

Continued Fractions

of numbers and series (by humans and machines)

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Seminar for PhD Students, and post-docs
LIP lab, Lyon

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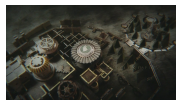
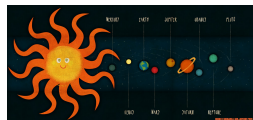
$$\sqrt{2} = 1 + \frac{1}{2 + \frac{1}{2 + \frac{1}{2 + \dots}}}$$

$$2 \frac{\pi - 3}{4 - \pi} = 1 + \frac{1}{2^2 + \frac{1}{3^2 + \dots}}$$

$$\exp(1/s) = 1 + \frac{2}{2s - 1 + \frac{1}{6s + \frac{1}{10s + \frac{1}{14s + \dots}}}}$$

How many gear teeth?

Huygens' planetarium (1682, published 1698)



$$2 + \frac{1}{3 + \frac{1}{1 + \frac{1}{\dots}}}$$



Figure : Huygens' work (in short)

Saturn/earth year ratio?

$$r := \frac{77\,708\,431}{2\,640\,858} \simeq ???$$

... demo!

Simple but powerful

Simple

- Geometry (on blackboard)
- Easy to compute (Euclid)

Real life

- Gregorian calendar
- Music...

Numerical approximation: of $r = p/q$ by truncations r_1, r_2, \dots

$$r_2 \leq \dots \leq r_{2n} \leq r \leq r_{2n+1} \leq \dots \leq r_1$$

Developing $x \in \mathbb{R}^+$

More generally?

- $r = n_0 + \epsilon_0 \in \mathbb{R}^+$, with $n_0 > 0$, $0 < \epsilon_0 < 1$

$$r = n_0 + \frac{1}{1/\epsilon_0} = n_0 + \frac{1}{n_1 + \frac{1}{1/\epsilon_1}} = n_0 + \frac{1}{n_1 + \frac{1}{n_2 + \dots}} =: [n_0, n_1, \dots]$$

- no radix dependency (*versus* $.1_3 = .333333333\dots_{10}$)
- nice distribution (Gauss 1800, Kutzmin 1929)

$$\Pr(a_i = k) = -\log_2 \left(1 - \frac{1}{(k+1)^2} \right)$$

Example (exercise)

Derive a continued fraction expression:

$$\sqrt{2} = n_1 + \frac{1}{n_2 + \frac{1}{n_3 + \dots}}$$

with $n_i \in \mathbb{N}^*$

Example

$$(\sqrt{2} - 1)(\sqrt{2} + 1) = 2 - 1 = 1$$

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$$\sqrt{2} = [1, 2, 2, 2, 2, 2, \dots]$$

Polynomials

First properties

- rational \iff finite expansion
- quadratic $(n + \sqrt{m}) \iff$ periodic expansion (at the limit)
- Pell-Fermat equation: $x^2 - yn = 1$
 - \sqrt{n} as continued fraction \implies all solutions! (Euclid, president!)

polynomials

Theorem (Galois 1828 (aged 17))

If $[a_0, a_1, a_2, \dots, a_n, a_1, a_2, \dots, a_n, \dots]$ is a zero of a polynomial, then $-[1, a_n, a_{n-1}, \dots, a_0, a_n, \dots, a_0, \dots]$ also is.

Application (Lenstra 2002, Hallgren 2007)

Real roots isolation of $P(X) \in \mathbb{R}[X]$, in exact arithmetic.

For numbers

Bestness:

- $r = [a_1, \dots, a_n, \dots]$:

$$\left| r - \frac{p}{q} \right| < \frac{1}{2q^2} \implies \frac{p}{q} = r_n \text{ for some truncation } r_n$$

Applications to transcendence

- first transcendental number (Liouville, 1851)
- e is transcendental (Hermite, 1873)

For numbers (Formal or numeric?)

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any pb?

$$(-\sqrt{2} - 1)(-\sqrt{2} + 1) = 1$$

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For functions

$$z = \frac{z}{1 - z + z}$$

$$z = \frac{z}{1 - z + \frac{z}{1 - z + \frac{z}{1 - z + \dots}}}$$

$$-1 = \frac{z}{1 - z + -1}$$

$$-1 = \frac{z}{1 - z + \frac{z}{1 - z + \frac{z}{1 - z + \dots}}}$$

Remark

$$f_n = z \frac{1 - (-z)^n}{1 - (-z)^{n+1}}, \text{ hence } \lim f_n(z) = \begin{cases} z, & |z| < 1 \\ -1, & |z| > 1 \end{cases}, z \in \mathbb{C}$$

Remark (Truncation, convergence acceleration)

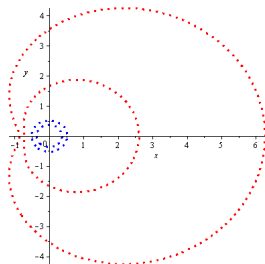
$$f(z) \simeq \frac{z}{1 - z + \frac{\dots}{1 - z + \frac{z}{1 - z + 0}}} \text{ but } f(z) = \frac{z}{1 - z + \frac{\dots}{1 - z + \frac{z}{1 - z + \{z/-1\}}}}$$

Padé vs Taylor

Padé vs Taylor

$$\begin{aligned}
 \ln(1+z) &= \sum_1^4 (-1)^{i+1} \frac{z^i}{i} + O(z^5) \\
 &= \frac{1/2 z^2 + z}{1/6 z^2 + z + 1} + O(z^5) \\
 &= \frac{z}{1 + \frac{z/2}{1 + \frac{z/6}{1 + z/3}}} + O(z^5)
 \end{aligned}$$

10 digits precision zones, $z \in \mathbb{C}$
(truncations with 20/30 coefficients)



- Taylor at order $2n$: convergence for $|z| < 1$ only,
- Padé at order n/n : convergence in \mathbb{C} except $z \in -1 - \mathbb{R}^+$.

Convergence speed

For complex functions

- Huge literature, and numerical applications (Bessel)
- Computationally meaningful

DEMO!

Conclusion

Continued Fractions:

- Nice formal tool,
- numerical approximation,
 - extraordinary convergence (Huygens, Hermite, and now!)
 - adapts to series,
- My PhD: Automatic derivation and study ;)

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Ad

- \LaTeX : emacs, AucTeX mode
- Bibliography: Zotero

Discussion

- Who is next?
- Was the format ok?
- Suggestions?