Minimum Spanning Trees and Prim's Algorithm

Version of October 23, 2014

Outline

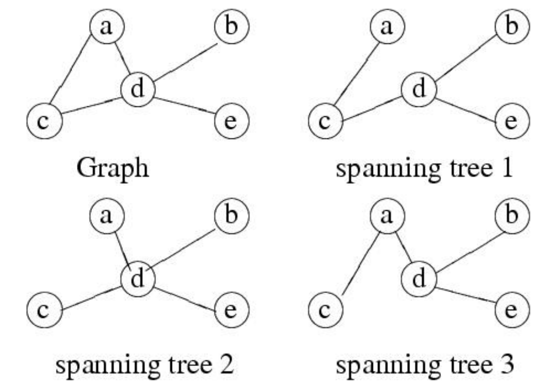
- Spanning trees and minimum spanning trees (MST).
- Tools for solving the MST problem.
- Prim's algorithm for the MST problem.
  - The idea
  - The algorithm
  - Analysis

Spanning Trees

Definition

A subgraph T of a undirected graph $G = (V, E)$ is a spanning tree of G if it is a tree and contains every vertex of G

Example



Theorem

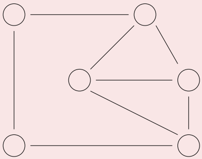
Every connected graph has a spanning tree.

Question

Why is this true?

Question

Given a connected graph G, how can you find a spanning tree of G?

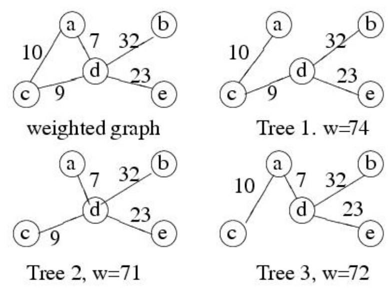


Weighted Graphs

Definition

A weighted graph is a graph, in which each edge has a weight (some real number) Could denote length, time, strength, etc.

Example



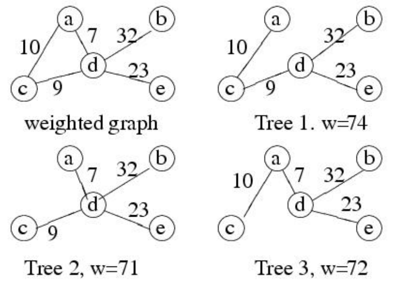
Definition

Weight of a graph: The sum of the weights of all edges

Definition

A Minimum spanning tree (MST) of an undirected connected weighted graph is a spanning tree of minimum weight (among all spanning trees).

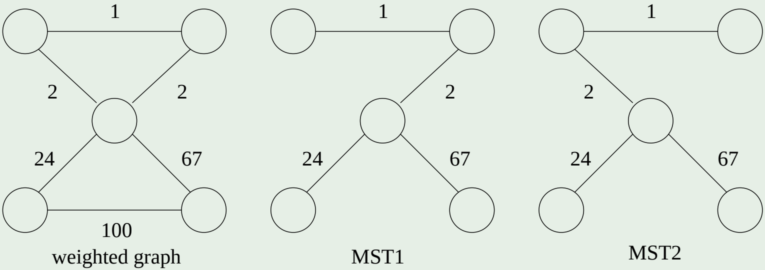
Example



Remark

The minimum spanning tree may not be unique

Example



Note: if the weights of all the edges are distinct, MST is provably unique (proof will follow from later results).

Minimum Spanning Tree Problem

Definition (MST Problem)

Given a connected weighted undirected graph G, design an algorithm that outputs a minimum spanning tree (MST) of G.

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Tools for solving the MST Problem

A tree is an acyclic graph

1. start with an empty graph
2. try to add edges one at a time, subject to not creating a cycle
3. if after adding each edge we are sure that the resulting graph is a subset of some minimum spanning tree, then, after $n-1$ steps we are done.

Hard part is ensuring (3)!

Generic Algorithm for MST problem

Definition

Let A be a set of edges such that $A \subseteq T$, where T is some MST. Edge (u, v) is safe edge for A, if $A \cup \{(u, v)\}$ is also a subset of some MST

- If at each step, we can find a safe edge (u, v), we can grow a MST

Generic-MST(G, w)

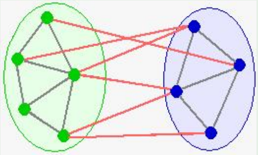
```
begin
    A= EMPTY;
    while A does not form a spanning tree do
        find an edge (u, v) that is safe for A;
        add (u, v) to A;
    end
    return A
end
```

Some Definitions

Definition

Let $G = (V, E)$ be a connected and undirected graph.
A cut $(S, V - S)$ of G is a partition of V.

Example



Definition

An edge $(u, v) \in E$ crosses the cut $(S, V - S)$ if one of its endpoints is in S, and the other is in $V - S$.
A cut respects a set A of edges if no edge in A crosses the cut.
An edge is a light edge crossing a cut if its weight is the minimum of any edge crossing the cut.

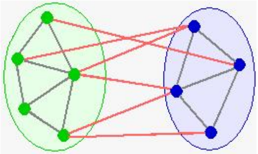
Lemma

- *Let $G = (V, E)$ be a connected, undirected graph with a real-valued weight function w defined on E*
- *A be a subset of E that is included in some minimum spanning tree for G.*

Let

- *$(S, V - S)$ be any cut of G that respects A*
- *(u, v) be a light edge crossing the cut $(S, V - S)$*

Then, edge (u, v) is safe for A.



This implies we can find a safe edge by

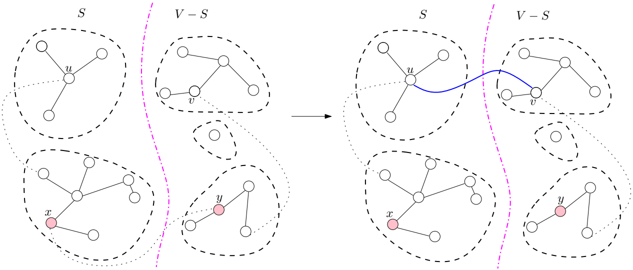
1. first finding a cut that respects A,
2. then finding a light edge crossing that cut.

That light edge is a safe edge.

Proof

- Let $A \subseteq T$, where T is a MST.
- Case 1: $(u, v) \in T$
  - $A \cup \{(u, v)\} \subseteq T$.
  - Hence (u, v) is safe for A.

- Case 2: $(u, v) \notin T$
  - *Idea: construct* another *MST T' s.t. $A \cup \{(u, v)\} \subseteq T'$.*
  - Consider the unique path P in T from u to v.
  - Since u and v are on opposite sides of the cut $(S, V - S)$,
    - There is at least one edge in P that crosses the cut.
    - Let (x, y) be such an edge.
  - Since the cut respects A, $(x, y) \notin A$.
  - Since (u, v) is a light edge crossing the cut, we have $w(u, v) \leq w(x, y)$.



- Adding (u, v) to T, creates a cycle with P.
  Removing any edge from this cycle gives a tree again.
  In particular, adding (u, v) and removing (x, y) creates a new tree $T^{'}$.

- The weight of $T^{'}$ is

$$\begin{aligned} w(T') &= w(T) - w(x, y) + w(u, v) \\ &\leq w(T) \end{aligned}$$

- Since T is a MST, $W(T) \leq W(T')$ so $W(T') = W(T)$ and T is also an MST.

- But $A \cup \{(u, v)\} \subseteq T^{'}$, so (u, v), is safe for A.
- The Lemma is proved.

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Prim's Algorithm

The generic algorithm gives us an idea how to 'grow' a MST.

- If you read the theorem and proof carefully, you will notice that the choice of a cut (and hence a corresponding light edge) in each iteration is arbitrary.
- We can select any cut (that respects current edge set A) and find a light edge crossing that cut to proceed.
- Different ways of chosing cuts correspond to different algorithms.
- The two major ones are Prim's algorithm and Kruskal's algorithm,

Prim's Algorithm

Prim's algorithm

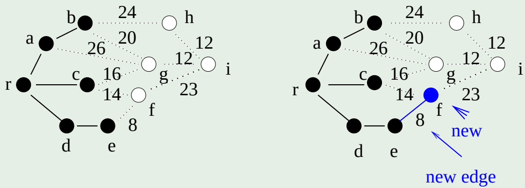
- grows a tree, adding a new light edge in each iteration, creating a new tree.

Growing a tree

- Start by picking any vertex r to be the root of the tree.
- While the tree does not contain all vertices in the graph: find shortest edge leaving tree and add it to the tree.

We will show that these steps can be implemented in total $O(E \cdot \log V)$.

Example



Step 0:

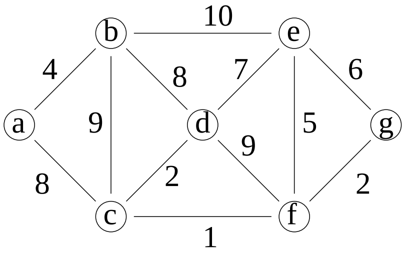
- Choose any element r; set $S = \{r\}$ and $A = \emptyset$.
- (Take r as the root of our spanning tree.)

Step 1:

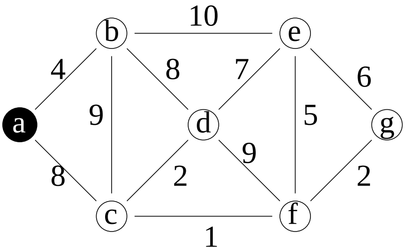
- Find a lightest edge such that one endpoint is in S and the other is in $V \setminus S$.
- Add this edge to A and its (other) endpoint to S.

Step 2:

- If $V \setminus S = \emptyset$, then stop and output (minimum) spanning tree (S, A);Otherwise, go to Step 1.



Connected graph

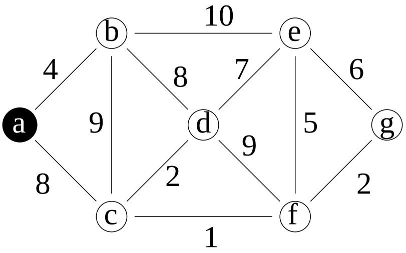


Step 0

S={a}

V \ S = {b,c,d,e,f,g}

lightest edge = {a,b}



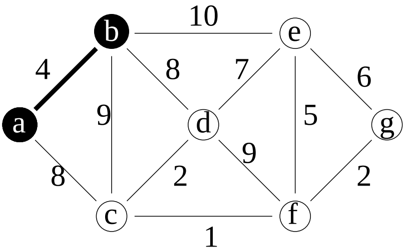
Step 1.1 before

S={a}

V \ S = {b,c,d,e,f,g}

A={}

lightest edge = {a,b}



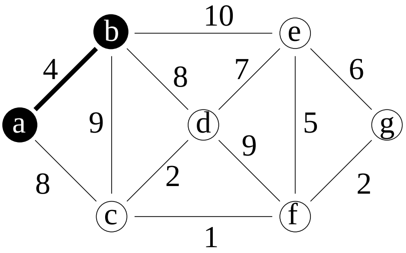
Step 1.1 after

S={a,b}

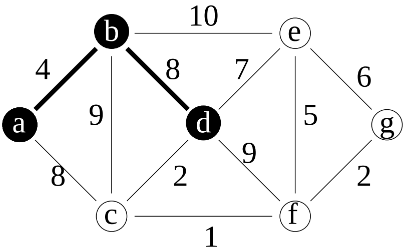
V \ S = {c,d,e,f,g}

A={{a,b}}

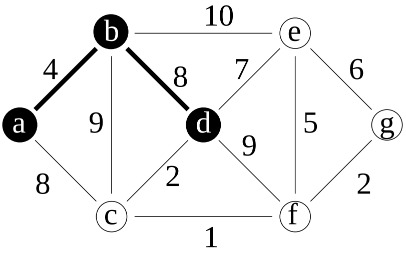
lightest edge = {b,d}, {a,c}



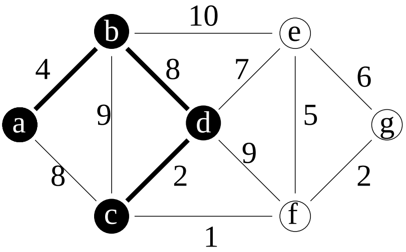
Step 1.2 before
S={a,b}
V \ S = {c,d,e,f,g}
A={{a,b}}
lightest edge = {b,d}, {a,c}



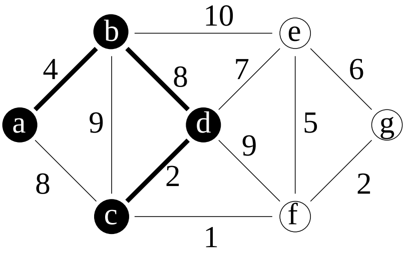
Step 1.2 after
S={a,b,d}
V \ S = {c,e,f,g}
A={{a,b},{b,d}}
lightest edge = {d,c}



Step 1.3 before
S={a,b,d}
V \ S = {c,e,f,g}
A={{a,b},{b,d}}
lightest edge = {d,c}



Step 1.3 after
S={a,b,c,d}
V \ S = {e,f,g}
A={{a,b},{b,d},{c,d}}
lightest edge = {c,f}



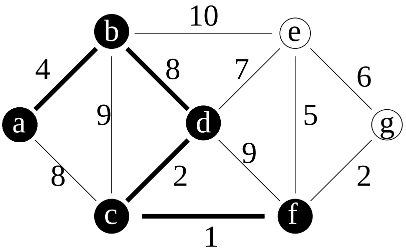
Step 1.4 before

S={a,b,c,d}

V \ S = {e,f,g}

A={{a,b},{b,d},{c,d}}

lightest edge = {c,f}



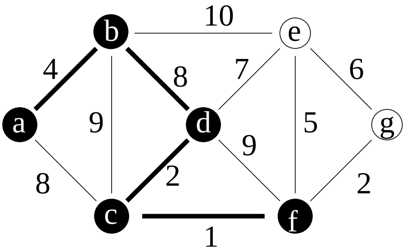
Step 1.4 after

S={a,b,c,d,f}

V \ S = {e,g}

A={{a,b},{b,d},{c,d},{c,f}}

lightest edge = {f,g}



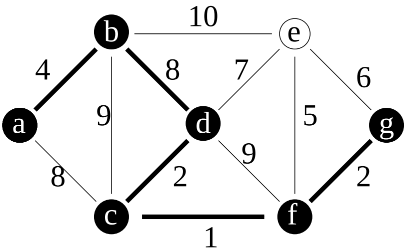
Step 1.5 before

S={a,b,c,d,f}

V \ S = {e,g}

A={{a,b},{b,d},{c,d},{c,f}}

lightest edge = {f,g}



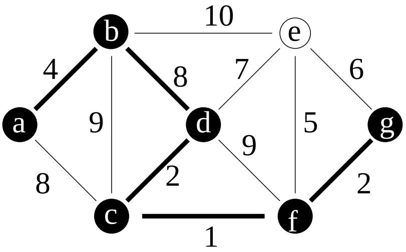
Step 1.5 after

S={a,b,c,d,f,g}

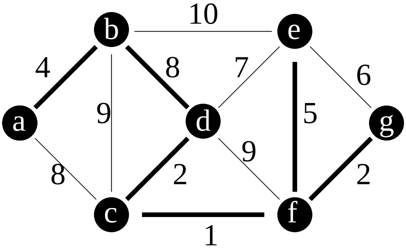
V \ S = {e}

A={{a,b},{b,d},{c,d},{c,f},
{f,g}}

lightest edge = {f,e}



Step 1.6 before
S={a,b,c,d,f,g}
V \ S = {e}
A={{a,b},{b,d},{c,d},{c,f},
{f,g}}
lightest edge = {f,e}



Step 1.6 after
S={a,b,c,d,e,f,g}
V \ S = {}
A={{a,b},{b,d},{c,d},{c,f},
{f,g},{f,e}}

MST completed

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Recall Idea of Prim's Algorithm

Step 0: Choose any element r and set $S = \{r\}$ and $A = \emptyset$. (Take r as the root of our spanning tree.)

Step 1: Find a lightest edge such that one endpoint is in S and the other is in $V \setminus S$.
Add this edge to A and its (other) endpoint to S.

Step 2: If $V \setminus S = \emptyset$, then stop and output the minimum spanning tree (S, A); Otherwise go to Step 1.

Questions

1. Why does this produce a minimum spanning tree?
2. How does the algorithm find the lightest edge and update A efficiently?
3. How does the algorithm update S efficiently?

Prim's Algorithm

Question

How does the algorithm update S efficiently?

Answer: Color the vertices.

- Initially all are white.
- Change the color to black when the vertex is moved to S.
- Use color$[v]$ to store color.

Question

How does the algorithm find a lightest edge and update A efficiently?

Answer:

1. Use a priority queue to find the lightest edge.
2. Use pred$[v]$ to update A.

Priority Queue is a data structure

- can be implemented as a heap

Supports the following operations:

Insert(u, key): Insert u with the key value key in Q.

u = Extract-Min(): Extract the item with minimum key value.

Decrease-Key(u, $new\text{-}key$): Decrease u's key value to $new\text{-}key$.

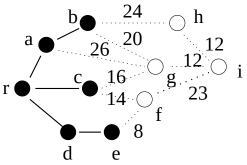
Remark: We already saw how to implement Insert and Extract-Min (and Delete) in $O(\log |Q|)$ time.
Same ideas can also be used to implement Decrease-Key in $O(\log |Q|)$ time.
Alternatively, can implement Decrease-Key using Delete followed by Insert.

Using a Priority Queue to Find the Lightest Edge

Each item of the queue is a pair $(u, key[u])$, where

- u is a vertex in $V \setminus S$,
- $key[u]$ is the weight of the lightest edge from u to any vertex in S.

(The endpoint of this edge in S is stored in $pred[u]$, which is used to build the MST tree.)



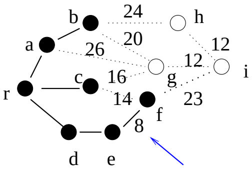
key[f] = 8, pred[f] = e

key[i] = infinity, pred[i] = nil

key[g] = 16, pred[g] = c

key[h] = 24, pred[h] = b

→ f has the minimum key



new edge

key[i] = 23, pred[i] = f

After adding the new edge and vertex f, update the key[v] and pred[v] for each vertex v adjacent to f

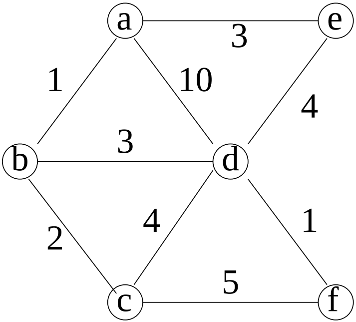
Description of Prim's Algorithm

```
begin
    foreach u ∈ V do
        color[u] = WHITE; key[u] = +∞; // initialize
    end
    key[r] = 0; pred[r] = NIL; // start at root
    Q = new PriQueue(V); // put vertices in Q
    while Q is nonempty do
        u= Q.Extract-Min(); // lightest edge
        foreach v ∈ adj[u] do
            if (color[v] = WHITE)&&(w[u, v] < key[v]) then
                key[v] = w[u, v]; // new lightest edge
                Q.Decrease-Key(v, key[v]);
                pred[v] = u;
            end
        end
        color[u] = BLACK;
    end
end
```

When the algorithm terminates, $Q = \emptyset$ and the MST is

$$T = \{\{v, pred[v]\} : v \in V \setminus \{r\}\}.$$

- The pred pointers define the MST as an inverted tree rooted at r.



u	a	b	c	d	e	f
key[u]						
pred[u]						

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```
begin
    foreach u ∈ V do
        key[u] = +∞; color[u] = WHITE; // O(V)
    end
    key[r] = 0; pred[r] = NIL;
    Q = new PriQueue(V); // O(V)
    while Q is nonempty do
        u= Q.Extract-Min(); // Do this for each vertex
        foreach v ∈ adj[u] do
            // Do the following for each edge twice
            if (color[v] = WHITE)&&(w[u, v] < key[v]) then
                key[v] = w[u, v]; pred[v] = u;
                Q.Decrease-Key(v, key[v]); // This is bottleneck
            end
        end
        color[u] = BLACK;
    end
end
```

Analysis of Prim's Algorithm

The data structure PriQueue (heap) supports the following two operations:

- ($O(|V|)$ for creating new Priority Queue
- $O(\log V)$ for Extract-Min on a PriQueue of size at most V.
  Total cost: $O(V \log V)$
- $O(\log V)$ time for Decrease-Key on a PriQueue of size at most V.
  Total cost: $O(E \log V)$.

Total cost is then $O((V + E) \log V) = O(E \log V)$

Going Further

A more advanced Priority Queue implementation called *Fibonnaci Heaps* allow

- $O(1)$ for inserting each item
- $O(\log |V|)$ for Extract-Min
- **$\mathbf{O}(1)$ (amortized) for each Decrease-Key**

Since algorithm performs $|V|$ Inserts, $|V|$ Extract-Mins and at most E Decrease-Keys this leads to a $O(|E| + |V| \log |V|)$ algorithm, improving upon the $O(E \log V)$ more naive implementation.