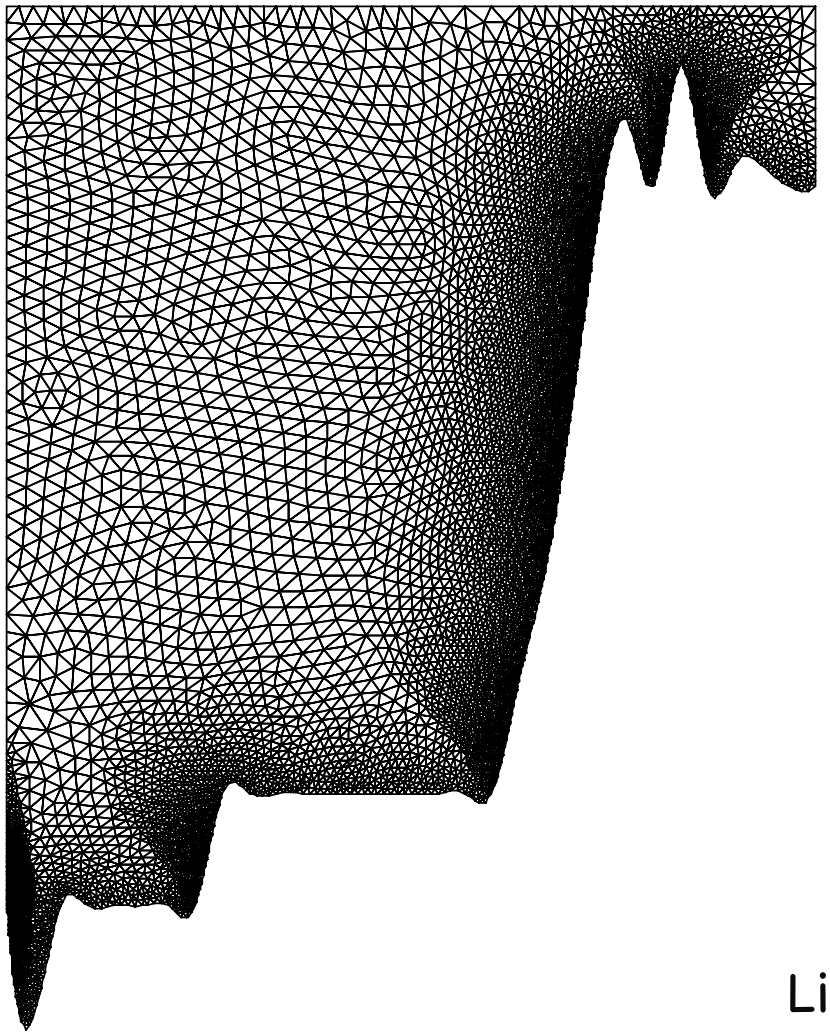
Unstructured mesh ...

really no different than a finite-element mesh.

indirect addressing is required, i.e. my neighbor to the right is not at i+1



Unstructured 3D meshes ... testing the idea in x-z



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Degrees of freedom and scaling ...

$$C \approx \alpha^D$$

where C is the cost
alpha is the scaling factor (min/max)
and D is the dimension.

for an alpha of .2, the cost goes as
.2, .04, .008 for D=1,2,3.

The trade-offs

Structured meshes

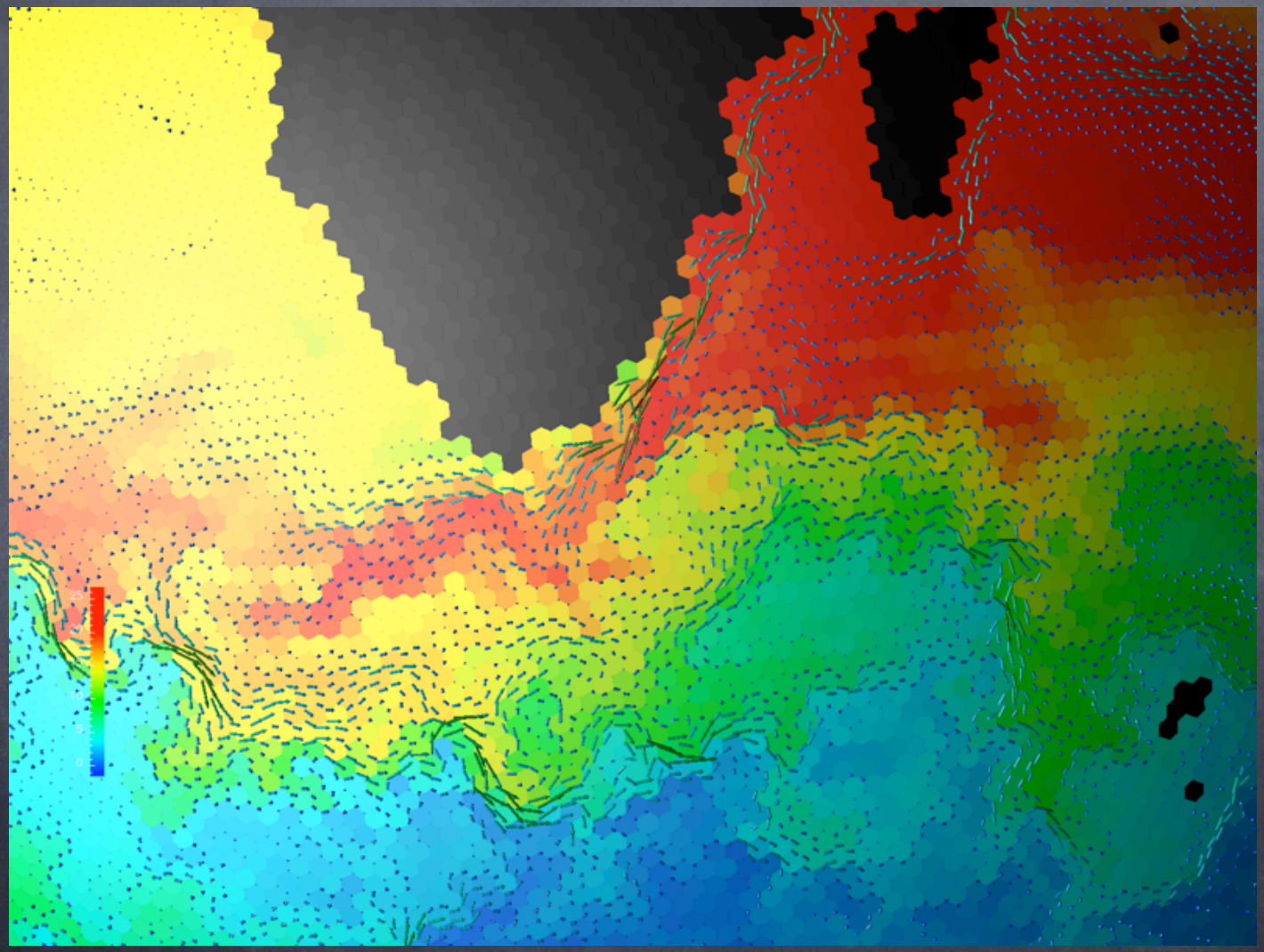
computationally efficient
quasi-uniform meshes only option
regional domains possible, but cumbersome

Unstructured meshes

computationally challenging
efficient allocation of resources (if we know where to put them)
regional domains possible
flexible and adaptive

Where is this all heading?

Voronoi Tessellations are viable, the question is are they better?



My very personal perspective

#1: Quasi-uniform Voronoi tessellations could be as useful as the traditional quadrilateral grids, given a commensurate effort as quads.

#2: If we think that non-uniform grids are potentially useful to global ocean modeling, then centroidal Voronoi tessellations are compelling.