

# Presentations and Extensions of Transversal Matroids

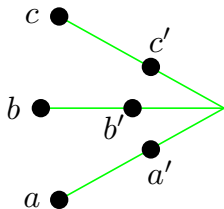
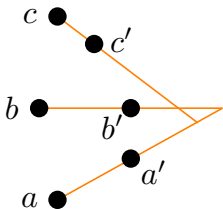
Joseph E. Bonin

The George Washington University

This includes joint work with  
Anna de Mier (Universitat Politècnica de Catalunya)

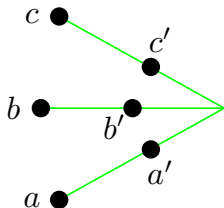
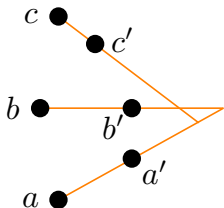
*The motivation, by analogy*

A matroid  $M$  can have inequivalent representations, so a matrix representation  $A$  can have extraneous information that limits which extensions of  $M$  can be represented by adding a column to  $A$ .



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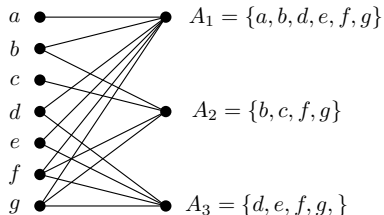
Transversal matroid often have many presentations.

How do the presentations relate to (e.g., limit) the transversal extensions?

## A brief review of transversal matroids

A set system,  $\mathcal{A} = (A_i : i \in [r])$ , is a multiset of sets.

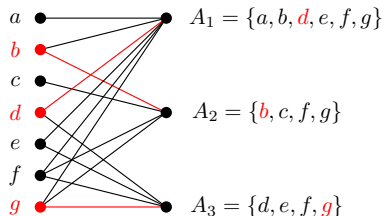
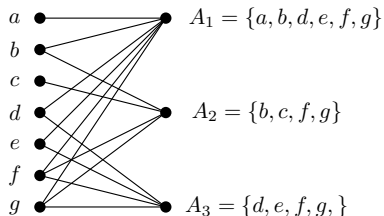
E.g.,  $A_1 = \{a, b, d, e, f, g\}$ ,  $A_2 = \{b, c, f, g\}$ ,  $A_3 = \{d, e, f, g, \}$ .



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The independent sets of the transversal matroid  $M[\mathcal{A}]$  are the partial transversals of  $\mathcal{A}$ .

The set system  $\mathcal{A}$  is a **presentation** of  $M[\mathcal{A}]$ .

*A brief review of transversal matroids*

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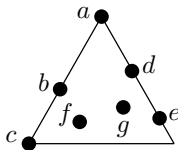
$$\begin{array}{c} \\ A_1 \\ A_2 \\ A_3 \end{array} \begin{array}{cccccc} a & b & c & d & e & f & g \\ \left( \begin{array}{cccccc} * & * & 0 & * & * & * & * \\ 0 & * & * & 0 & 0 & * & * \\ 0 & 0 & 0 & * & * & * & * \end{array} \right)$$

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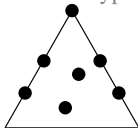


$$A_2 = \{b, c, f, g\} \quad A_3 = \{d, e, f, g\}$$

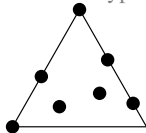
We can replace \*s with positive reals so that (i) determinants that aren't forced to be zero for generic reasons are nonzero and (ii) each non-zero column sums to 1.

## The multitude of presentations

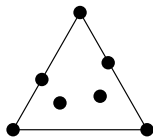
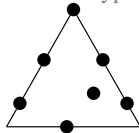
1 of this type



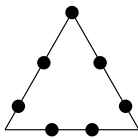
4 of this type



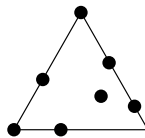
2 of this type



4 of this type



1 of this type

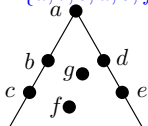


8 of this type



## The order on presentations

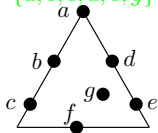
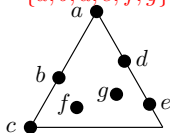
$$A_1 = \{a, b, c, d, e, f, g\}$$



$$A_2 = \{b, c, f, g\} \quad A_3 = \{d, e, f, g\}$$

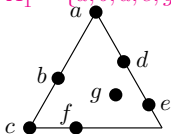
$$\{a, b, d, e, f, g\}$$

$$\{a, b, c, d, e, g\}$$



$$\{b, c, f, g\} \quad \{d, e, f, g\} \quad \{b, c, f, g\} \quad \{d, e, f, g\}$$

$$A'_1 = \{a, b, d, e, g\}$$



$$A_2 = \{b, c, f, g\} \quad A_3 = \{d, e, f, g\}$$

A partial order on the presentations of  $M$ :

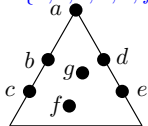
$$(A_1, \dots, A_r) \leq (A'_1, \dots, A'_r)$$

if, up to re-indexing the sets,

$$A_i \subseteq A'_i \quad \text{for all } i \in [r].$$

## The order on presentations

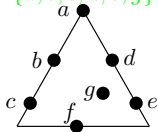
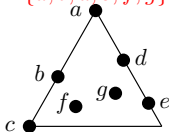
$$A_1 = \{a, b, c, d, e, f, g\}$$



$$A_2 = \{b, c, f, g\} \quad A_3 = \{d, e, f, g\}$$

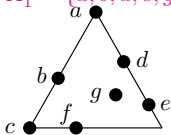
$$\{a, b, d, e, f, g\}$$

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A partial order on the presentations of  $M$ :

$$(A_1, \dots, A_r) \leq (A'_1, \dots, A'_r)$$

if, up to re-indexing the sets,

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A transversal matroid has a unique maximal presentation.

(Mason, 1971.)

Typically there are many minimal presentations.

## Extensions of a presentation of a transversal matroid

We consider only

- (a) presentations of  $M$  with  $r = r(M)$  sets, and
- (b) single-element transversal extensions of  $M$  of rank  $r$ .

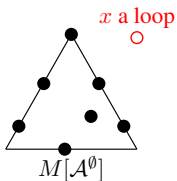
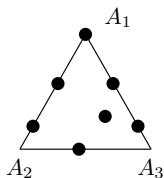
Let  $\mathcal{A} = (A_i : i \in [r])$  be a presentation of  $M$ .

For  $x \notin E(M)$  and  $J \subseteq [r]$ , set  $\mathcal{A}^J = (A_i^J : i \in [r])$  where

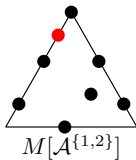
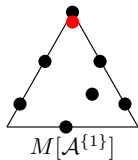
$$A_i^J = \begin{cases} A_i \cup x, & \text{if } i \in J, \\ A_i, & \text{otherwise.} \end{cases}$$

The extension  $\mathcal{A}^J$  of  $\mathcal{A}$  yields the extension  $M[\mathcal{A}^J]$  of  $M$ .

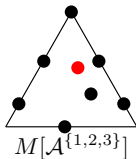
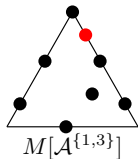
## Examples of extensions



$$M[\mathcal{A}^{\{2\}}] = M[\mathcal{A}^{\{1,2\}}]$$



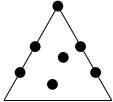
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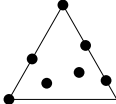
$$M[\mathcal{A}^{\{2,3\}}] = M[\mathcal{A}^{\{1,2,3\}}]$$

New results on the number of extensions of  $M$  from one presentation

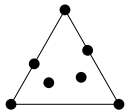
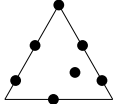
5 extensions



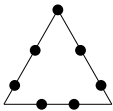
6 extensions



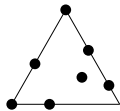
5 extensions



8 extensions



8 extensions



8 extensions

Let  $\mathcal{T}_{\mathcal{A}}$  be the set of transversal extensions of  $M$  obtained by extending  $\mathcal{A}$ .

### Theorem

Let  $r = r(M)$ .

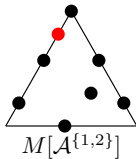
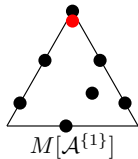
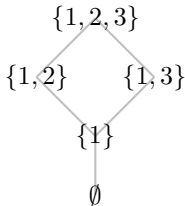
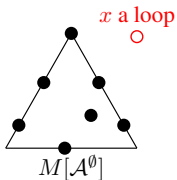
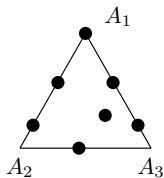
$\mathcal{A}$  is minimal iff  $|\mathcal{T}_{\mathcal{A}}| = 2^r$ ;  
otherwise  $|\mathcal{T}_{\mathcal{A}}| \leq \frac{3}{4} \cdot 2^r$ .

If  $\mathcal{A}$  neither is minimal nor covers a minimal presentation, then

$$|\mathcal{T}_{\mathcal{A}}| \leq \frac{5}{8} \cdot 2^r.$$

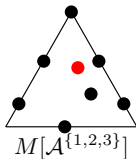
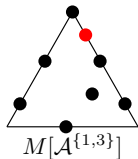
(2014)

## A lattice



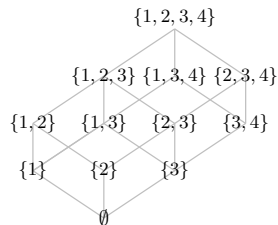
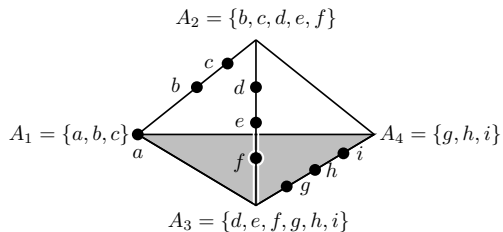
$$M[\mathcal{A}^{\{2\}}] = M[\mathcal{A}^{\{1,2\}}]$$

$$M[\mathcal{A}^{\{3\}}] = M[\mathcal{A}^{\{1,3\}}]$$



$$M[\mathcal{A}^{\{2,3\}}] = M[\mathcal{A}^{\{1,2,3\}}]$$

*A second example of such a lattice*



## Closure operators

A **closure operator** on a set  $S$  is a map  $\sigma : 2^S \rightarrow 2^S$  for which

1.  $X \subseteq \sigma(X)$  for all  $X \subseteq S$ ,
2. if  $X \subseteq Y \subseteq S$ , then  $\sigma(X) \subseteq \sigma(Y)$ , and
3.  $\sigma(\sigma(X)) = \sigma(X)$  for all  $X \subseteq S$ .

A  **$\sigma$ -closed set** is a subset  $X$  of  $S$  with  $\sigma(X) = X$ .

The  $\sigma$ -closed sets, ordered by inclusion, form a lattice;  
the lattice operations are  $X \vee Y = \sigma(X \cup Y)$  and  $X \wedge Y = X \cap Y$ .



*A closure operator related to extensions*

Let  $\mathcal{A} = (A_i : i \in [r])$  be a presentation of  $M$ , where  $r = r(M)$ .

**Lemma**

For  $e \in E(M) - A_i$ ,  $(A_1, A_2, \dots, A_{i-1}, A_i \cup e, A_{i+1}, \dots, A_r)$   
is a presentation of  $M$  iff  $e$  is a coloop of  $M \setminus A_i$ . *(Bondy & Welsh, 1971)*

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So, for  $J \subseteq [r]$ , there is a maximum  $K \subseteq [r]$  with  $M[\mathcal{A}^J] = M[\mathcal{A}^K]$ .

Define  $\sigma_{\mathcal{A}} : 2^{[r]} \rightarrow 2^{[r]}$  by  $\sigma_{\mathcal{A}}(J) = K$ .

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### Theorem

The map  $\sigma_{\mathcal{A}}$  is a closure operator on  $[r]$ . The join of  $I$  and  $J$  in the lattice  $L_{\mathcal{A}}$  of  $\sigma_{\mathcal{A}}$ -closed sets is  $I \cup J$ , so  $L_{\mathcal{A}}$  is distributive.

If  $\mathcal{B} = (B_i : i \in [r])$  is a presentation of  $M$  with  $A_i \subseteq B_i$  for all  $i$ , then  $L_{\mathcal{B}}$  is a sublattice of  $L_{\mathcal{A}}$ . (2014)

*$L_{\mathcal{A}}$  and  $\mathcal{T}_{\mathcal{A}}$  are isomorphic lattices*

Order  $\mathcal{T}_{\mathcal{A}}$ , the set of extensions of  $M$  obtained by extending  $\mathcal{A}$ , by the weak order.

For matroids  $M$  and  $N$  on  $E$ , we have  $M \leq_w N$  in the weak order if  $r_M(X) \leq r_N(X)$  for all  $X \subseteq E$ ;  
equivalently, every independent set of  $M$  is independent in  $N$ .

### Theorem

*The map  $J \mapsto M[\mathcal{A}^J]$  is an isomorphism from  $L_{\mathcal{A}}$  onto  $\mathcal{T}_{\mathcal{A}}$ , so  $\mathcal{T}_{\mathcal{A}}$  is a distributive lattice.*

(2014)

*Minimal presentations are most important for extensions*

**Theorem**

*If  $\mathcal{A} \leq \mathcal{B}$ , then  $\mathcal{T}_{\mathcal{B}} \subseteq \mathcal{T}_{\mathcal{A}}$ .*

*If  $A_i \subseteq B_i$  for all  $i \in [r]$ , and if  $J \subseteq [r]$ , then  $M[\mathcal{A}^{\sigma_{\mathcal{B}}(J)}] = M[\mathcal{B}^J]$ .*

*(2014)*

**Corollary**

*For any single-element transversal extension  $M'$  of  $M$ , there is a minimal presentation  $\mathcal{A}$  of  $M$  and a set  $J$  with  $M' = M[\mathcal{A}^J]$ . (2013)*

## *A characterization of minimal presentations*

### Theorem

Let  $r = r(M)$ . A presentation  $\mathcal{A}$  of  $M$  is minimal iff  $|\mathcal{T}_{\mathcal{A}}| = 2^r$ .

(2013)

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If  $\mathcal{A}$  is not minimal, then  $r(M \setminus A_i) < r - 1$  for some  $A_i \in \mathcal{A}$ .

Set  $J = [r] - i$ .

So  $x$  is a coloop of  $M[\mathcal{A}^J] \setminus A_i$ , so  $M[\mathcal{A}^J] = M[\mathcal{A}^{J \cup i}]$ , so  $|\mathcal{T}_{\mathcal{A}}| < 2^r$ .



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If  $\mathcal{A}$  is minimal and  $i \in [r]$ , then  $r(M \setminus A_i) = r - 1$ .

If  $J \subseteq [r]$  and  $i \notin J$ , then  $x$  is not a coloop of  $M[\mathcal{A}^J] \setminus A_i$ ;  
since  $M[\mathcal{A}^J] \setminus A_i$  extends  $M \setminus A_i$ , so  $J$  is closed. Thus,  $L_{\mathcal{A}} = 2^{[r]}$ .

*Non-minimal presentations lose at least a quarter of the extensions*

## Theorem

*If  $\mathcal{A}$  is not minimal, then  $|\mathcal{T}_{\mathcal{A}}| = |L_{\mathcal{A}}| \leq \frac{3}{4} \cdot 2^r$ . (2014)*

This follows by finding the maximal proper sublattices of  $2^{[r]}$  that contain  $\emptyset$  and  $[r]$ , which are

$$L_{ij} = \{X \subseteq [r] : X \cap \{i, j\} \neq \{i\}\}$$

for any distinct  $i, j$  in  $[r]$ ; also,  $|L_{ij}| = \frac{3}{4} \cdot 2^r$ .

*Non-minimal presentations lose at least a quarter of the extensions*

## Theorem

*If  $\mathcal{A}$  is not minimal, then  $|\mathcal{T}_{\mathcal{A}}| = |L_{\mathcal{A}}| \leq \frac{3}{4} \cdot 2^r$ . (2014)*

This follows by finding the maximal proper sublattices of  $2^{[r]}$  that contain  $\emptyset$  and  $[r]$ , which are

$$L_{ij} = \{X \subseteq [r] : X \cap \{i, j\} \neq \{i\}\}$$

for any distinct  $i, j$  in  $[r]$ ; also,  $|L_{ij}| = \frac{3}{4} \cdot 2^r$ .

Finding the next-largest sublattices (two types), and showing that the number of extensions goes down, gives:

## Theorem

*If  $\mathcal{A}$  neither is minimal nor covers a minimal presentation, then  $|\mathcal{T}_{\mathcal{A}}| = |L_{\mathcal{A}}| \leq \frac{5}{8} \cdot 2^r$ . (2014)*

All sublattices of  $2^{[r]}$  can be  $L_{\mathcal{A}}$

### Theorem

Let  $L$  be a sublattice of  $2^{[r]}$  (so  $I \cap J \in L$  and  $I \cup J \in L$  for all  $I, J \in L$ ) with both  $\emptyset$  and  $[r]$  in  $L$ .

There is a transversal matroid  $M$  of rank  $r$  with  $L = L_{\mathcal{A}}$ , where  $\mathcal{A}$  is the maximal presentation of  $M$ .

For each integer  $n \geq r$ , there is a presentation  $\mathcal{A}$  of the uniform matroid  $U_{r,n}$  on  $[n]$  with  $L = L_{\mathcal{A}}$ . (2014)

*How many extensions can be common to two presentations?*

### Theorem

*If  $\mathcal{A}$  and  $\mathcal{B}$  are different presentations of  $M$ , then*

$$2 \leq |\mathcal{T}_{\mathcal{A}} \cap \mathcal{T}_{\mathcal{B}}| \leq \frac{3}{4} \cdot 2^r.$$

(2014)

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### Theorem

The set  $L_{\mathcal{A},\mathcal{B}} = \{J \in L_{\mathcal{A}} : M[\mathcal{A}^J] = M[\mathcal{B}^K] \text{ for some } K \in L_{\mathcal{B}}\}$   
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Closure under unions follows from:

### Theorem

*For all subsets  $J$  and  $K$  of  $[r]$ , the join of  $M[\mathcal{A}^J]$  and  $M[\mathcal{A}^K]$  in the lattice of all extensions of  $M$  is transversal and is  $M[\mathcal{A}^{J \cup K}]$ .* (2013)

The corresponding statement for meets is false.

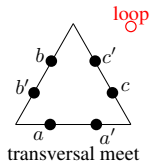
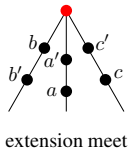
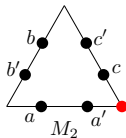
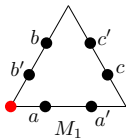
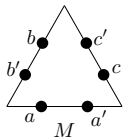
## A question

Under the weak order, the set of all single-element extensions of a matroid  $M$  is a lattice.

### Question

Is the set of all single-element *transversal* extensions of  $M$ , ordered by the weak order, a lattice?

As we saw,  $\mathcal{T}_A$  is a lattice, but this question concerns the union of  $\mathcal{T}_A$  over all presentations  $A$  of  $M$ .





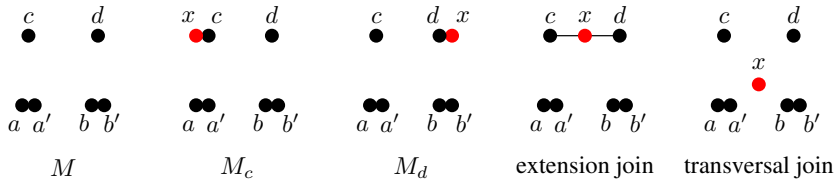
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Thank you for listening.