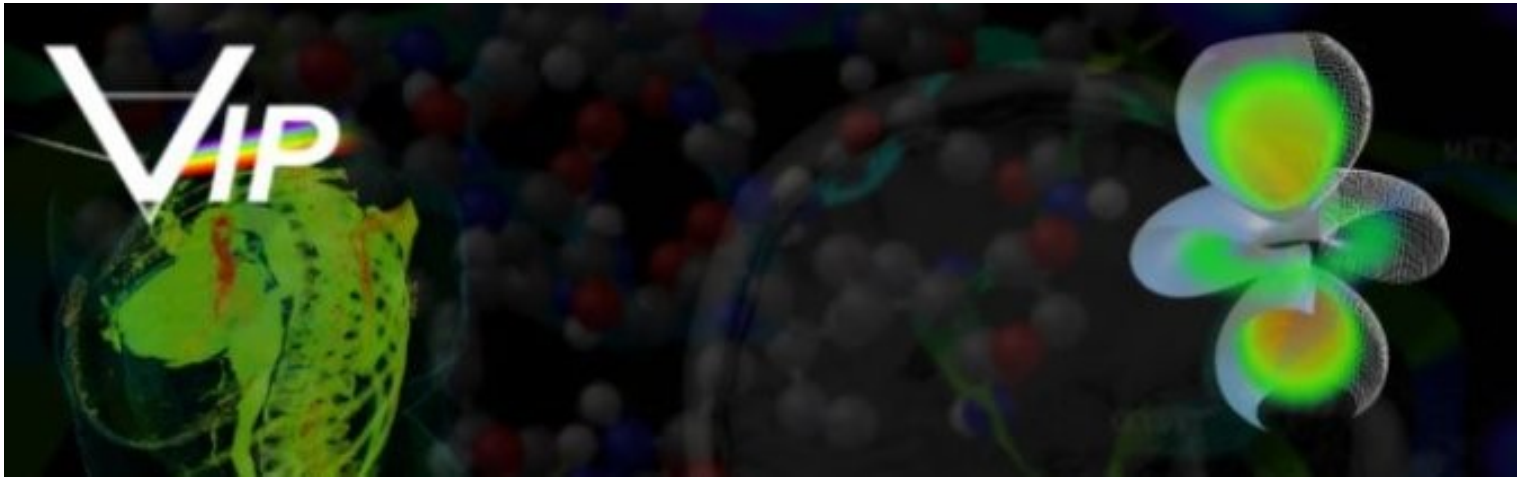


# Partial Differential Equations in Image Processing

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# Overview

- Introduction
- Perona-Malik type equations
- Image inpainting
- Geometrical image segmentation
- Piece of art ..

# Introduction

Quick overview of PDE methods in image processing.

Processed image can be modeled by a real function

$$u^0(x,\theta) \qquad u^0 : \Omega \times [0, T_A] \to \mathbb{R}$$

$$\Omega \subset \mathbb{R}^d$$

Some examples:

- nonlinear image filtration
- edge detection
- deblurring and image enhancement
- restoration
- image inpainting
- shape extraction and analysis
- image segmentation
- motion analysis

K.Mikula, "*Numerical solution, analysis and application of geometrical nonlinear diffusion equations*", Edition of Scientific Publications, No. 34, Publishing House of the Slovak University of Technology, Bratislava, 2006, ISBN 80-227-2392-4 PDF

# Introduction

Gauss function

$$G_\sigma(x) = \frac{1}{(4\pi\sigma)^{d/2}} \mathrm{e}^{\frac{-|x|^2}{4\sigma}}$$

is a fundamental solution of the linear heat (diffusion) equation.

Replace Gaussian smoothing (with variance $v = \sqrt{2\sigma}$) by solving the linear heat equation for a corresponding time $t = \sigma$ with initial condition given by the processed image.

A.P. Witkin, "*Scale-space filtering*", in: Proceedings Eight International Conference on Artificial Intelligence, Vol. 2 (1983) pp. 1019-1022

# Introduction - basic notations

Application of any PDE to an initially given image is understood as its embedding in the so-called **scale-space**.

Depending on types of systems we consider **linear** scale spaces and **nonlinear** scale spaces.

**Image multiscale analysis**.

**Scale** – the abstract time parameter.

Function **g()**

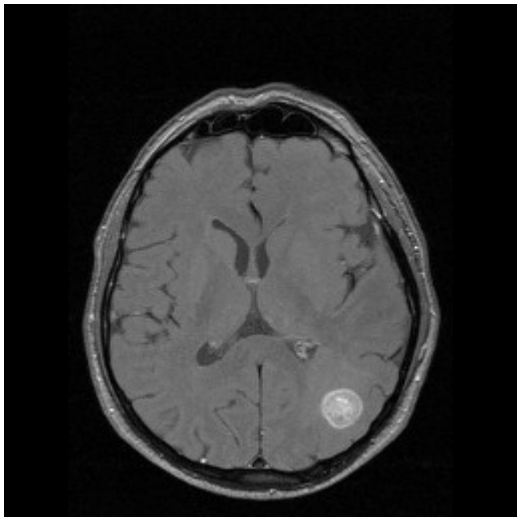
$g : \mathbb{R}_0^+ \to \mathbb{R}^+$ is a nonincreasing function

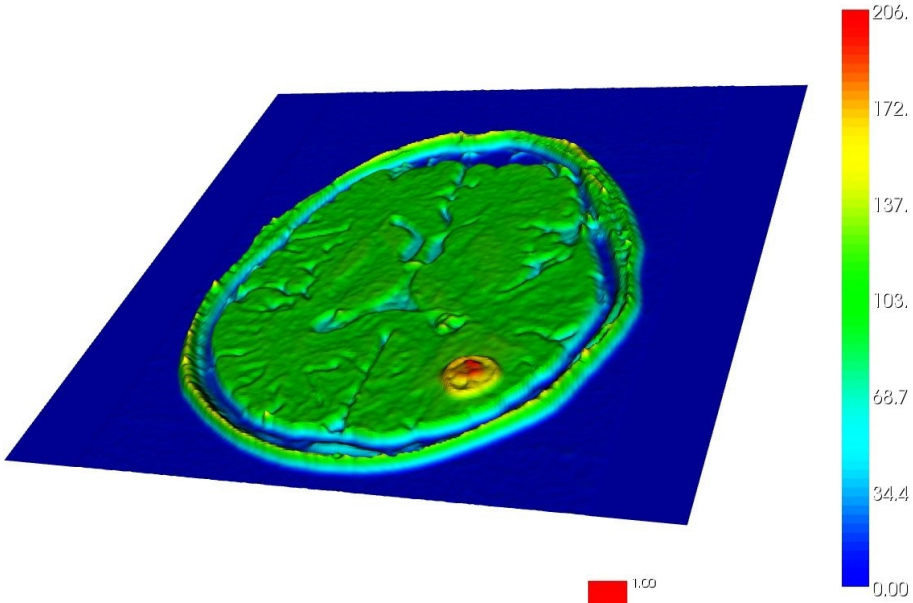
$g(\sqrt{s})$ is smooth

$g(0) = 1$

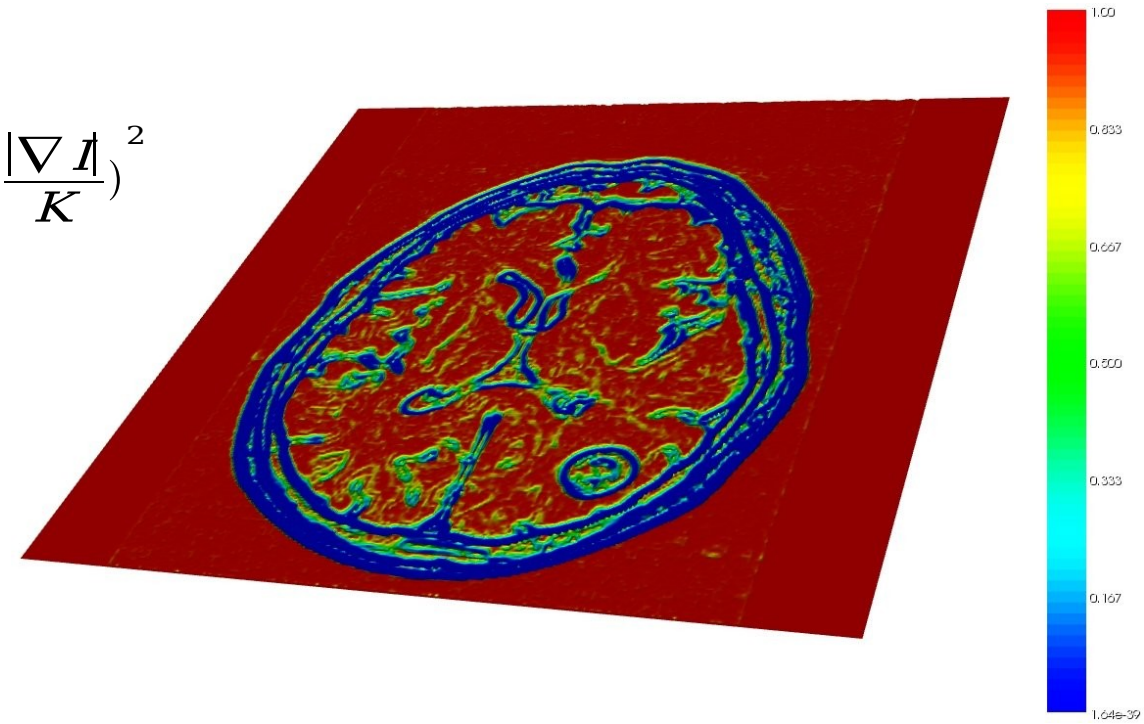
$g(s) \to 0$ for $s \to \infty$

# Introduction – function g()






$$g(|\nabla I|) = \mathrm{e}^{-\left(\frac{|\nabla I|}{K}\right)^2}$$



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# Perona-Malik – anisotropic diffusion

Gaussian smoothing blurs edges in the images and moves their positions.

P.Perona and J.Malik proposed a nonlinear diffusion process governed by the shape of the diffusion coefficient given by the function **g()**.

$$u_t - \nabla . (g(|\nabla u|) \nabla u) = 0$$

$$\frac{\delta u}{\delta n} = 0 \quad \text{on} \quad I \times \delta \Omega$$

$$u(0, x) = u^0(x) \quad \text{in} \quad \Omega$$

P.Perona, J.Malik, "*Scale-space and edge detection using anisotropic diffusion*" Proc. IEEE Computer Society Workshop on Computer Vision (1987)

# Perona-Malik – anisotropic diffusion

From a mathematical point of view, for practical choices of g(), the system proposed by P.Perona and J.Malik is often ill-posed.

V.Catte, P.L.Lions, J.M.Morel and T.Coll introduced the convolution with the Gaussian kernel $G_\sigma$.

$$u_t - \nabla . ( g( |\nabla G_\sigma * u| ) \nabla u ) = 0$$

$$\frac{\delta u}{\delta n} = 0 \quad \text{on} \quad I \times \delta \Omega$$

$$u(0,x) = u^0(x) \quad \text{in} \quad \Omega$$


S.Kichenassamy, "*The Perona-Malik paradox*", SIAM J. Appl. Math., Vol.57, No.5 (1997) pp. 1328-1342

V.Catte, P.L.Lions, J.M.Morel, T.Coll, "*Imageselective smoothing and edge detection by nonlinear diffusion*", SIAM J.Numer.Anal. 29 (1992) pp. 182-193

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- existence
- uniqueness

$$u_t - \nabla . (g(|\nabla G_\sigma * u|) \nabla u) = 0$$

$$\frac{\delta u}{\delta n} = 0 \quad \text{on} \quad I \times \delta \Omega$$

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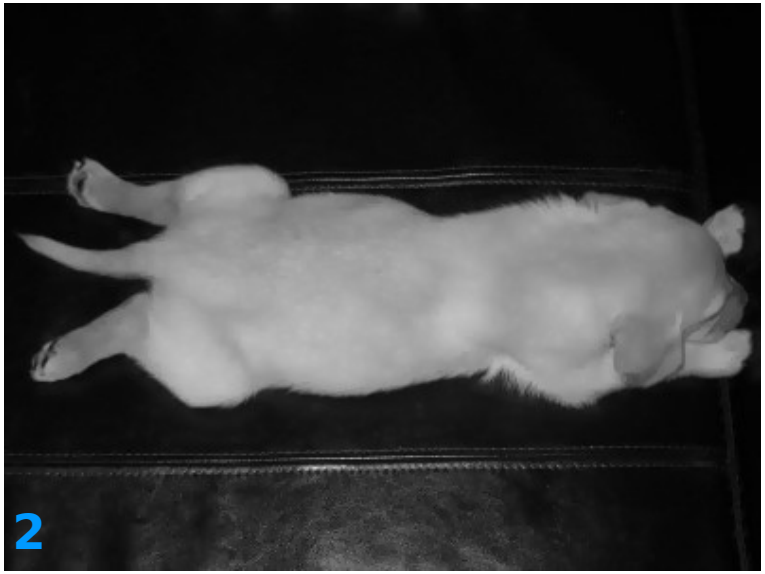
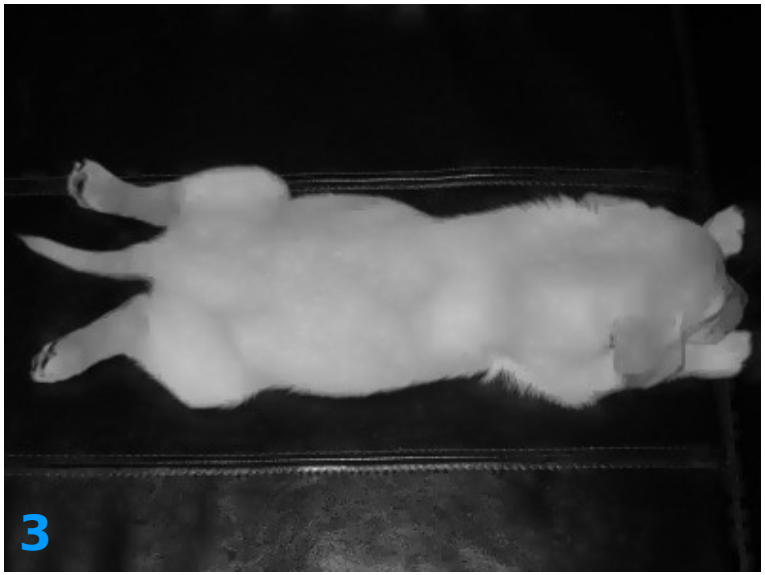
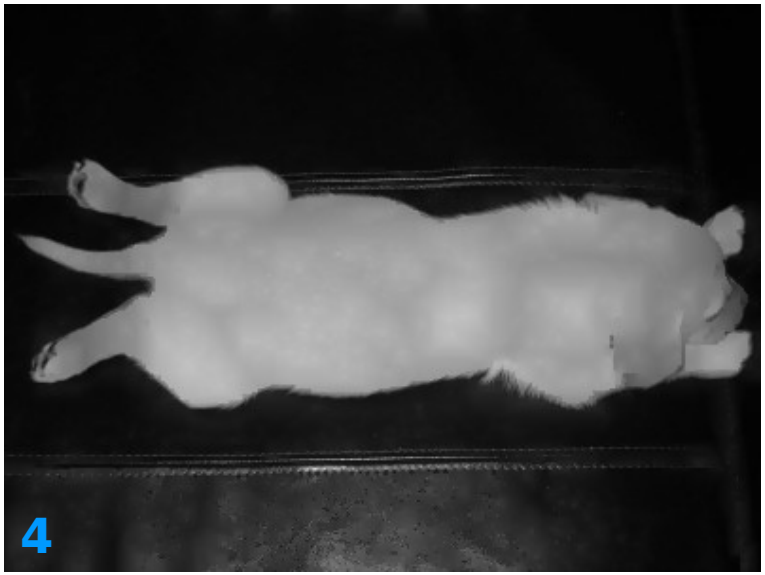
# Perona-Malik – example

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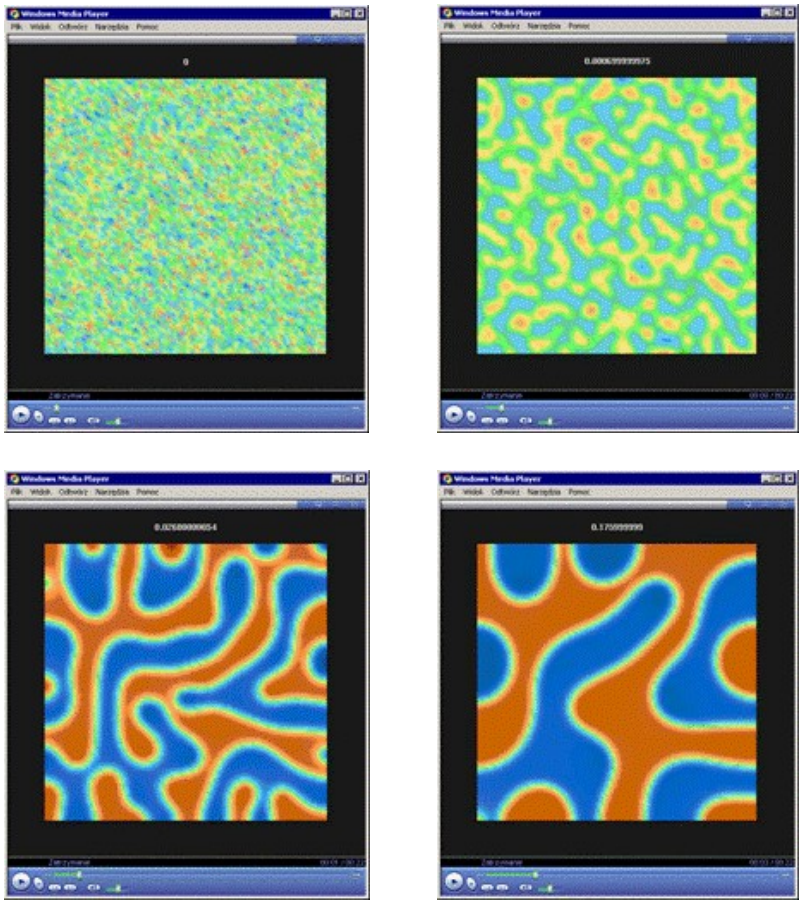
# Image inpainting – C-H equation

Cahn-Hiliard simulations by
dr Kerstin Kantiem



Modified Cahn-Hiliard Equation
Inpainting Model

$$u_t = -\Delta\left(\epsilon \Delta u - \frac{1}{\epsilon} W'(u)\right) + \lambda(\vec{x})(f-u)$$

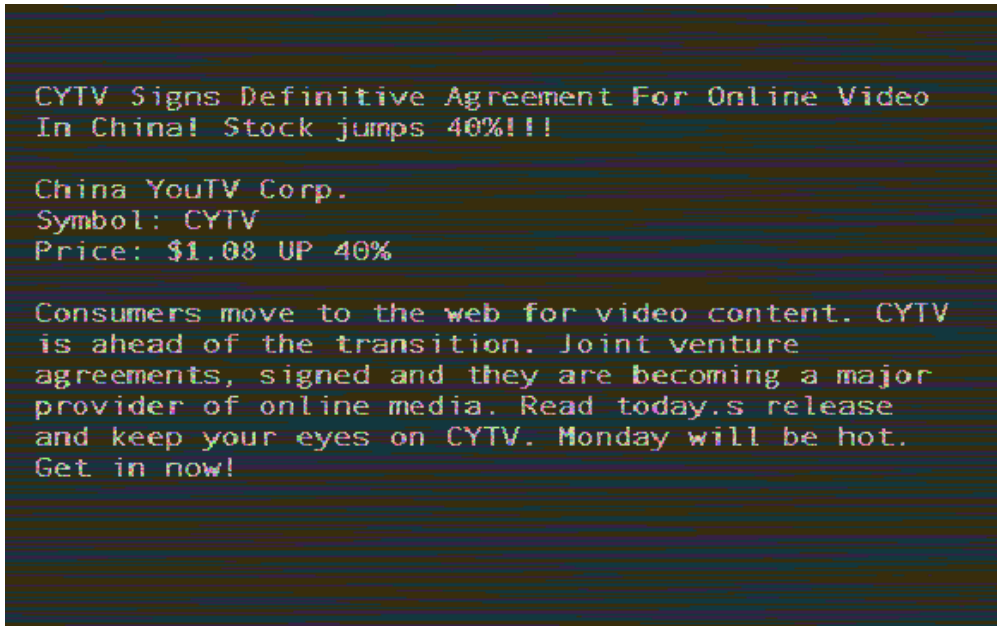
$\lambda(\vec{x})=0$ if $\vec{x}\in D$ $\qquad W(u)=u^2(u-1)^2$

$\lambda(\vec{x})=\lambda_0$ if $\vec{x}\in \Omega\setminus D$ $\qquad f(\vec{x})$ - image

A.Bertozzi, S.Esedoglu, A.Gillette, "*Inpainting of Binary Images Using the Cahn-Hiliard Equation*", IEEE Transactions in Image Processing (submitted)

# Image inpainting


Spam filtering



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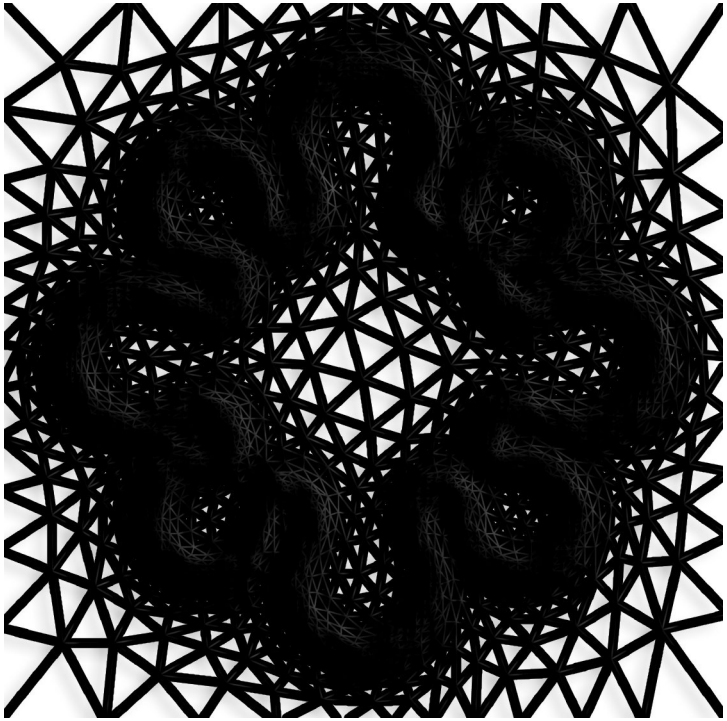
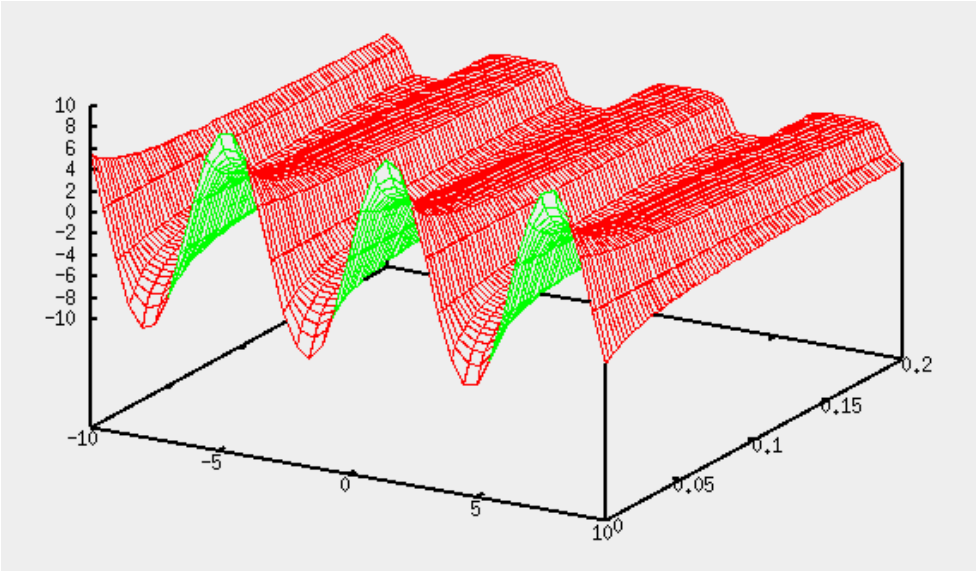
# Geometrical image segmentation

Allen-Cahn equation:

$$\epsilon \frac{\delta u}{\delta t} - \epsilon \Delta u + \frac{1}{\epsilon}(u^3 - u) = 0 \quad \text{in} \quad \Omega$$

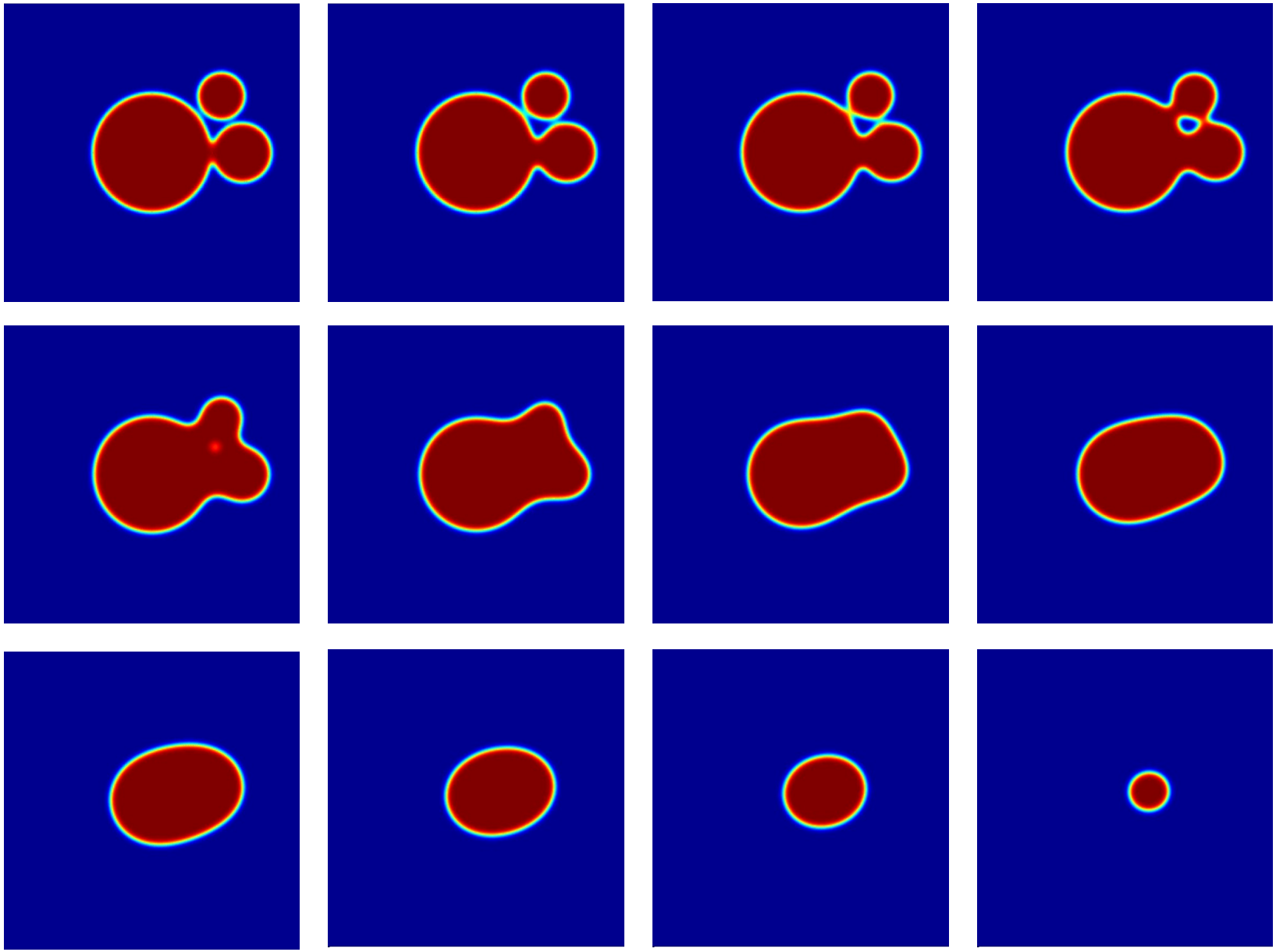
$$\frac{\delta u}{\delta n} = 0 \quad \text{on} \quad \delta \Omega$$






S.Allen, J.W.Cahn, "*A microscopic theory for anthiphase boundary motion and its applications to antiphase domain coarsening*", Acta Metall. 27 (1979), 1084-1095

# Geometrical image segmentation



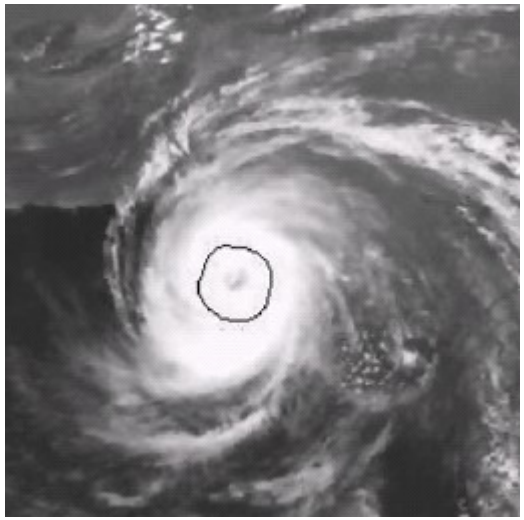
# Geometrical image segmentation

1. Create an initial guess $u_{ini}$

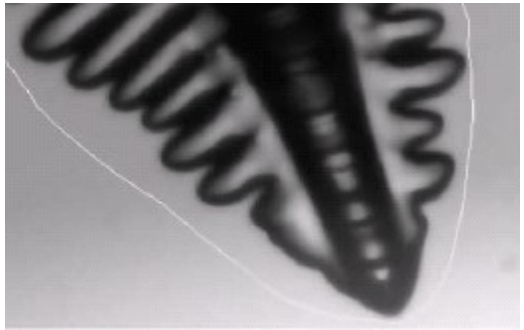
2. Track the level set of function $u$

$$\epsilon \frac{\delta u}{\delta t} = \epsilon \cdot \nabla \cdot (g(|\nabla G_\sigma * I_0|) \nabla u) + g(|\nabla G_\sigma * I_0|) \cdot (\frac{1}{\epsilon} f(u) + \epsilon C |\nabla u|) \quad \text{in } \Omega$$

$$\frac{\delta u}{\delta n} = 0 \quad \text{on } \delta \Omega \qquad\qquad u_{t=0} = u_{ini}$$



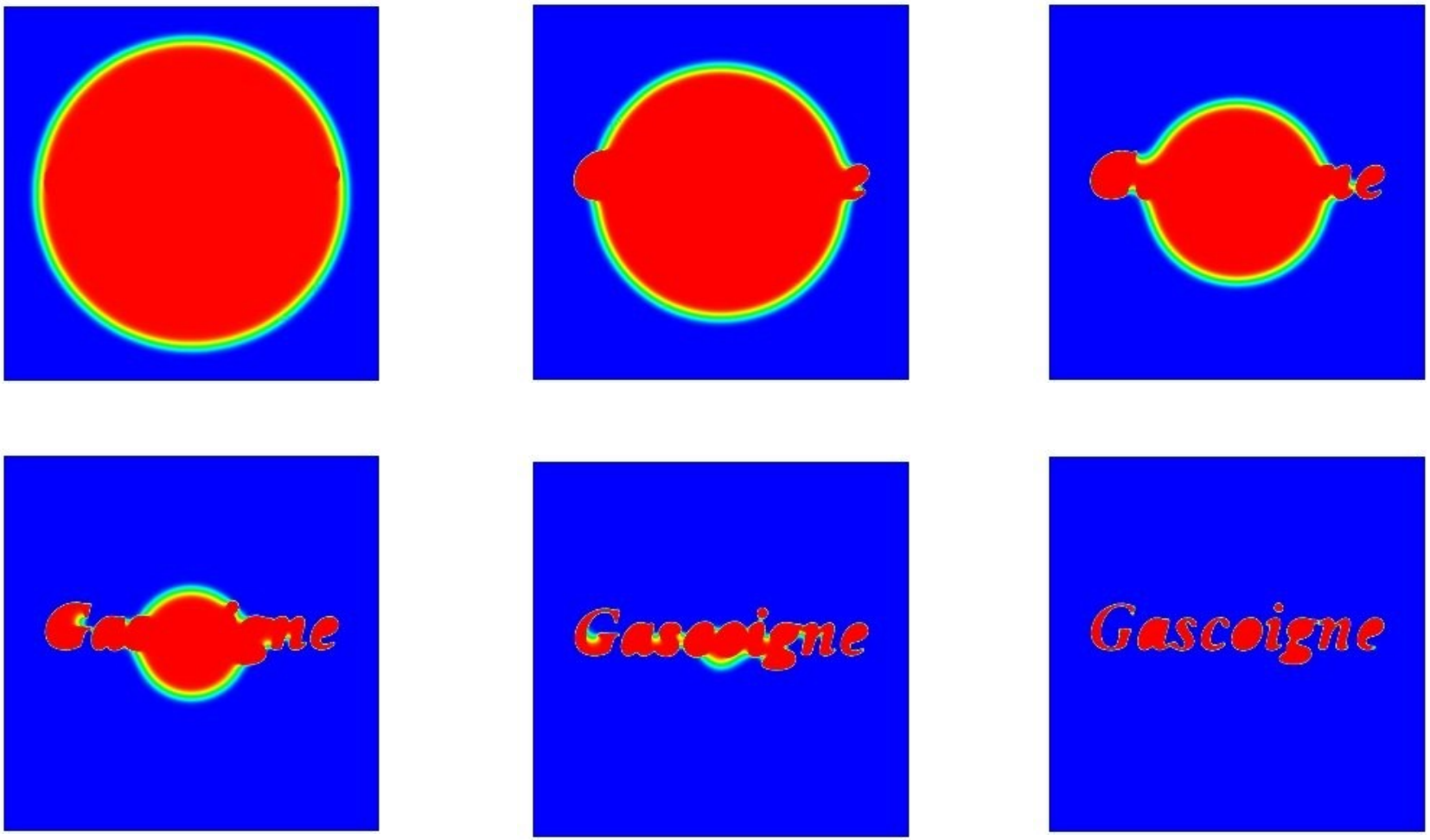
M.Benes, V.Chalupecky, K.Mikula



M.Benes, V.Chalupecky, K.Mikula

M.Benes, V.Chalupecky, K.Mikula, "*Geometrical image segmentation by the Allen-Cahn equation*", Applied Numerical Mathematics, Vol. 51 (2004) pp. 187- 205 PDF

# Geometrical image segmentation



# Geometrical image segmentation

Some good properties:

- nice geometrical properties (handles topological cases)
- easy to apply in 3D
- shape modeling (by controlling the epsilon parameter)

Some bad properties:

- time consuming (interface have to be refined)

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# “The Art of Scale-Space”

J.A.Bangham, S.Gibson, R.Harvey, “*The Art of Scale-Space*” School of Information Systems, University of East Anglia, Norwitch

# Questions?