Continued Fractions

of numbers and series (by humans and machines)

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Seminar for PhD Students, and post-docs
LIP lab, Lyon

8 December 2014

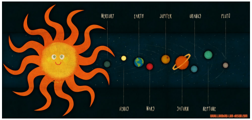
$$\sqrt{2} = 1 + \cfrac{1}{2 + \cfrac{1}{2 + \cfrac{1}{2 + \cdots}}}$$

$$2\frac{\pi - 3}{4 - \pi} = 1 + \cfrac{1}{2^2 + \cfrac{1}{3^2 + \cdots}}$$

$$\exp(1/s) = 1 + \cfrac{2}{2s - 1 + \cfrac{1}{6s + \cfrac{1}{10s + \cfrac{1}{14s + \ldots}}}}$$

How many gear teeth?

Huygens' planetarium (1682, published 1698)




$$2 + \cfrac{1}{3 + \cfrac{1}{1 + \cfrac{1}{\cdots}}}$$

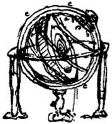


Figure : Huygens' work (in short)

Saturn/earth year ratio?

$$r := \frac{77\ 708\ 431}{2\ 640\ 858} \simeq ???$$

. . . demo!

Simple but powerful

Simple

- Geometry (on blackboard)
- Easy to compute (Euclid)

Real life

- Gregorian calendar
- Music...

Numerical approximation: of $r = p/q$ by truncations $r_1, r_2, \ldots$.

$$r_2 \leq \ldots \leq r_{2n} \leq r \leq r_{2n+1} \leq \ldots \leq r_1$$

Developing $x \in \mathbb{R}^+$

More generally?

- $r = n_0 + \epsilon_0 \in \mathbb{R}^+$, with $n_0 > 0$, $0 < \epsilon_0 < 1$
  $$r = n_0 + \frac{1}{1/\epsilon_0} = n_0 + \frac{1}{n_1 + \frac{1}{1/\epsilon_1}} = n_0 + \frac{1}{n_1 + \frac{1}{n_2 + \dots}} =: [n_0, n_1, \dots]$$
- no radix dependency (*versus* $.1_3 = .333333333\dots_{10}$)
- nice distribution (Gauss 1800, Kutzmin 1929)

$$\Pr(a_i = k) = -\log_2\left(1 - \frac{1}{(k+1)^2}\right)$$

Example (exercise)

Derive a continued fraction expression:

$$\sqrt{2} = n_1 + \cfrac{1}{n_2 + \cfrac{1}{n_3 + \dots}}$$

with $n_i \in \mathbb{N}^\star$

Example

$$(\sqrt{2}-1)(\sqrt{2}+1) = 2-1 = 1$$

$$\sqrt{2} = 1 + \frac{1}{1+\sqrt{2}}$$

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$$\sqrt{2} = 1 + \cfrac{1}{2 + \cfrac{1}{2 + \cfrac{1}{2+\cdots}}}$$

$$\sqrt{2} = [1, 2, 2, 2, 2, 2, \ldots]$$

Polynomials

First properties

- rational $\iff$ finite expansion
- quadratic $(n+\sqrt{m})$ $\iff$ periodic expansion (at the limit)
- Pell-Fermat equation: $x^2 - yn = 1$
  - $\sqrt{n}$ as continued fraction $\implies$ all solutions! (Euclid, president!)

polynomials

Theorem (Galois 1828 (aged 17))

If $[a_0,\ a_1, a_2, \ldots, a_n,\ a_1, a_2, \ldots, a_n,\ \ldots]$ *is a zero of a polynomial, then* $-[1,\ a_n, a_{n-1}, \ldots, a_0,\ a_n, \ldots, a_0,\ \ldots]$ *also is.*

Application (Lenstra 2002, Hallgren 2007)

Real roots isolation of $P(X) \in \mathbb{R}[X]$*, in exact arithmetic.*

For numbers

Bestness:

- $r = [a_1, \ldots, a_n, \ldots]$:

$$|r - \frac{p}{q}| < \frac{1}{2q^2} \implies \frac{p}{q} = r_n \text{ for some truncation } r_n$$

Applications to transcendence

- first transcendental number (Liouville, 1851)
- e is transcendental (Hermite, 1873)

For numbers (Formal or numeric?)

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any pb?

$$(-\sqrt{2}-1)(-\sqrt{2}+1) = 1$$
$$-\sqrt{2} = 1 + \frac{1}{1-\sqrt{2}}$$
$$-\sqrt{2} = 1 + \cfrac{1}{1 + 1 + \cfrac{1}{1+-\sqrt{2}}}$$
$$-\sqrt{2} = 1 + \cfrac{1}{2 + \cfrac{1}{2 + \cfrac{1}{2 + \cdots}}}$$

For functions

$$\textcolor{red}{z} = \cfrac{z}{1 - z + \textcolor{red}{z}} \qquad\qquad -1 = \cfrac{z}{1 - z + -1}$$

$$z = \cfrac{z}{1 - z + \cfrac{z}{1 - z + \cfrac{z}{1 - z + \dots}}} \qquad\qquad -1 = \cfrac{z}{1 - z + \cfrac{z}{1 - z + \cfrac{z}{1 - z + \dots}}}$$

Remark

$$f_n = z\frac{1-(-z)^n}{1-(-z)^{n+1}}, \textit{ hence } \lim f_n(z) = \begin{cases} z, & |z| < 1 \\ -1, & |z| > 1 \end{cases}, z \in \mathbb{C}$$

Remark (Truncation, convergence acceleration)

$$f(z) \simeq \cfrac{z}{1 - z + \cfrac{\dots}{1 - z + \cfrac{z}{1 - z + \textcolor{red}{0}}}} \textit{ but } f(z) = \cfrac{z}{1 - z + \cfrac{\dots}{1 - z + \cfrac{z}{1 - z + \textcolor{red}{\{z / -1\}}}}}$$

Padé *vs* Taylor

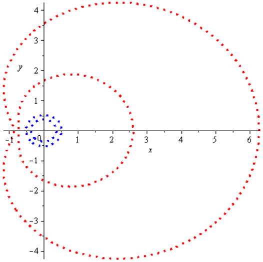
Padé *vs* Taylor

$$\ln(1+z) = \sum_{1}^{4} (-1)^{i+1} \frac{z^i}{i} + O(z^5)$$

$$= \frac{1/2\, z^2 + z}{1/6\, z^2 + z + 1} + O(z^5)$$

$$= \cfrac{z}{1 + \cfrac{z/2}{1 + \cfrac{z/6}{1 + z/3}}} + O(z^5)$$

10 digits precision zones, $z \in \mathbb{C}$
(truncations with 20/30 coefficients)



- Taylor at order $2n$: convergence for $|z| < 1$ only,
- Padé at order n/n : convergence in $\mathbb{C}$ except $z \in -1 - \mathbb{R}^+$.

Convergence speed

For complex functions

- Huge literature, and numerical applications (Bessel)
- Computationally meaningful

DEMO!

Conclusion

Continued Fractions:

- Nice formal tool,
- numerical approximation,
    - extraordinary convergence (Huygens, Hermite, and now!)
    - adapts to series,
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Ad

- LaTeX: emacs, AucTeX mode
- Bibliography: Zotero

Discussion

- Who is next?
- Was the format ok?
- Suggestions?