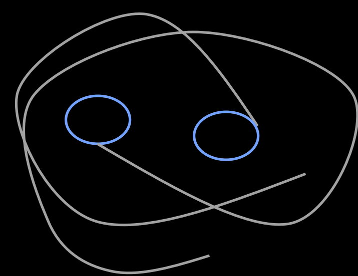
A Periodic Table for Black Hole Orbits



Ultimate observation

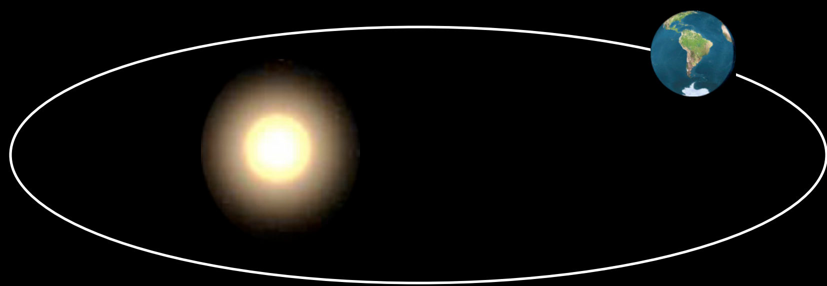
Gabe Perez-Giz

NSF

JL

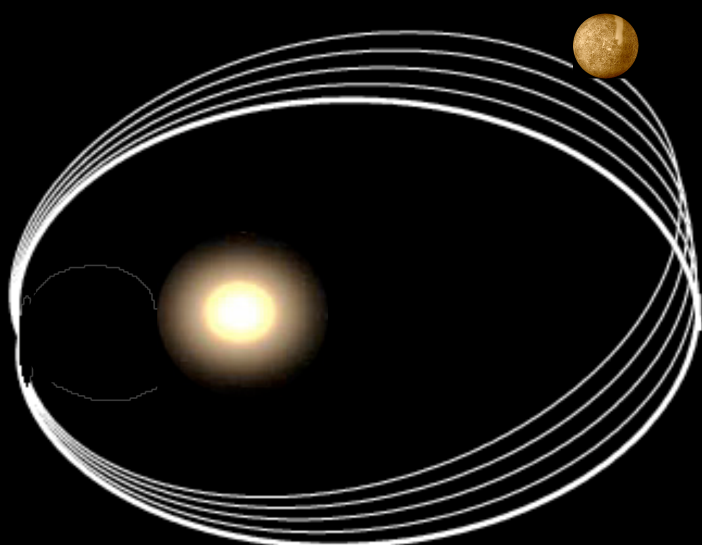
Janna Levin

2body problem is insoluble



Newtonian elliptical orbits

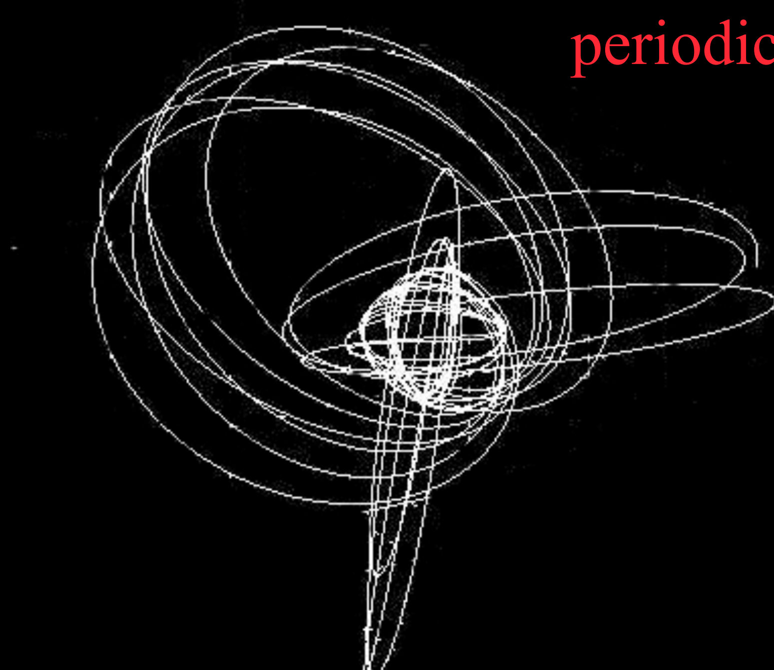
regular



Relativistic precession

test particle regular

In the strong-field regime periodic table of BH orbits



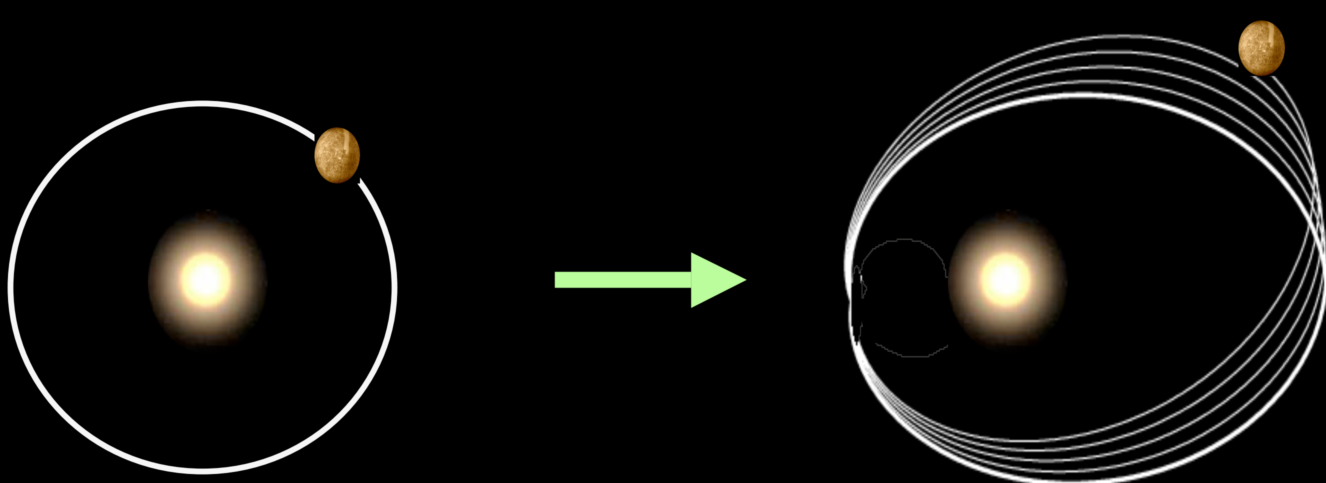
Relativistic 2body with spins - chaotic

Periodic Orbits are Special

Some orbits are more equal than others

For example, Circular Orbits are special

1. Easy to handle
2. Some orbits look like small perturbations to circular ones



Circular orbit

Nearby low-eccentricity precessing orbit

Periodic Orbits are Special

Some orbits are more equal than others

For example, Circular Orbits are special

1. Easy to handle
2. Some orbits look like small perturbations to circular ones

Yet, most orbits are not near circular orbits
circulars cannot encode entire BH dynamics

Periodic Orbits are even more special

1. Easy to handle
2. ***All*** orbits look like small perturbations to periodic ones

all orbits are near periodic orbits
periodics ***can*** encode entire BH dynamics

Periodic Orbits Encode Entire Black Hole dynamics

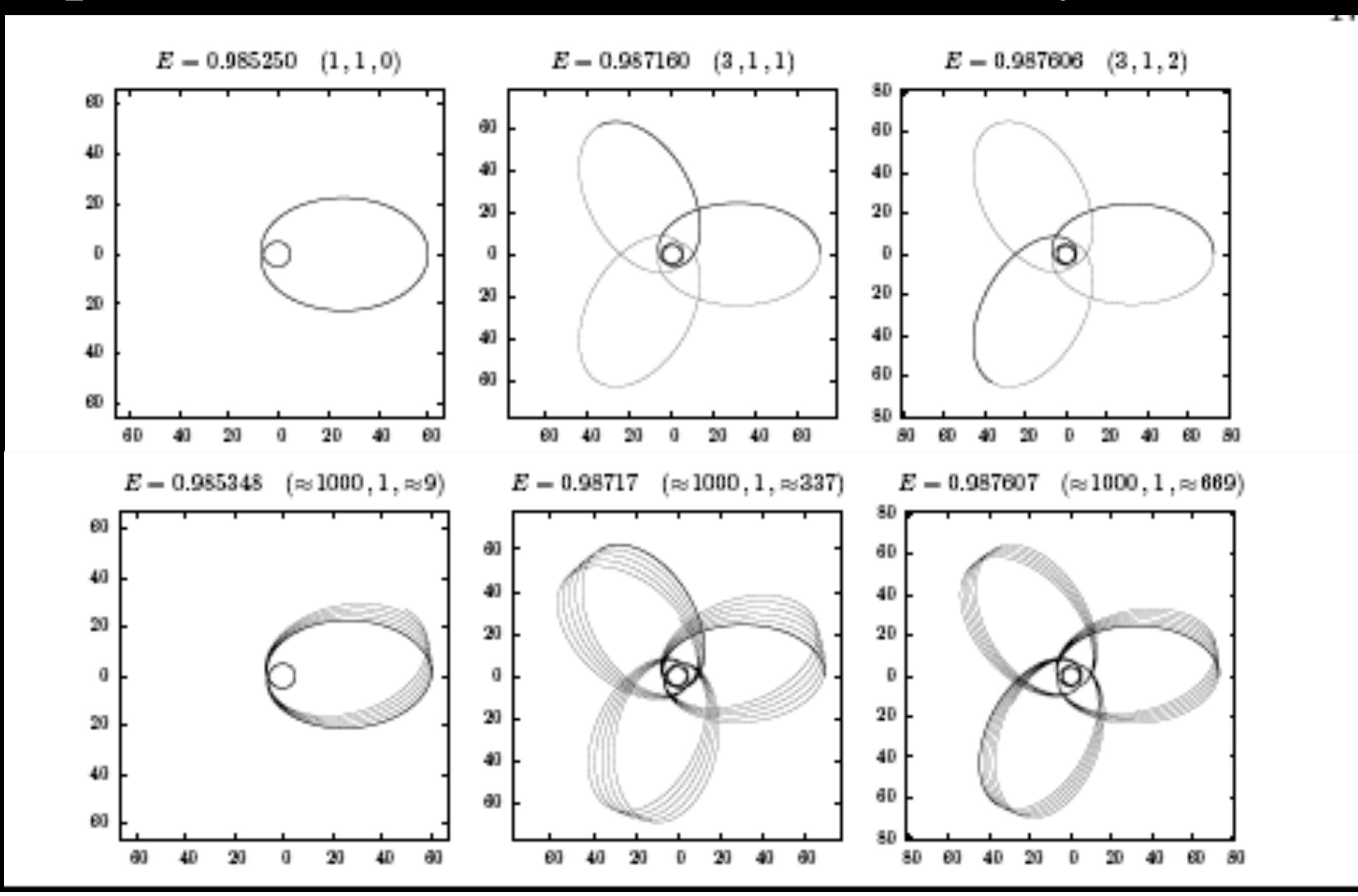
Periodic Orbits are even more special

1. Easy to handle

2. ***All*** orbits look like small perturbations to periodic ones

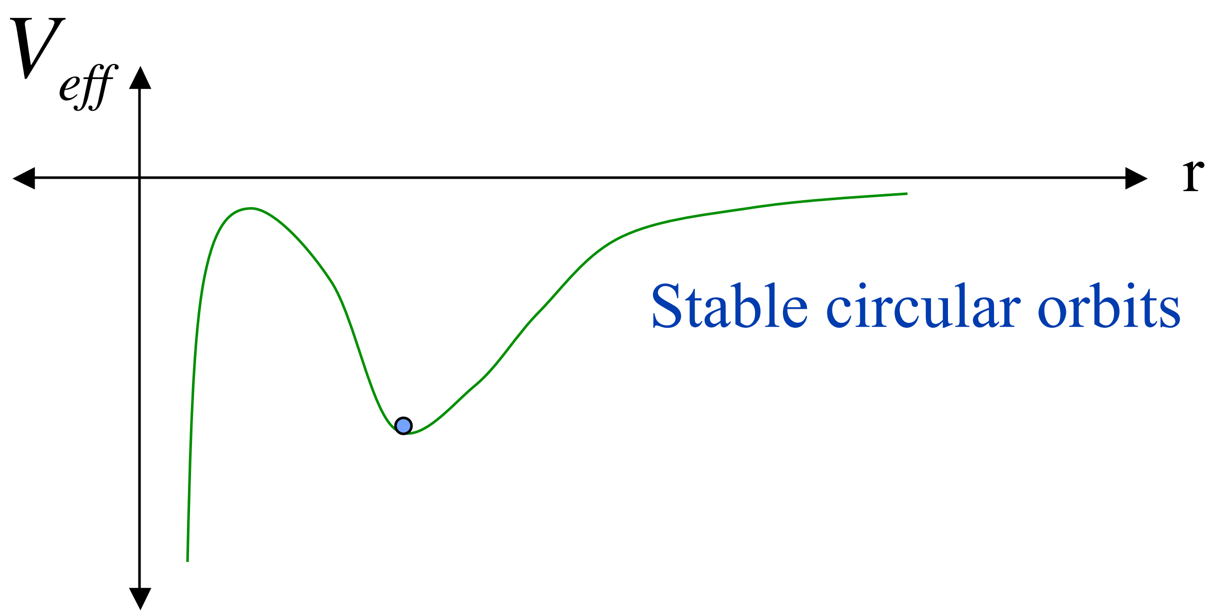
all orbits are near periodic orbits

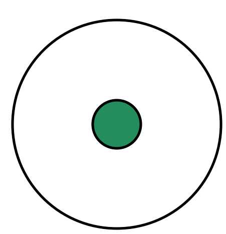
periodics ***can*** encode entire BH dynamics



Schwarzschild reduces to motion in one dimension

$$\frac{1}{2}\dot{r}^2 + \underbrace{\frac{1}{2}\left(1-\frac{2M}{r}\right)\left(\frac{L^2}{r^2}+1\right)}_{V_{eff}} = \frac{E^2}{2}$$





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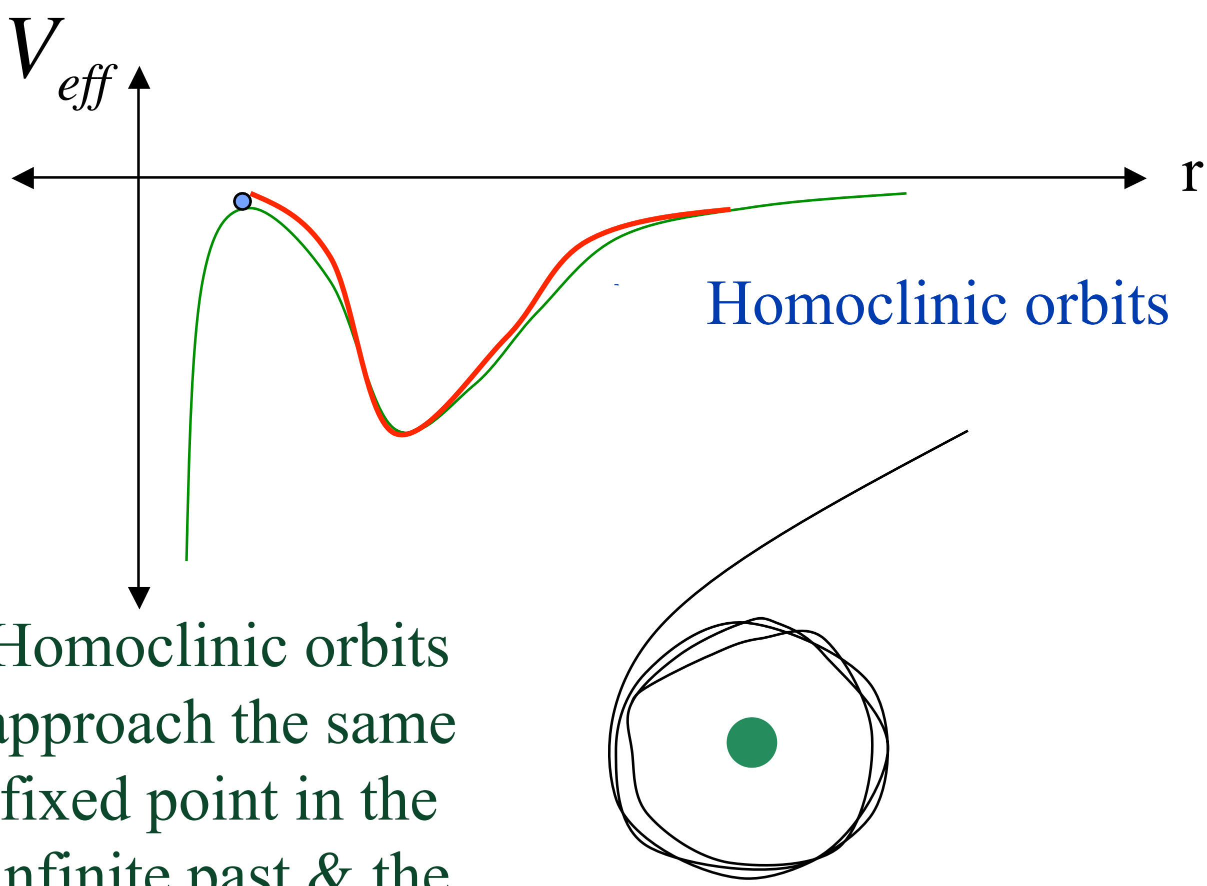
$$\frac{1}{2}\dot{r}^2 + \underbrace{\frac{1}{2}\left(1 - \frac{2M}{r}\right)\left(\frac{L^2}{r^2} + 1\right)}_{V_{eff}} = \frac{E^2}{2}$$



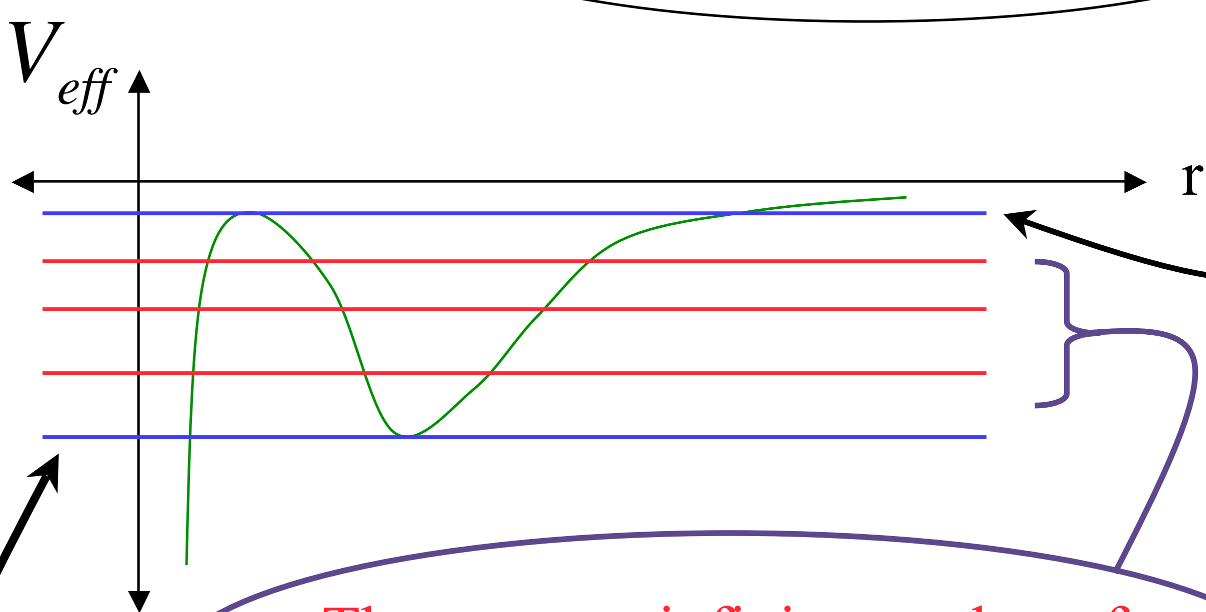
Homoclinic orbits approach the same fixed point in the infinite past & the infinite future: mark intersection of stable and unstable manifolds

A Periodic Table of Black Hole Orbits

Energy Level Diagram

$$\frac{1}{2}\dot{r}^2 + V_{eff} = \frac{E^2}{2}$$

Highest energy orbit

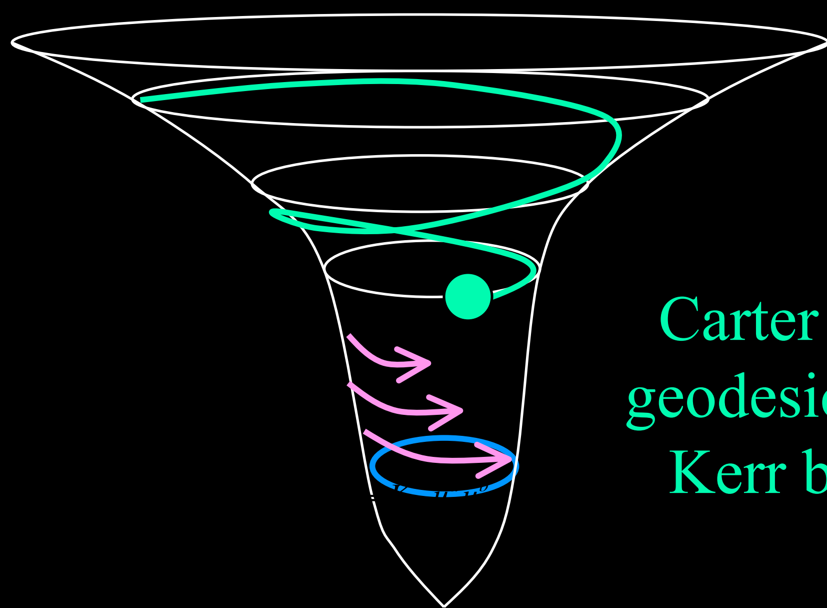


There are an infinite number of periodic orbits

Lowest energy orbit is the stable circular orbit

Begin Our Story Here

We know orbits around a spinning black hole



Carter found the geodesics around a Kerr black hole

Timelike Killing field, an axial Killing field, and orbits still timelike

E,L,μ,Q

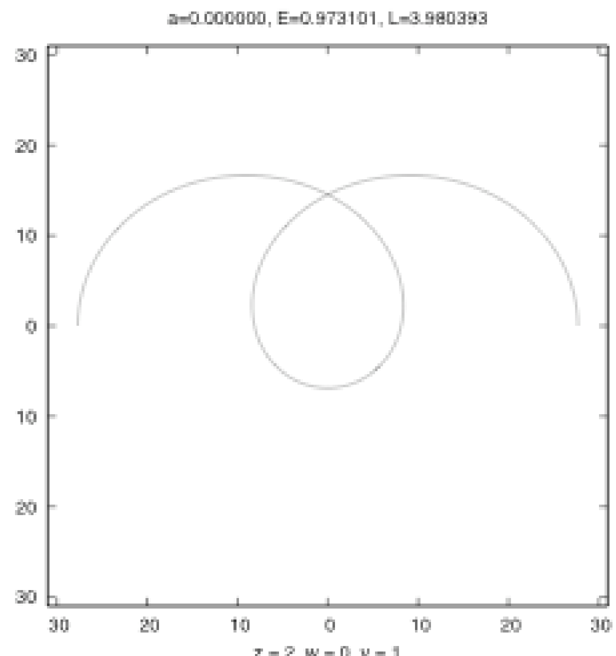
Where Q is the Carter constant generated by a Killing tensor

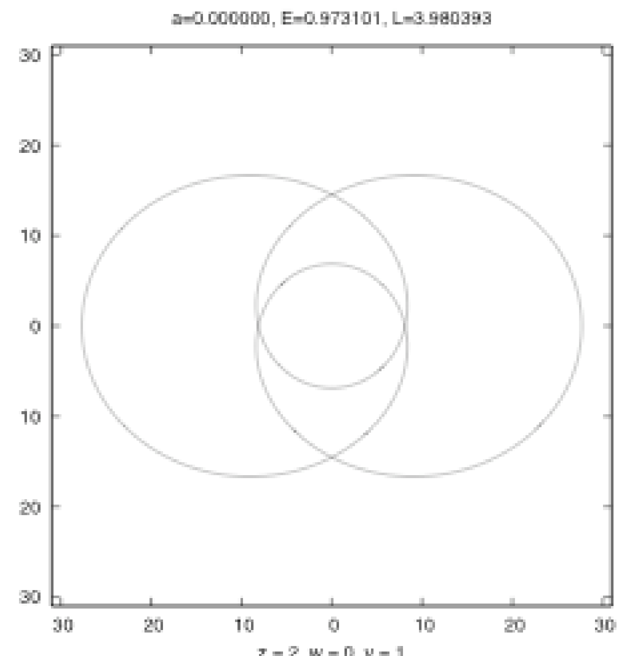
Therefore motion lies on tori and integrable

Taxonomy of Periodic Orbits

Kerr
Not Chaotic
Equatorial
Bound, Non-plunging

z = zooms
(leaves)



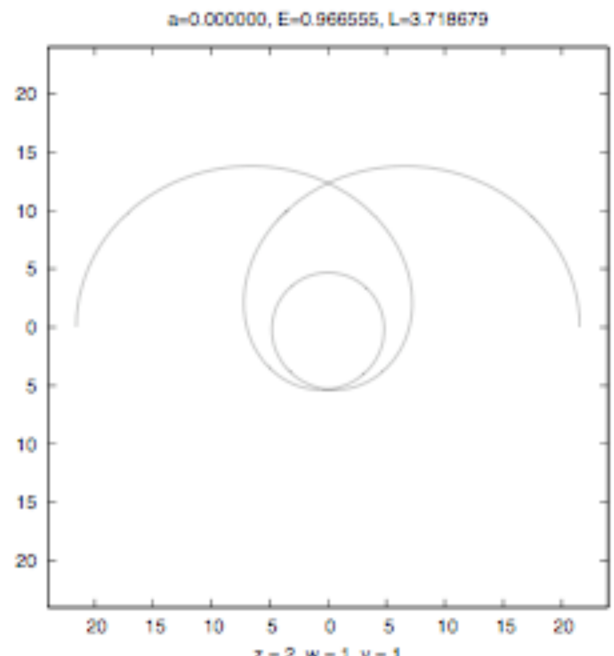


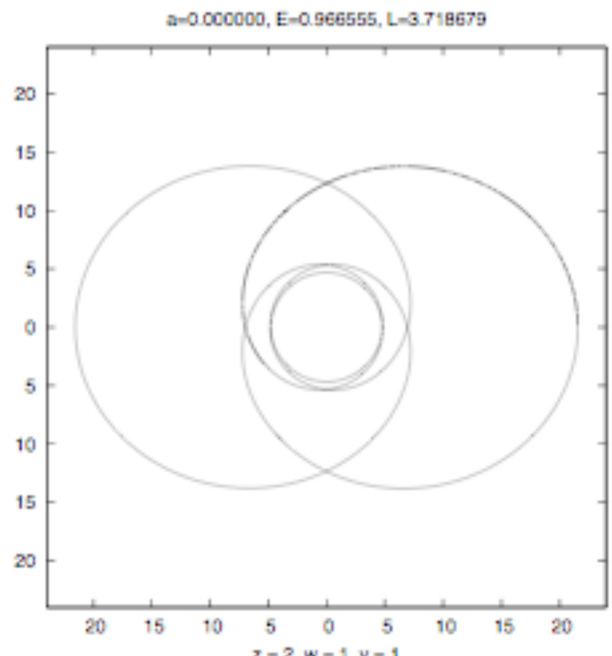
$$\Delta\phi_r = 3\pi$$

$$\Delta\phi = 3\pi z = 6\pi$$

Taxonomy of Periodic Orbits

$z = 2$
w = whirls

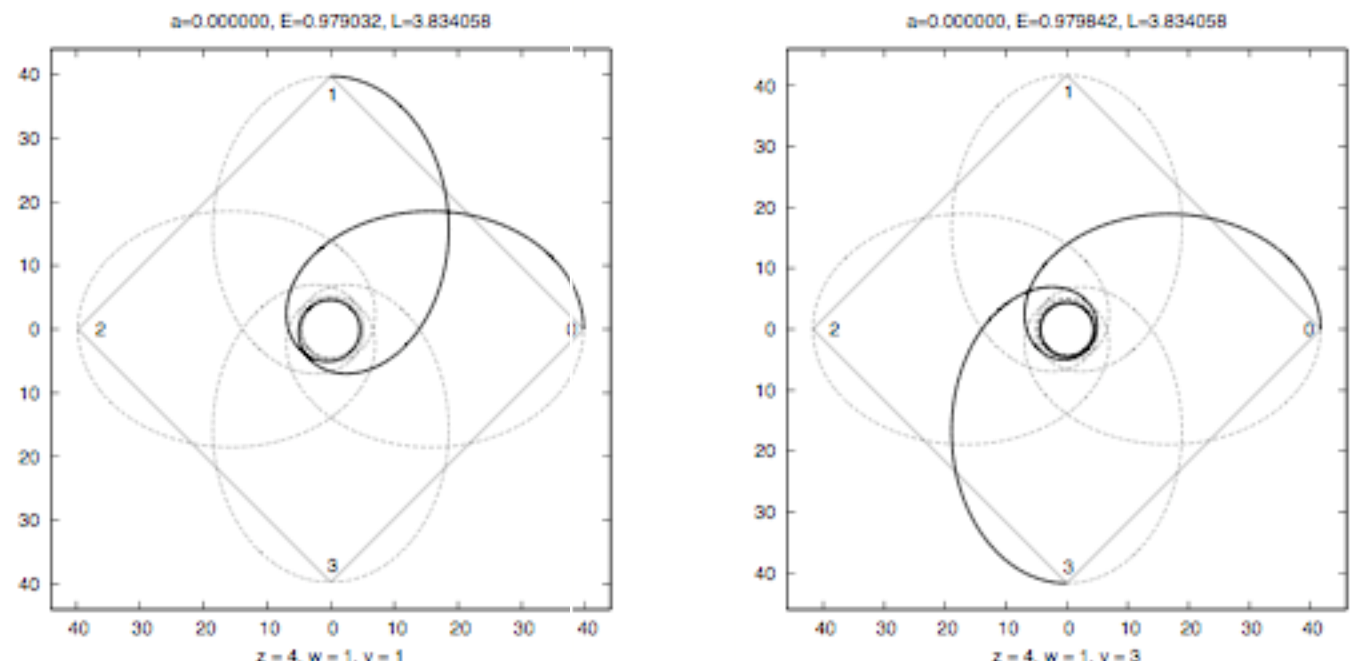




$$\Delta\phi_r = 5\pi \qquad \Delta\phi = 5\pi z = 10\pi$$

Taxonomy of Periodic Orbits

z = 4
w = 1
v = vertices

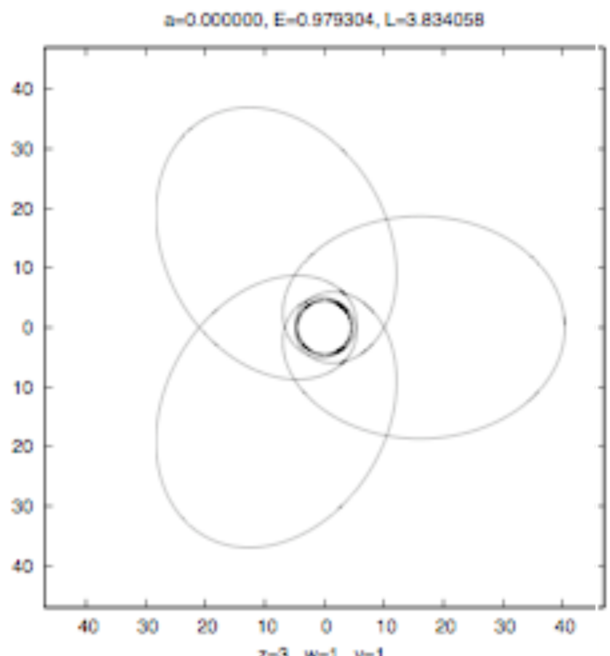


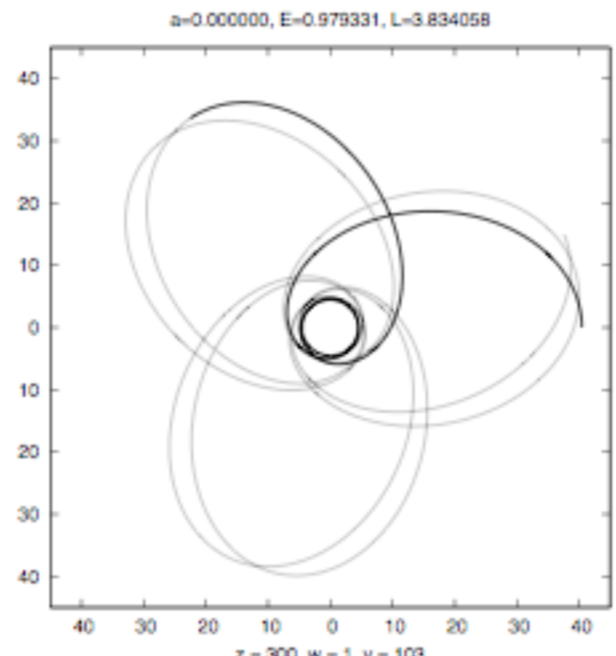
$$\Delta\phi_r = 2\pi\left(1 + w + \frac{v}{z}\right)$$

$4\pi+\pi/2$ $4\pi+3\pi/2$

High z orbits as precession of low z orbits

$(z = 3, w = 1, v = 1)$





$(z = 300, w = 1, v = 103)$

Finding periodic orbits with rational numbers

$$(z, w, v)$$

$$\Delta\varphi_r = 2\pi\left(1 + w + \frac{v}{z}\right)$$

Writing this another way, every periodic orbit is associated with a rational number:

$$\boxed{q = w + \frac{v}{z} = \frac{\Delta\varphi_r}{2\pi} - 1}$$

We can find all of the periodic orbits by integrating the Kerr equations subject to the condition that the accumulated angle is fixed by the rational number *q*

$$\Delta\varphi_r = 2\int_{t(r_p)}^{t(r_a)} \frac{d\varphi}{dt}\,dt$$

Another derivation of *q*

Every equatorial orbit has two fundamental frequencies

Radial frequency; T_r is the radial period

$$\omega_r = \frac{2\pi}{T_r}$$

Angular frequency; average of $\frac{d\varphi}{dt}$ over T_r

$$\omega_\varphi = \frac{1}{T_r}\int \frac{d\varphi}{dt}dt = \frac{\Delta\phi_r}{T_r}$$

For periodic orbits, their ratio is rationally related

$$\frac{\omega_\varphi}{\omega_r} = \frac{\Delta\varphi_r}{2\pi} = 1 + w + \frac{v}{z} = 1 + q$$

ω_r and ω_φ are the same frequencies derived from an action-angle formulation of the dynamics

Circulars

Can still define this ratio for stable circulars

$$\omega_\varphi = \frac{d\varphi}{dt}$$

$\omega_r \longrightarrow$ Frequency of radial oscillations about the ciruclar

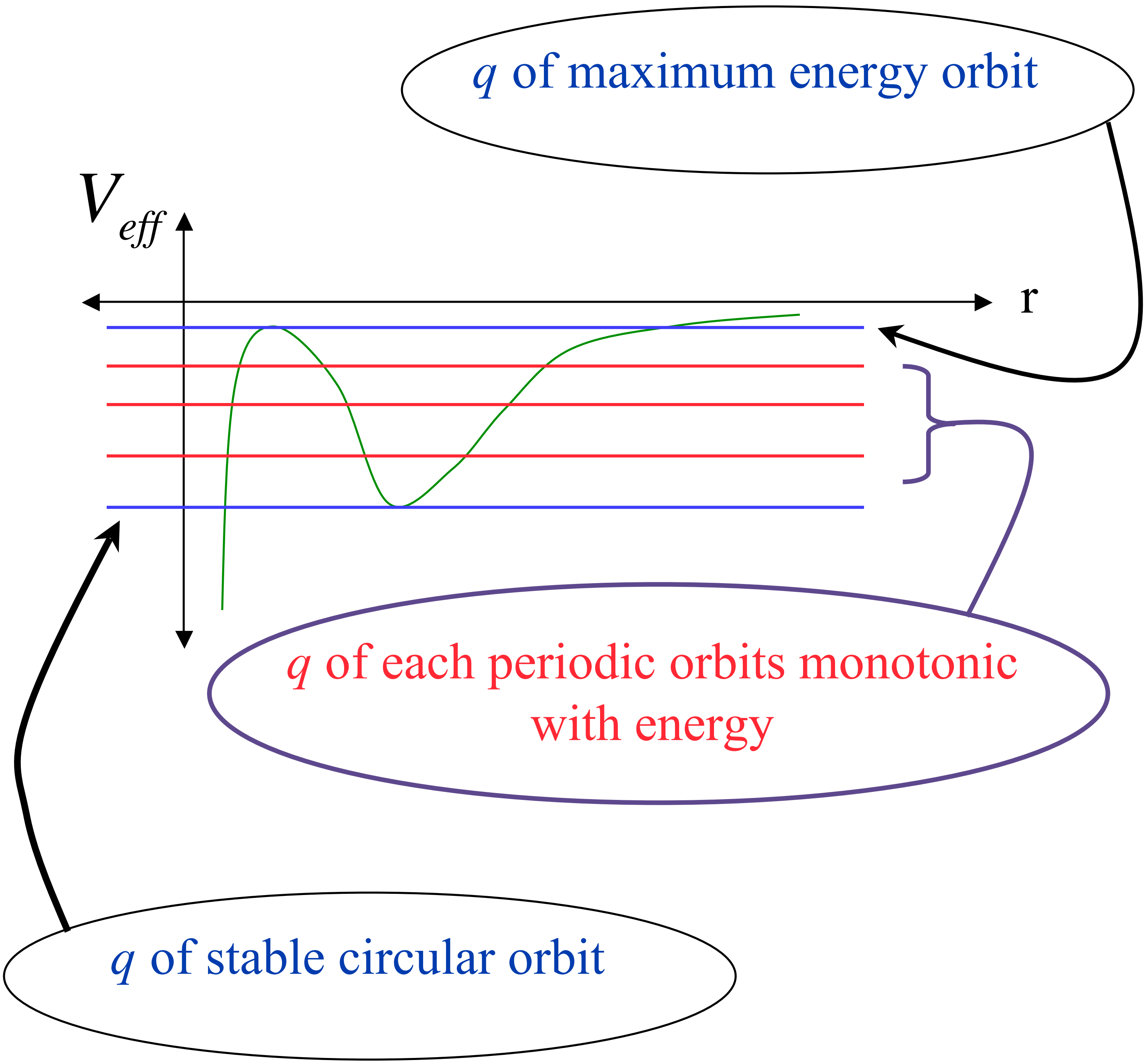
Can also define as the limit as the eccentricity *e* goes to zero

$$q_c = \lim_{e\to 0}\left(\frac{\omega_\varphi}{\omega_r} - 1\right)$$

This allows us to build a periodic table

Revisit the Energy Level Diagram

$$q_c < q < q_{max}$$

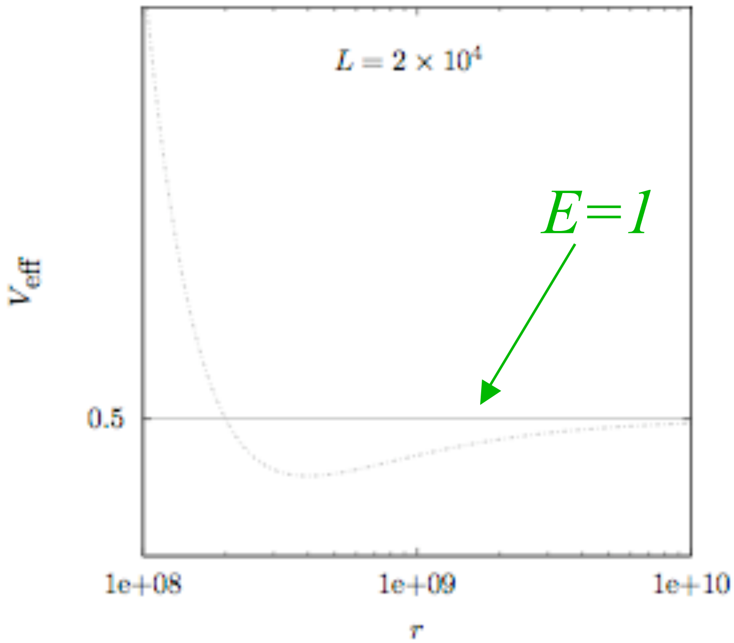


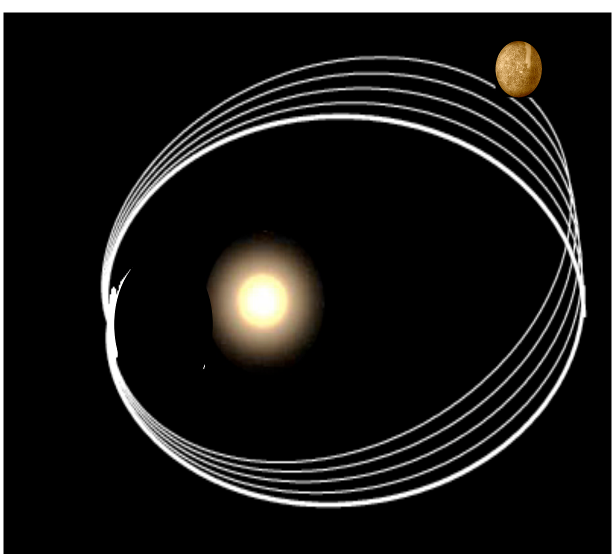
$$q_c < q < q_{max}$$

The weak-field regime

Effective potential for solar system values

$$q_{max} = q_{(E=1)}$$





$$q_c < q_{max} \ll 1$$

$$w + \frac{v}{z} \ll 1$$

$$w = 0$$

$$z \to \infty$$

$$q_c < q < q_{max}$$

The weak-field regime

Mercury, for instance, precesses 43"/century. With an orbital period of 88 days this comes to ~0.1"/orbit or $360^o/(3600^2)$

$$q_c < q_{max} \ll 1$$
$$w + \frac{v}{z} \ll 1$$
$$w = 0$$
$$z \rightarrow \infty$$

Mercury is close to an orbit with

$$z \approx (3600^2) = 1.296 \times 10^7$$

leaves and no whirls

$$(1.296 \times 10^7, 0, 1)$$

In the weak-field limit, orbits are Mercury-type precessing ellipses

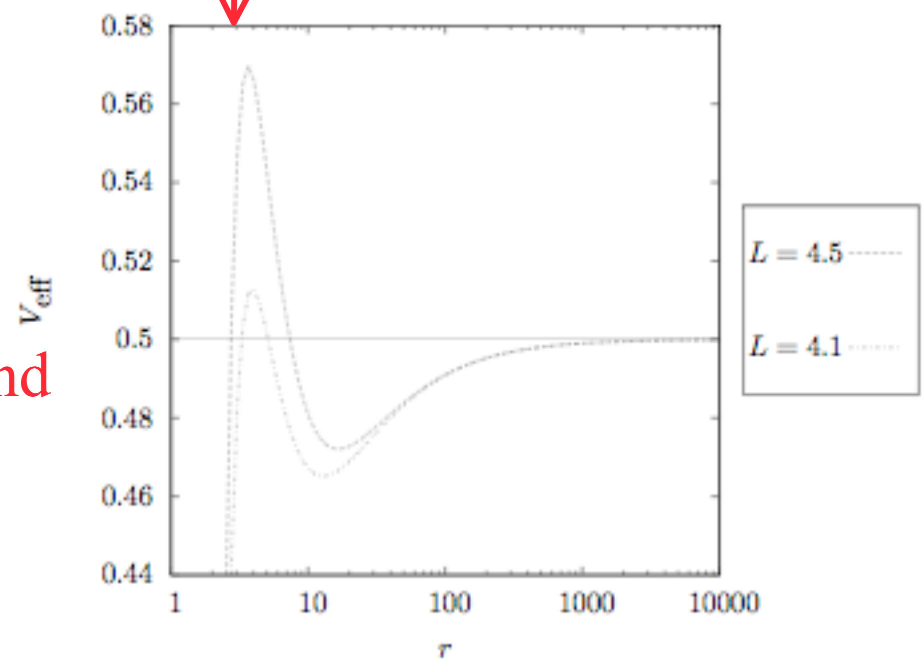
$q_c < q < \infty$

The very strong-field regime

E=

IBCO

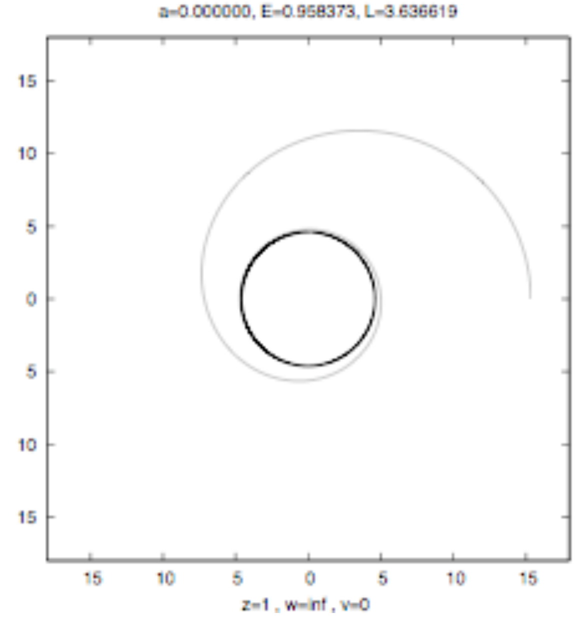
Innermost bound circular orbit



Every unstable bound circular orbit has a homoclinic orbit

Homoclinic oribts whirl an inifinite number of times

$$q_{max} = w + \frac{v}{z} = \infty$$



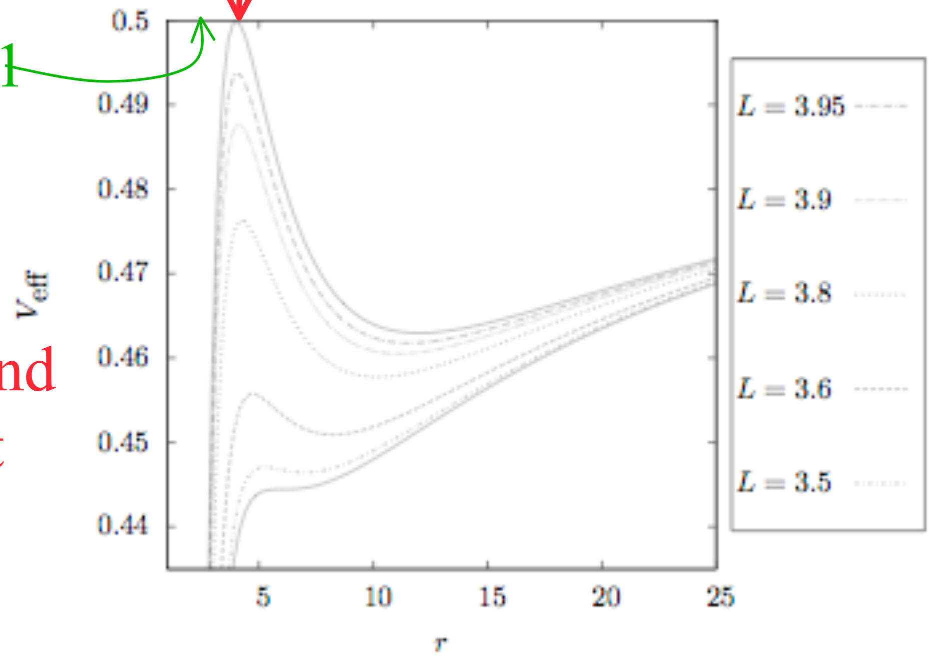
$$q_c < q < \infty$$

The very strong-field regime

IBCO

E=1

Innermost bound circular orbit



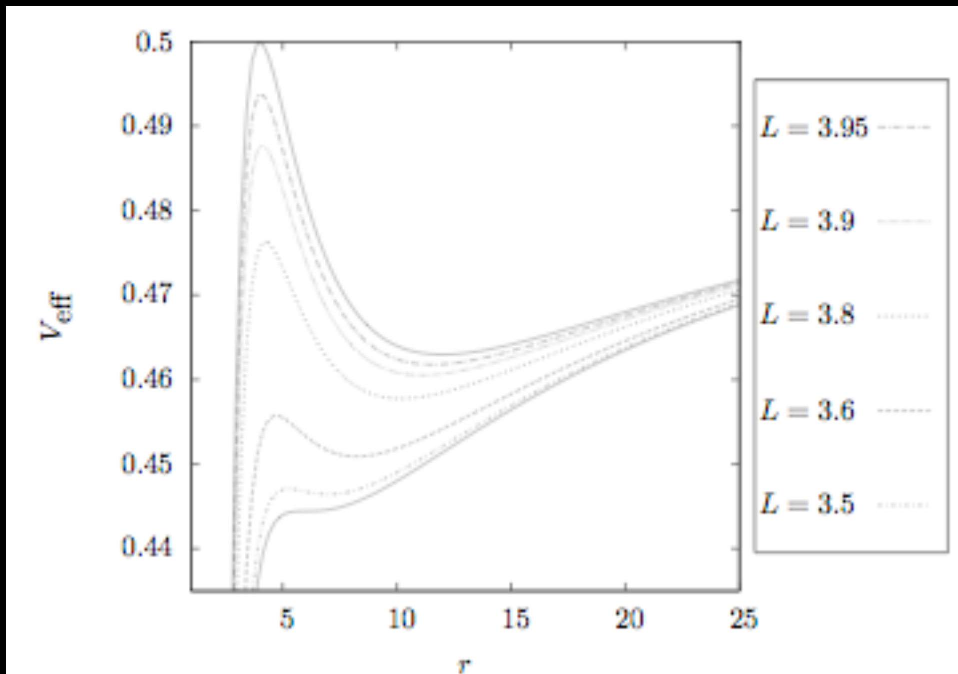
What is q_c ?

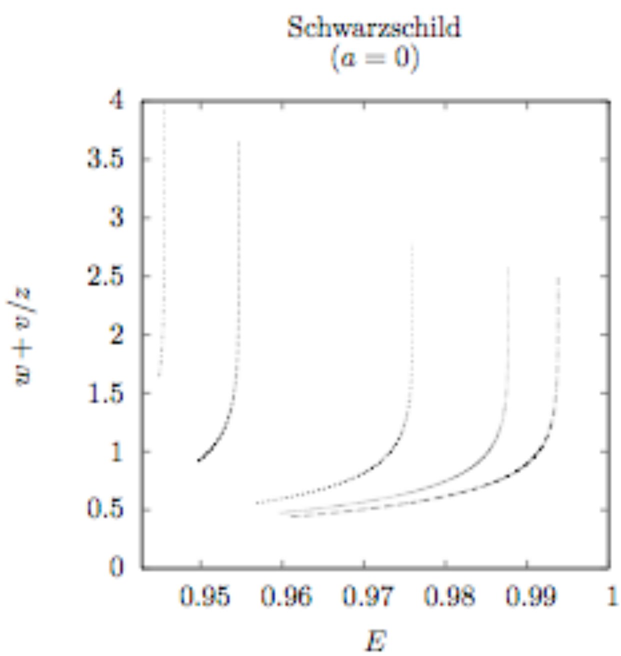
Is $q_c = 0$?

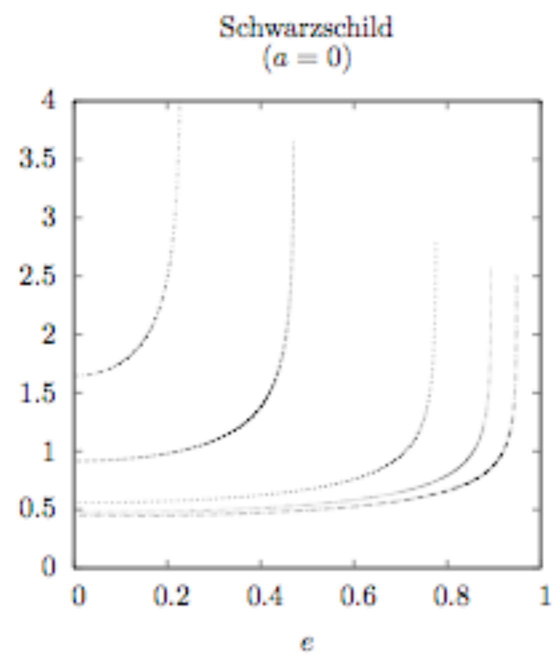
In other words, are all q, and therefore all periodic orbits, allowed?

No!

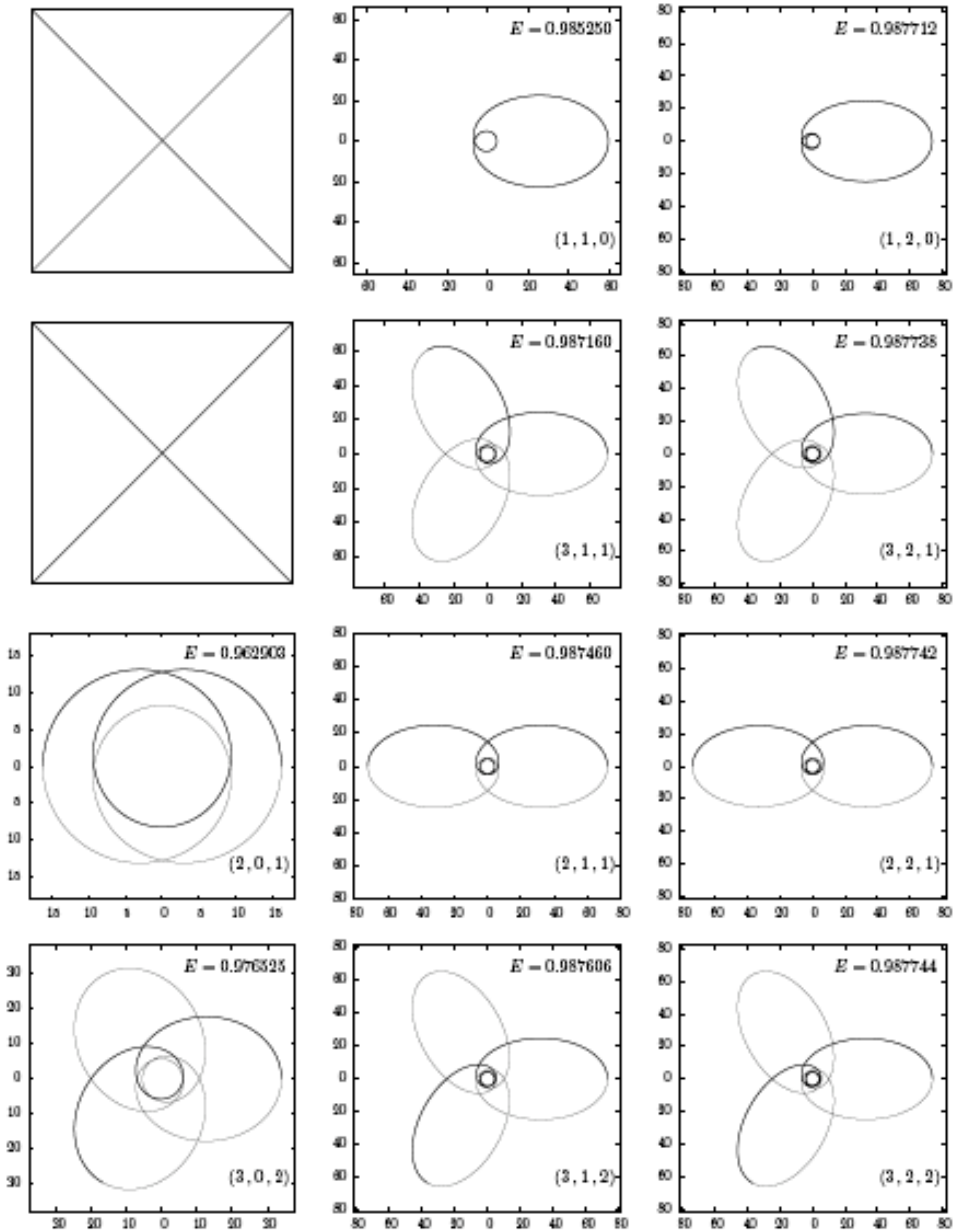
Range of Periodic Orbits



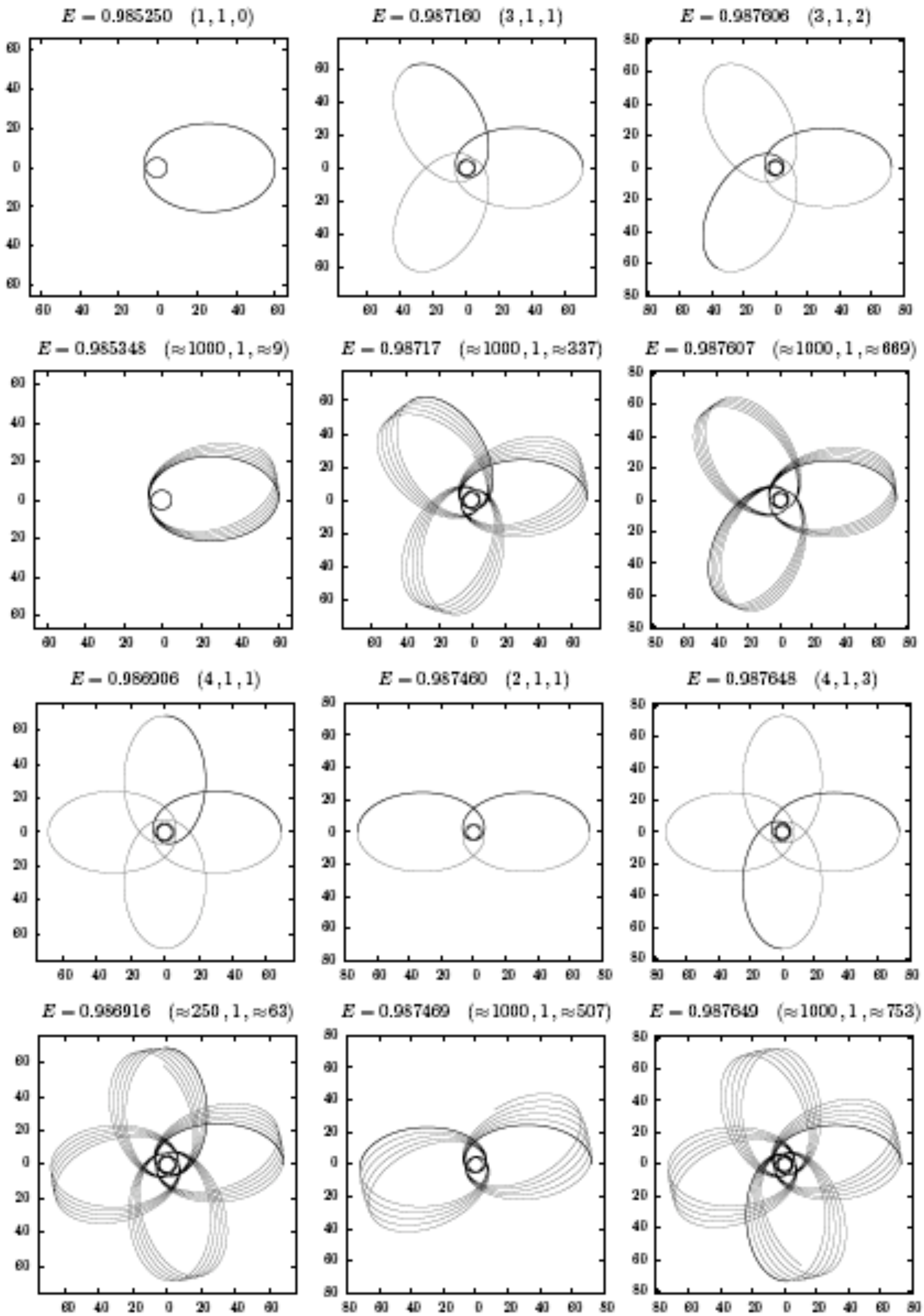




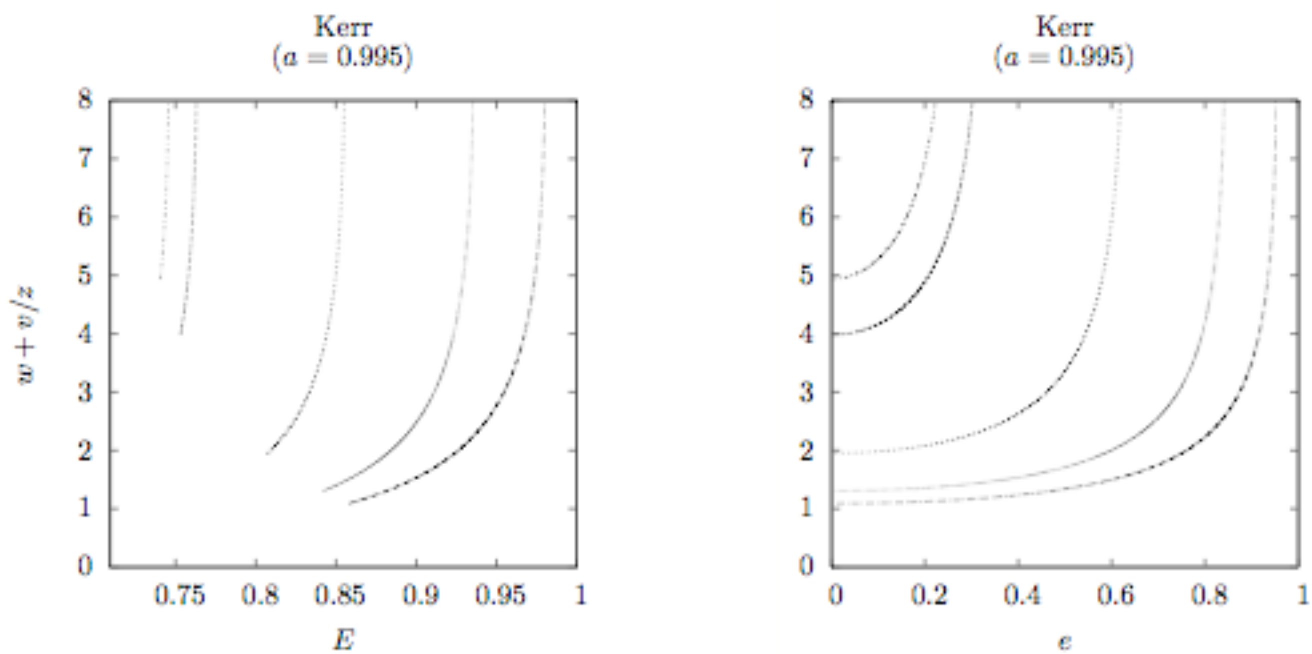
$$\left(w + \frac{v}{z}\right)_{stable\ circ} \leq \left(w + \frac{v}{z}\right) \leq \left(w + \frac{v}{z}\right)_{max}$$



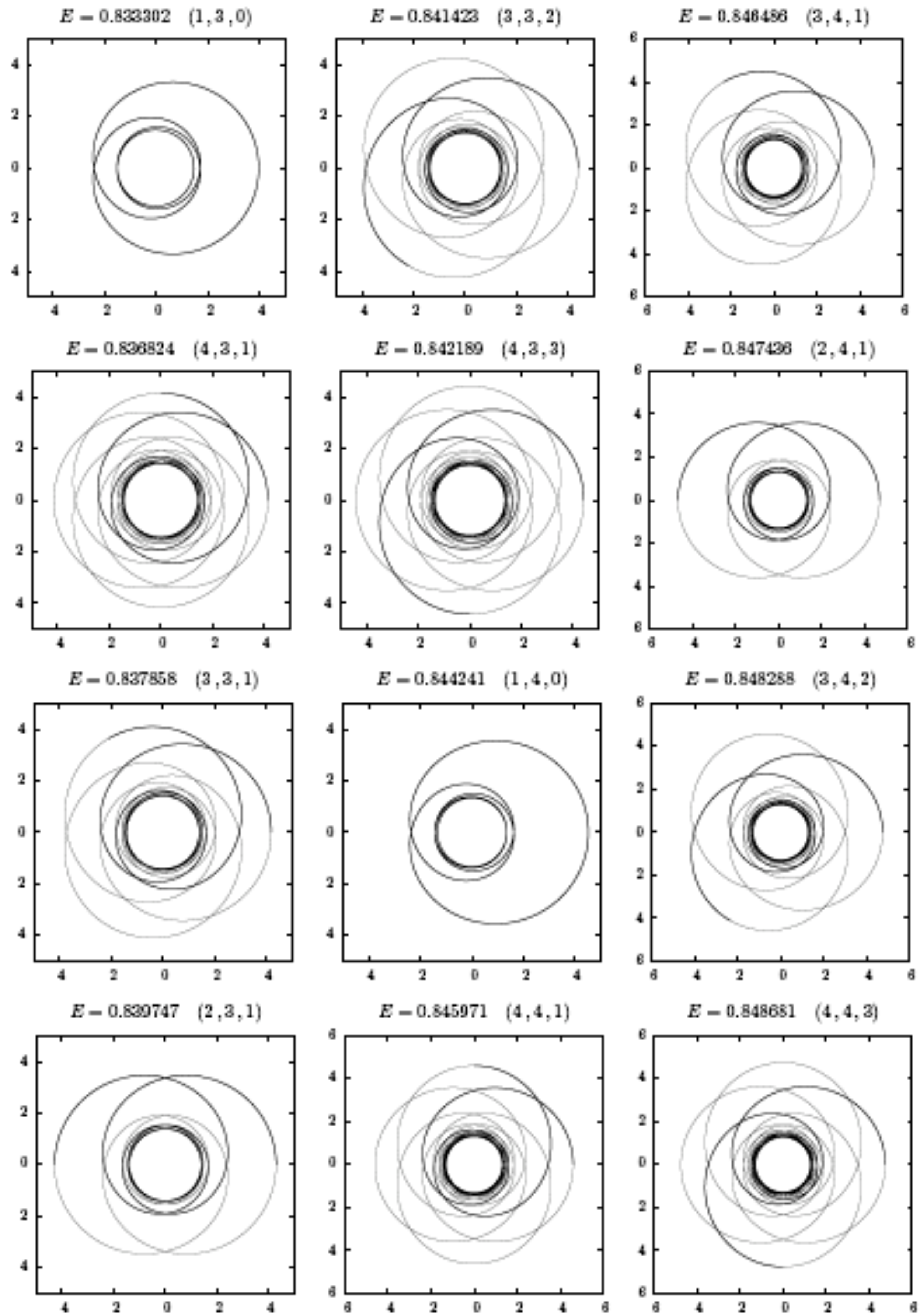
$E = 0.985250$
$(1, 1, 0)$
$E = 0.987712$
$(1, 2, 0)$
$E = 0.987160$
$(3, 1, 1)$
$E = 0.987738$
$(3, 2, 1)$
$E = 0.962903$
$(2, 0, 1)$
$E = 0.987460$
$(2, 1, 1)$
$E = 0.987742$
$(2, 2, 1)$
$E = 0.976525$
$(3, 0, 2)$
$E = 0.987606$
$(3, 1, 2)$
$E = 0.987744$
$(3, 2, 2)$


E = 0.985250 (1,1,0)
E = 0.987160 (3,1,1)
E = 0.987606 (3,1,2)
E = 0.985348 (≈1000,1,≈9)
E = 0.98717 (≈1000,1,≈337)
E = 0.987607 (≈1000,1,≈669)
E = 0.986906 (4,1,1)
E = 0.987460 (2,1,1)
E = 0.987648 (4,1,3)
E = 0.986916 (≈250,1,≈63)
E = 0.987469 (≈1000,1,≈507)
E = 0.987649 (≈1000,1,≈753)

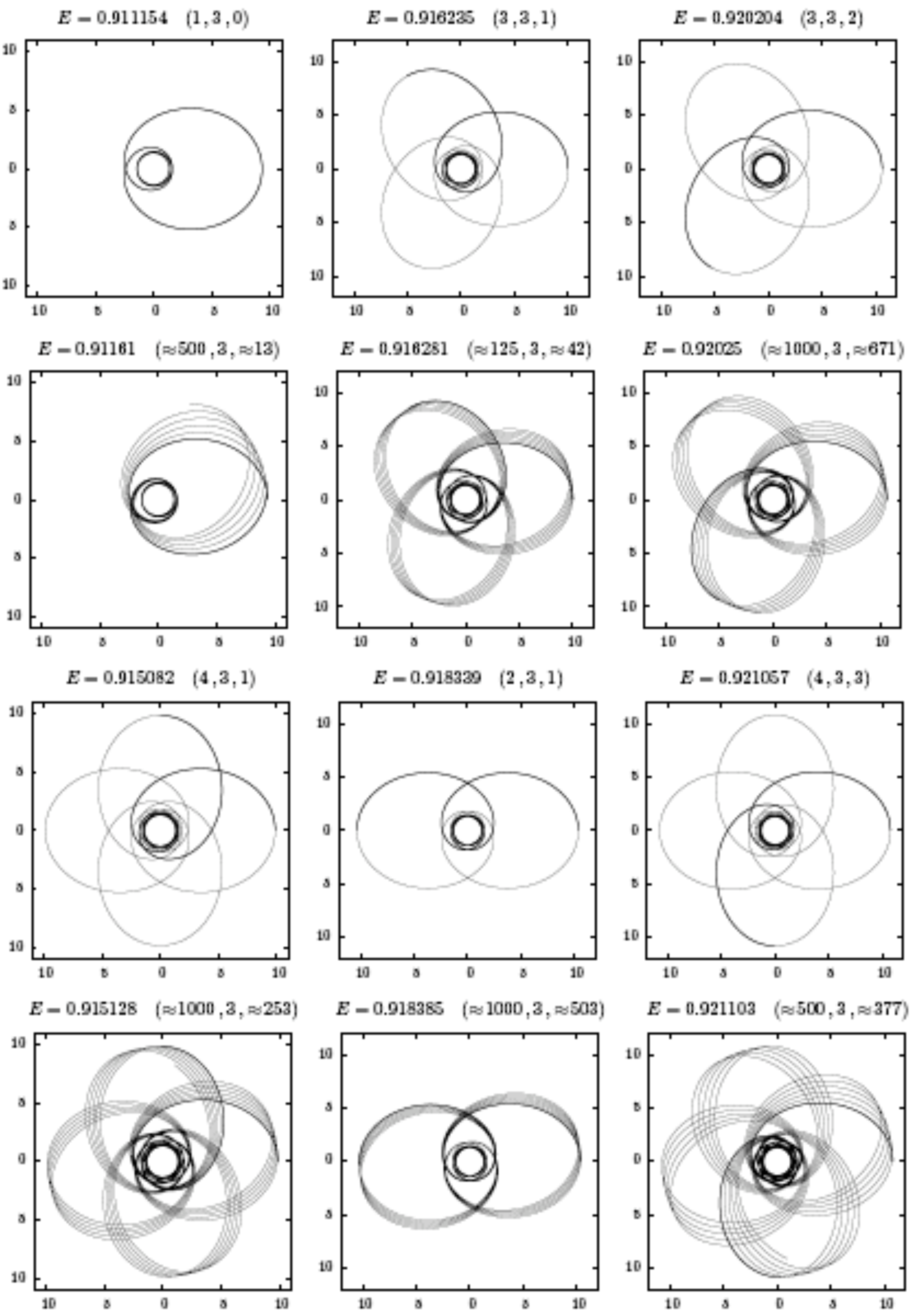
Range of Periodic Orbits
(a=0.995)



$$\left(w + \frac{v}{z}\right)_{stable\ circ} \leq \left(w + \frac{v}{z}\right) \leq \left(w + \frac{v}{z}\right)_{max}$$



$E = 0.833302 \quad (1,3,0)$
$E = 0.841423 \quad (3,3,2)$
$E = 0.846486 \quad (3,4,1)$
$E = 0.836824 \quad (4,3,1)$
$E = 0.842189 \quad (4,3,3)$
$E = 0.847436 \quad (2,4,1)$
$E = 0.837858 \quad (3,3,1)$
$E = 0.844241 \quad (1,4,0)$
$E = 0.848288 \quad (3,4,2)$
$E = 0.839747 \quad (2,3,1)$
$E = 0.845971 \quad (4,4,1)$
$E = 0.848681 \quad (4,4,3)$


$E = 0.911154 \quad (1, 3, 0)$
$E = 0.916235 \quad (3, 3, 1)$
$E = 0.920204 \quad (3, 3, 2)$
$E = 0.91161 \quad (\approx 500, 3, \approx 13)$
$E = 0.916281 \quad (\approx 125, 3, \approx 42)$
$E = 0.92025 \quad (\approx 1000, 3, \approx 671)$
$E = 0.915082 \quad (4, 3, 1)$
$E = 0.918339 \quad (2, 3, 1)$
$E = 0.921057 \quad (4, 3, 3)$
$E = 0.915128 \quad (\approx 1000, 3, \approx 253)$
$E = 0.918385 \quad (\approx 1000, 3, \approx 503)$
$E = 0.921103 \quad (\approx 500, 3, \approx 377)$

Recap of Dynamical Results

• For a given L,

$$q_c < q < q_{max}$$

• In the Newtonian limit, $q_c \sim q_{max} \sim 0$; Keplerian orbits are ellipses

• No zoom-whirl behavior in the weak-field regime

• At the ISCO, $q_c \sim q_{max} \sim \infty$

• In the very strong-field regime ($L < L_{IBCO}$)

$$q_c < q < \infty$$

• In the strong-field regime, the simple precessing ellipse familiar from planetary orbits is forbidden

• In the strong-field, all aperiodic eccentric orbits are precessions of low-leaf clovers

• As the ISCO is approached, all orbits zoom and whirl for any a

Utility of Taxonomy

Gravitational wave astronomy

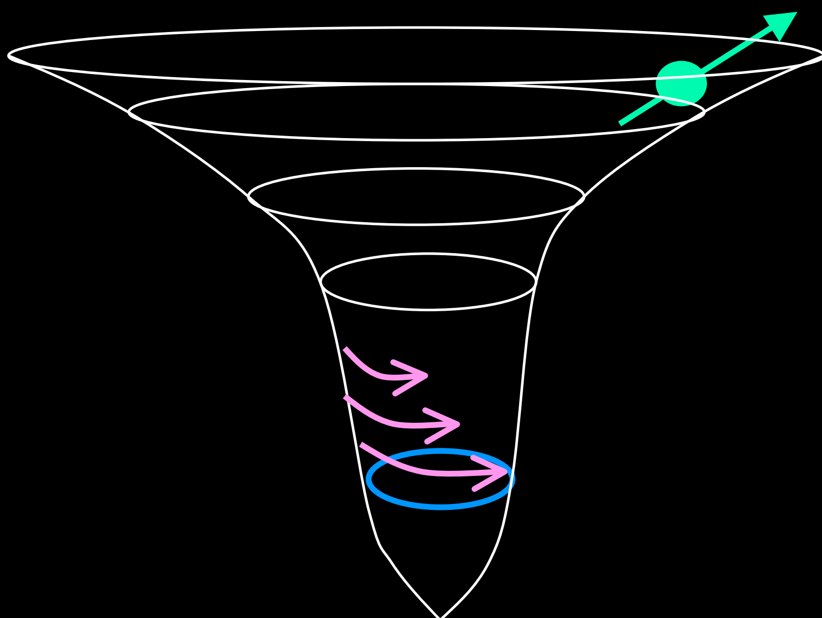
- EMRI’s on eccentric inner orbits
- All eccentric orbits will show zooms and all high spin show whirls
- Data analysis

Computation of Inspiral

- Inspiral through periodic orbits
- Fourier decomposition in 1 frequency instead of 3

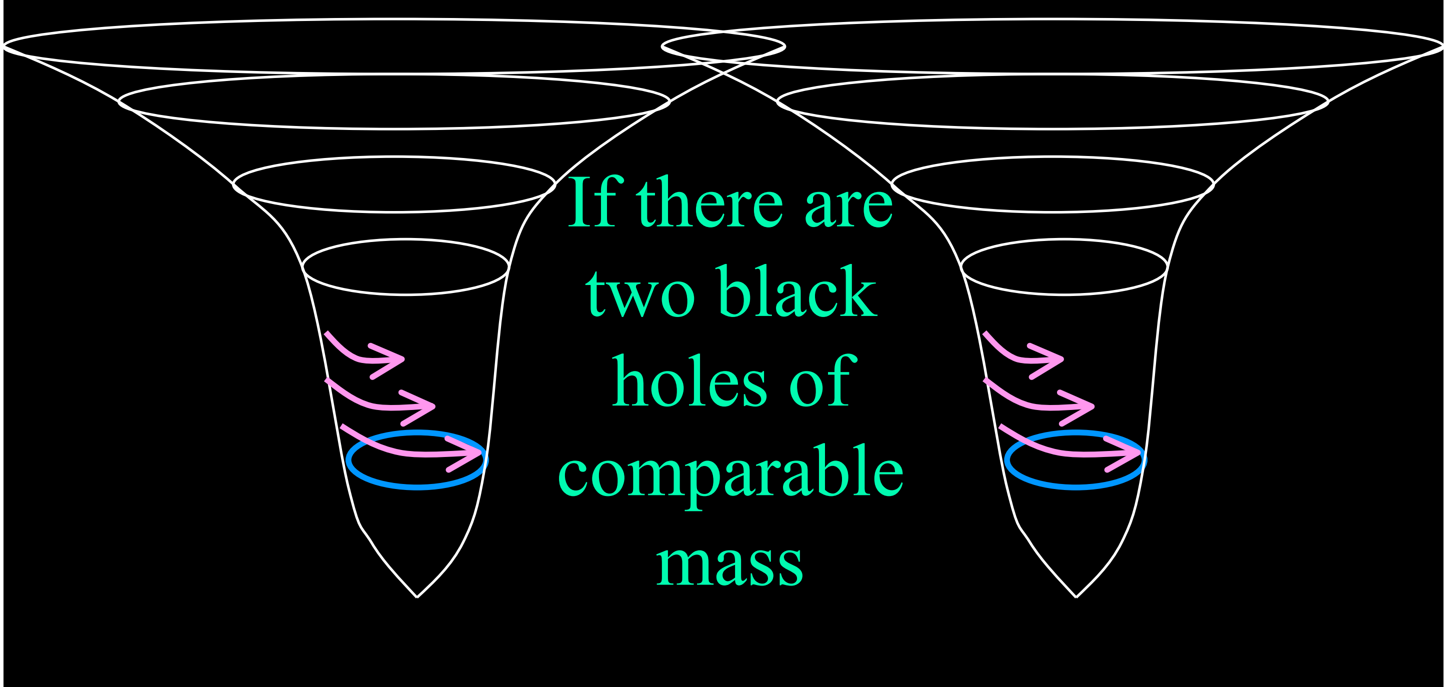
Spin including transition to chaos

What don’t we Know



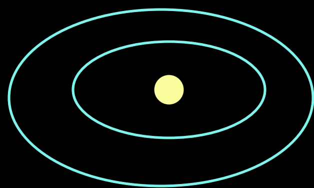
If the companion spins - orbits are strongly altered

There are no known exact solutions to the orbits of these BH binaries

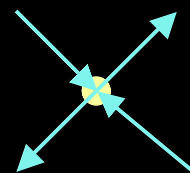


If there are two black holes of comparable mass

The periodic orbits



Elliptical/stable

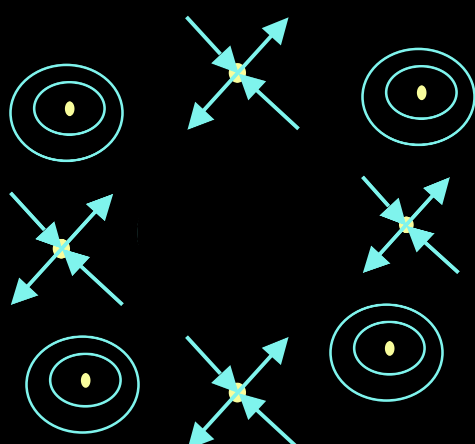


Hyperbolic/unstable

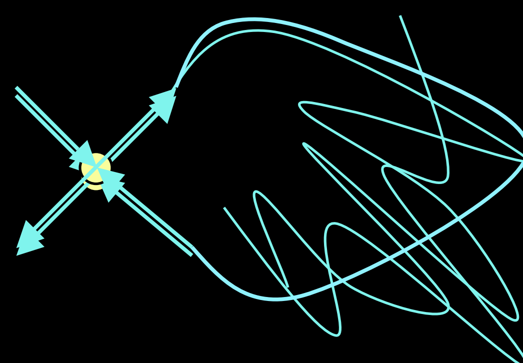
Periodic orbits define the phase space

KAM theorem

Chaos develops around certain unstable periodic orbits



Proliferate into pairs of
elliptical and hyperbolic

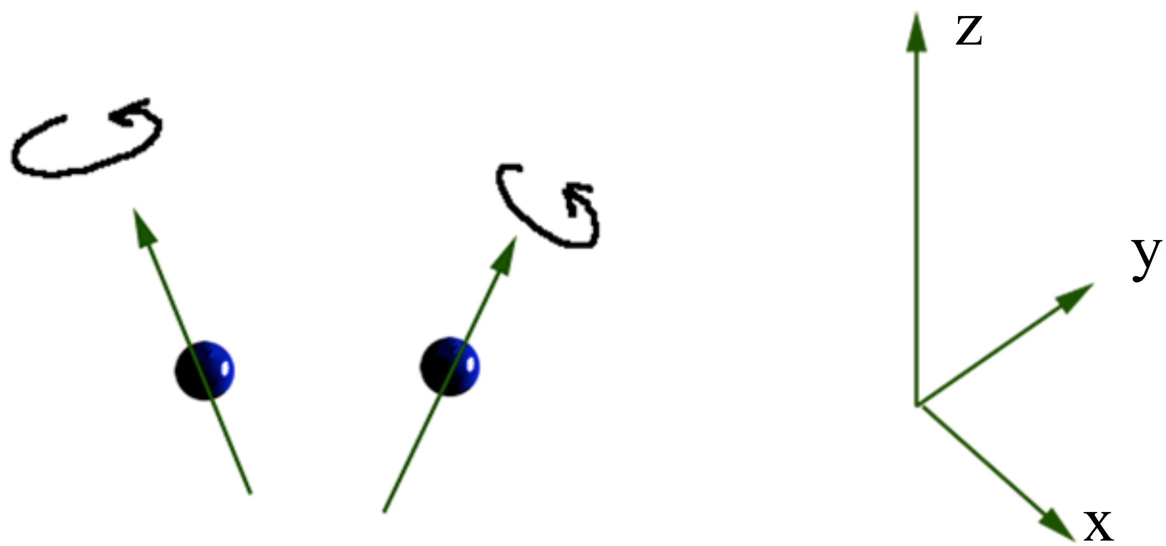


Homoclinic tangle

To pack a proliferating infinity of orbits
in a finite phase space form a
Fractal

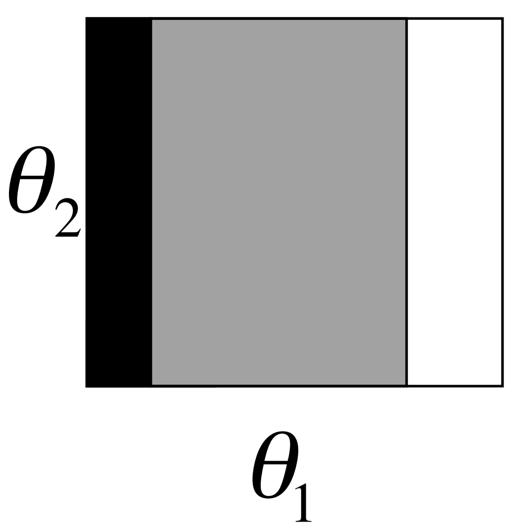
Fractal Basin Boundaries

Post-Newtonian approximation to the 2body problem



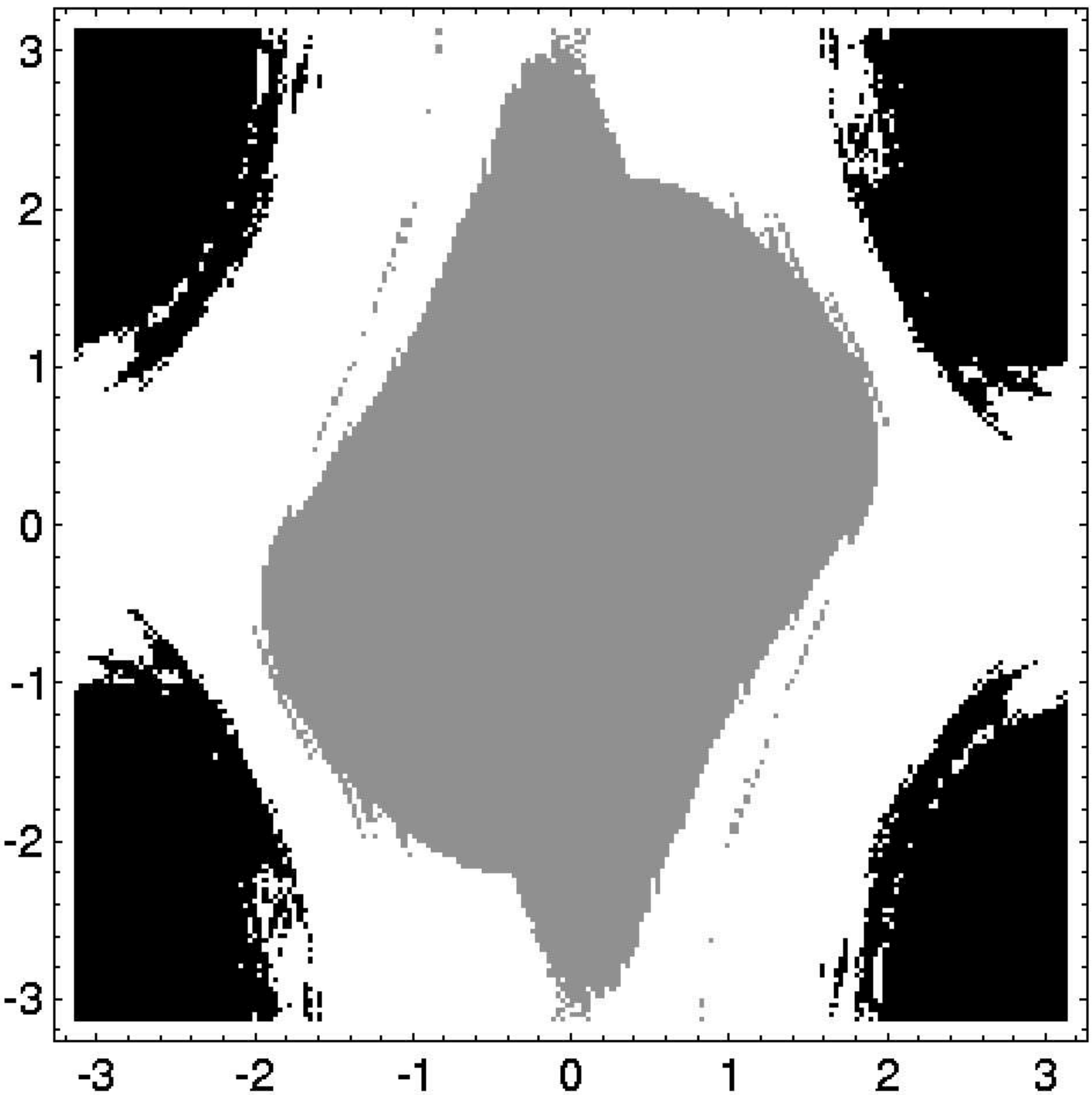
$$\theta = \cos^{-1}\left(\vec{S} \cdot \hat{z}\right)$$

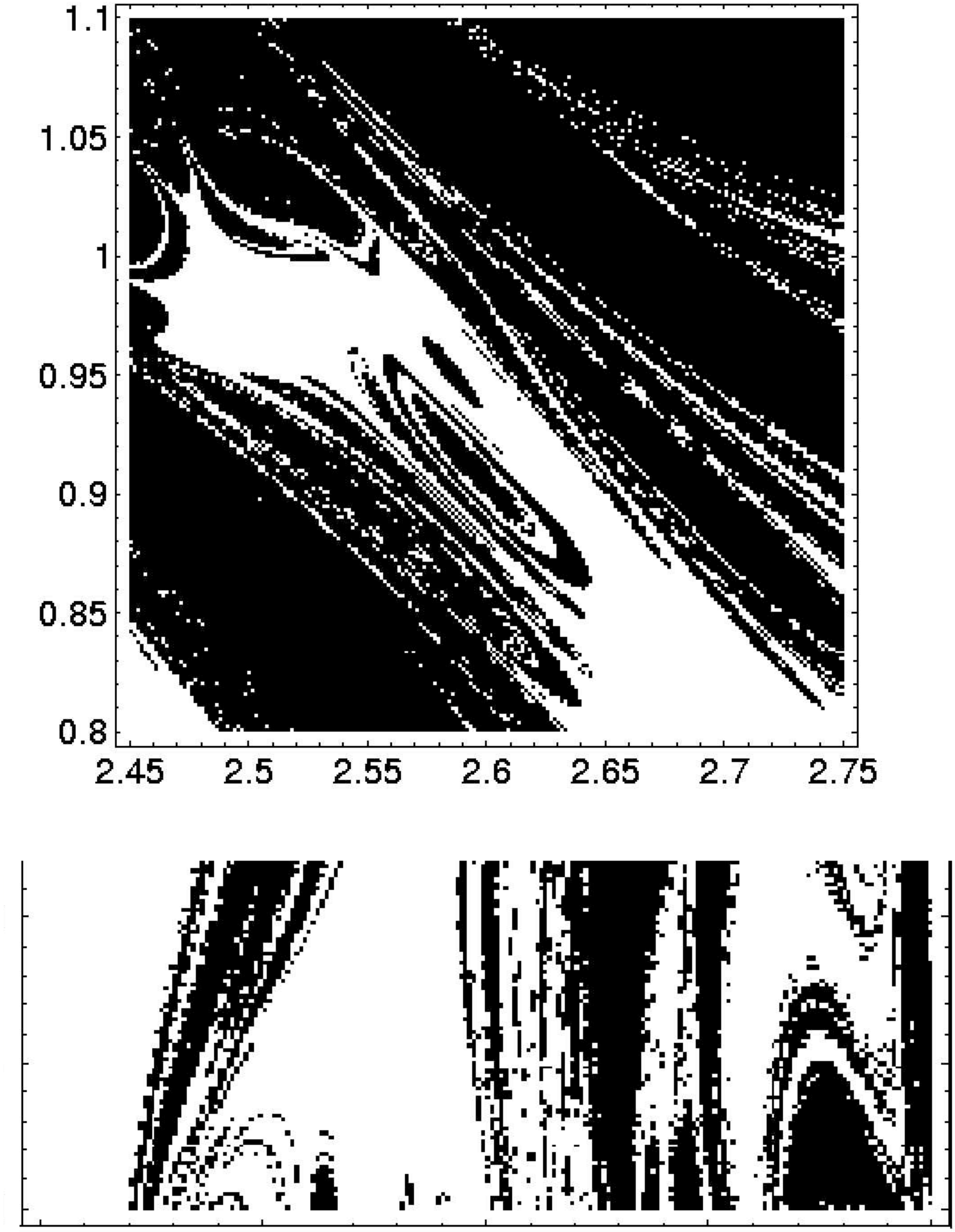
look at initial basins as the spin angles are varied
find they're fractal



- Black $\Rightarrow$ merger
- White $\Rightarrow$ stable
- Grey $\Rightarrow$ escape

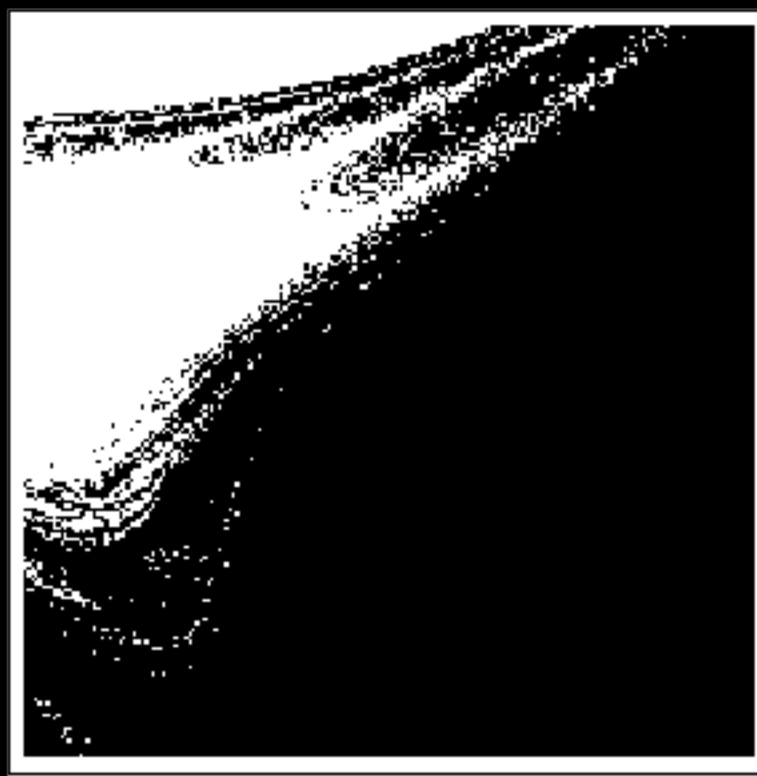
Fractal Basin Boundary

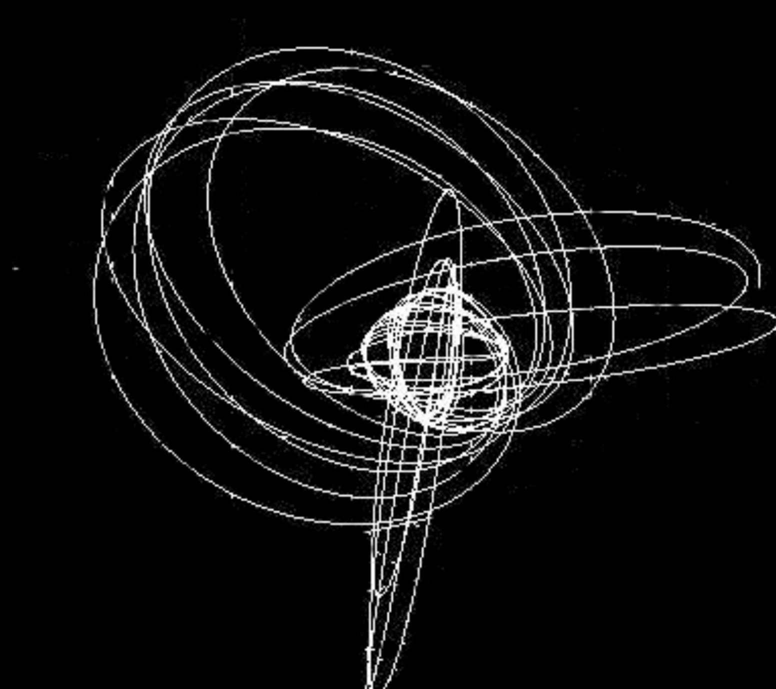




Chaotic scattering off an a fractal set of periodic orbits

$$\frac{m_1}{m_2} = 3, S_1 = 1, S_2 = 0, \theta_1 = 95^o, r = 5M$$





Enough Chaos for Now

Difficult to Proceed:

1. Chaos in Conservative Dynamics

2. Damped by Gravitational Waves

3. Whether completely damped or observable in grey area

2PN not relativistic enough

Kerr with spinning test particle doesn't include mass of companion

In Brief

First test of GR the precession of the perhelion of mercury

Almost a century later, looking for precession of multi-leaf orbits

Zoom-whirls

Might even observe transition to chaos