
Localization Error Analysis in Wireless Sensor Networks

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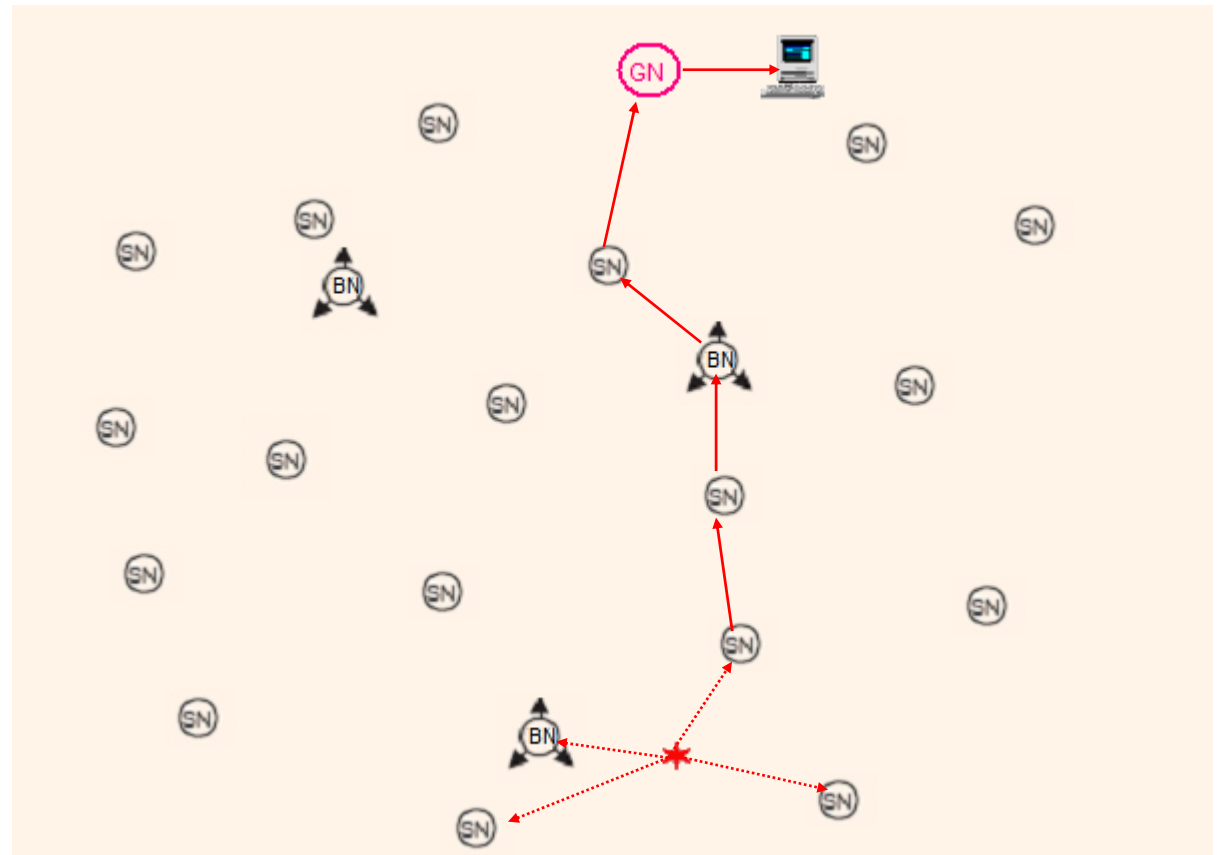
What is a Wireless Sensor Network ?

■ Entities

- Sensor Nodes
- Beacon Nodes
- Gateway Nodes

■ Function

- Sample
- Process
- Communicate



Wireless Sensor Networks

- **Why WSNs ?**

Small size, Low cost, High Reliability and Accessibility

- **Unique challenges**

Power, Random Deployment, Unreliable Communication

- **Applications**

Habitat monitoring, Battle fields, Surveillance, Nuclear power plants, etc.

- **Functions**

Parameter measurement, Target localization and tracking.

Localization

- Node Localization
 - Source Localization
 - Differences
 - Cooperative/Non-cooperative
 - Node Density
 - Computational Complexity
-

Localization – A generic definition

- Given a set of entities (nodes) with known locations and a source entity, the problem is to estimate the location of the source.
 - Location aware --- Sensor node
 - Location unaware --- Source
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Localization

■ Measurements Modalities

- Received Signal Strength (RSS)
- Time of Flight (TOF)
- Time of Arrival (TOA)
- Time Difference of Arrival (TDOA)
- Direction of Arrival (DOA)

■ Measurements

- Range
 - Range Difference
-

Localization Algorithms

- Mostly non iterative
 - Range Difference
 - Locus is a hyperbola
 - At least four nodes are required
 - Least Square Estimation
 - Range
 - Locus is a circle
 - At least three nodes are required
 - Also Least Square Estimation
-

Localization Error

■ Network Parameters

- Node density
- Available energy resources
- Circuit noise
- Location errors

■ Environmental Parameters

- Sensing modality and its propagation model
 - Terrain's geographical topology
 - Ambient noise levels
-

Problem Formulation

- *To characterize the localization error with respect to the network and environmental parameters in an algorithm independent manner*
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Notation

- (x_s, y_s) --- source location
- (\hat{x}_s, \hat{y}_s) --- estimated source location
- (x_i, y_i) --- i_{th} sensor node
- r_i --- distance between the source and i_{th} node
- m_i --- range measurement at the i_{th} sensor node.
- m_{ij} --- range-difference measurement between i_{th} and the j_{th} sensor nodes

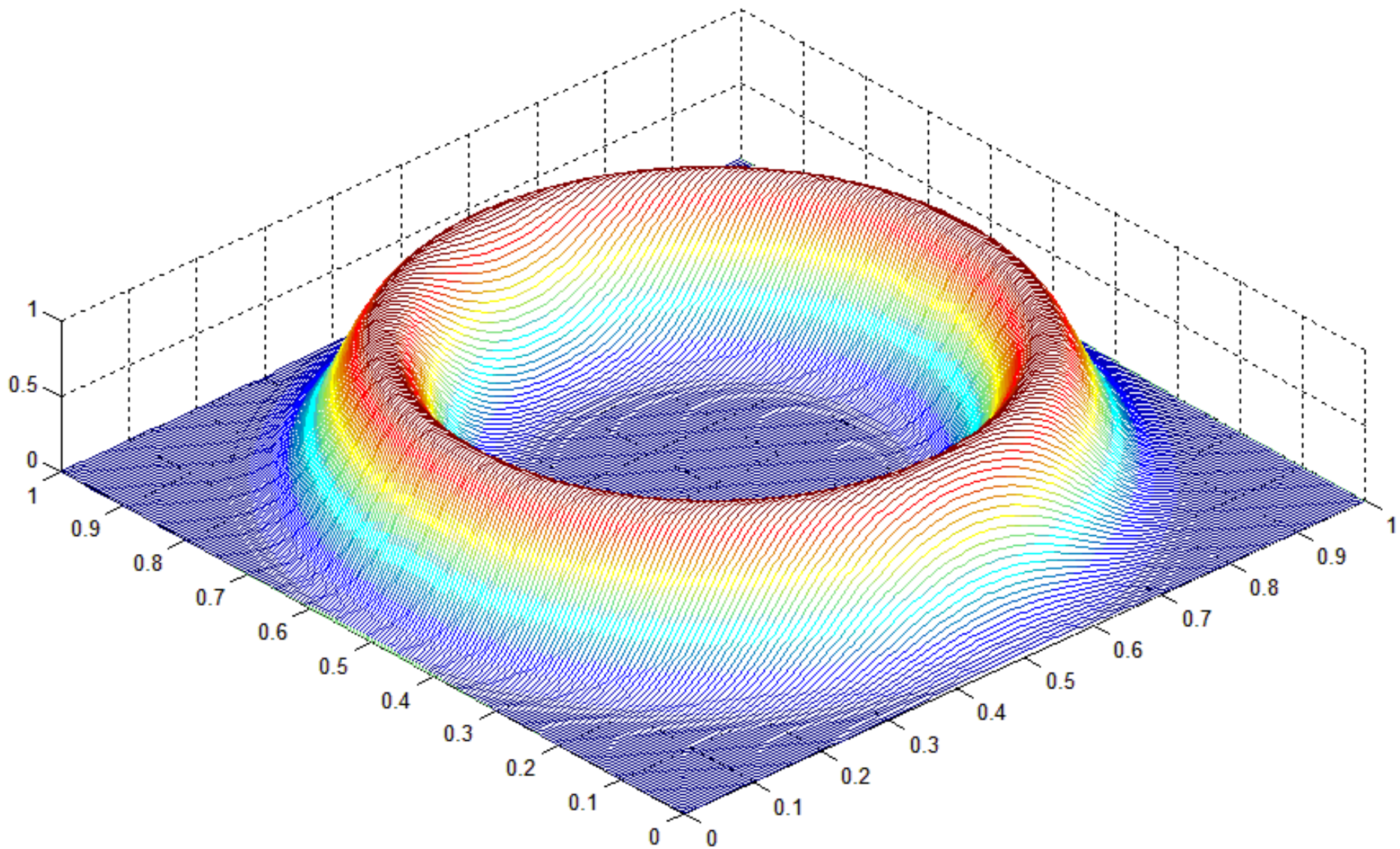
Error models

■ Range Measurements

- Gaussian error

$$f(x, y; \mu, \sigma) =$$

$$\frac{1}{\sqrt{2\pi}} \exp\left(-\frac{\left(\mu - \sqrt{(x - \mu)^2 + (y - \mu)^2}\right)^2}{2\sigma^2}\right)$$



Error models

- Range Difference Measurements
 - Joint Gaussian error

$$f(\bar{m}, \bar{r}) = \frac{1}{(\sqrt{2\pi})^N \det(C)} \exp\left(-\frac{1}{2} \bar{m}^T C^{-1} \bar{m}\right)$$

$\bar{m} = [m_1 \dots m_N]$ $\bar{r} = [r_1 \dots r_N]$

Error models

- Range Difference Measurements
 - Derived Gaussian error

$$f(m_{21}, m_{31}, \dots, m_{N1} | (x_1, y_1), \dots, (x_N, y_N); (x_s, y_s))$$

$$= c \exp \left\{ \left(\frac{(r_1 - (m_2 - r_2) - (m_3 - r_3) - \dots - (m_N - r_N))^2}{2N\sigma^2} \right) - \left(\frac{r_1^2 + (m_2 - r_2)^2 + (m_3 - r_3)^2 + \dots + (m_N - r_N)^2}{2\sigma^2} \right) \right\}$$

Data Collection Techniques

- Closest N Activation Model (CNAM)
 - Fixed Radius Activation Model (FRAM)
-

Post-deployment and a priori error performance

- Given sensor network

- Random network (Poisson points)

- CNAM $f((x_1, y_1), \dots, (x_N, y_N)) = (\lambda)^N e^{-\lambda\pi((x_s - x_N)^2 + (y_s - y_N)^2)}$

- FRAM $f((x_1, y_1), (x_2, y_2), \dots, (x_N, y_N) | \mathbf{N}) = \frac{1}{(\pi R^2)^N}$

Cramer-Rao Lower Bound



$$J_f = E \left[\frac{\partial \log(\mathcal{L})}{\partial \theta} \right]^2$$

Cramer-Rao Lower Bound

$$\begin{bmatrix} \text{var } \hat{\alpha} \\ \text{cov } \hat{\alpha}, \hat{\beta} \end{bmatrix} \geq \frac{1}{E} \begin{bmatrix} J_{11} & J_{12} \\ J_{21} & J_{22} \end{bmatrix}$$

$$J_1 = E \left[\frac{\partial^2 \log \mathcal{L}}{\partial \alpha^2} \right]$$

$$J_2 = E \left[\frac{\partial^2 \log \mathcal{L}}{\partial \beta^2} \right]$$

$$J_{12} = E \left[\frac{\partial^2 \log \mathcal{L}}{\partial \alpha \partial \beta} \right] = J_{21}$$

Range Measurements – Post-deployment CRLB

$$J_{IF} = \sum_{i=1}^N \frac{(A + 2\alpha) (xx)^2}{\alpha r_i^2 (K + r_i)^2}$$

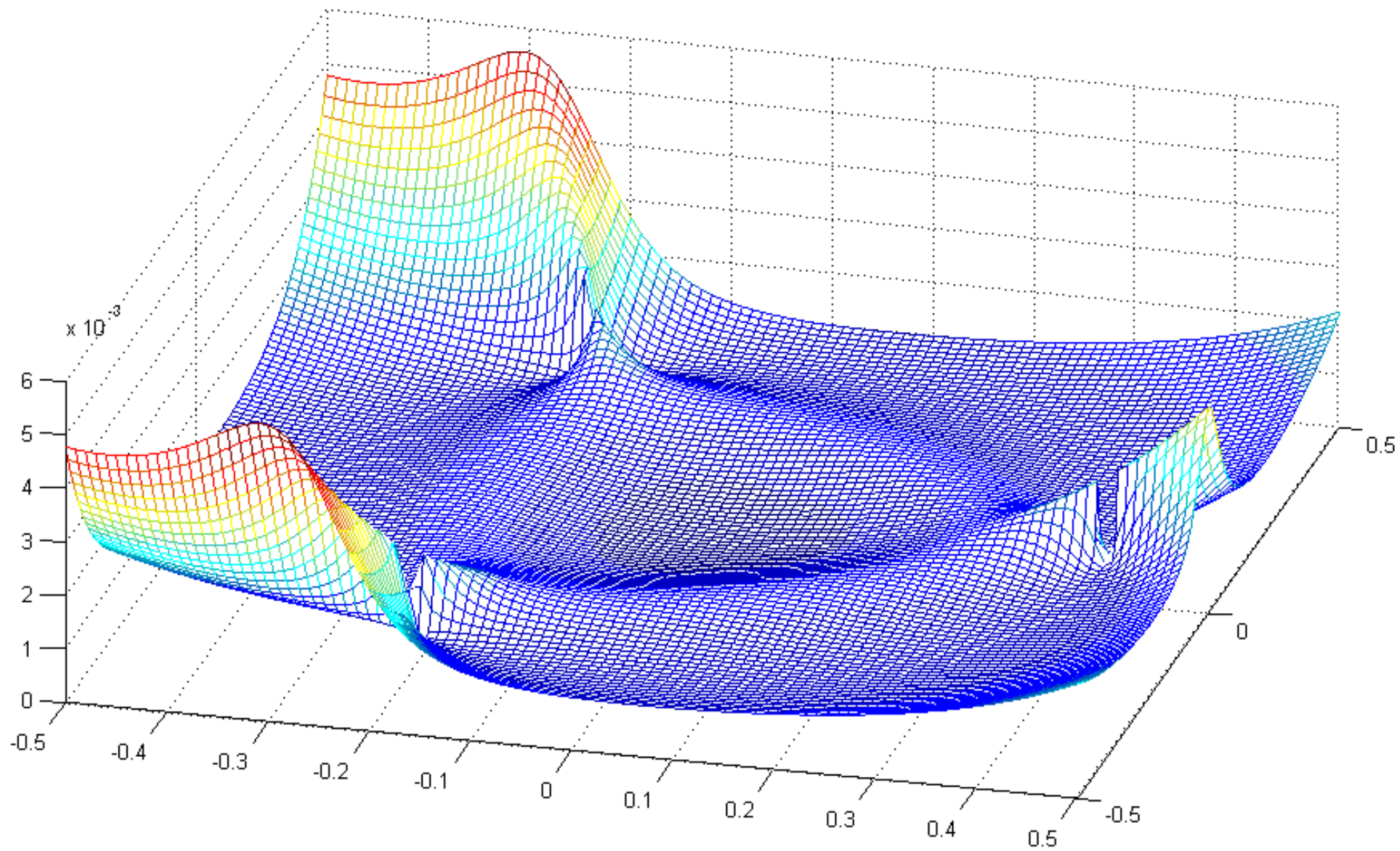
~~$$J_{IF} = \sum_{i=1}^N \frac{(A + 2\alpha) (xx)^2}{\alpha r_i^2 (K + r_i)^2}$$~~

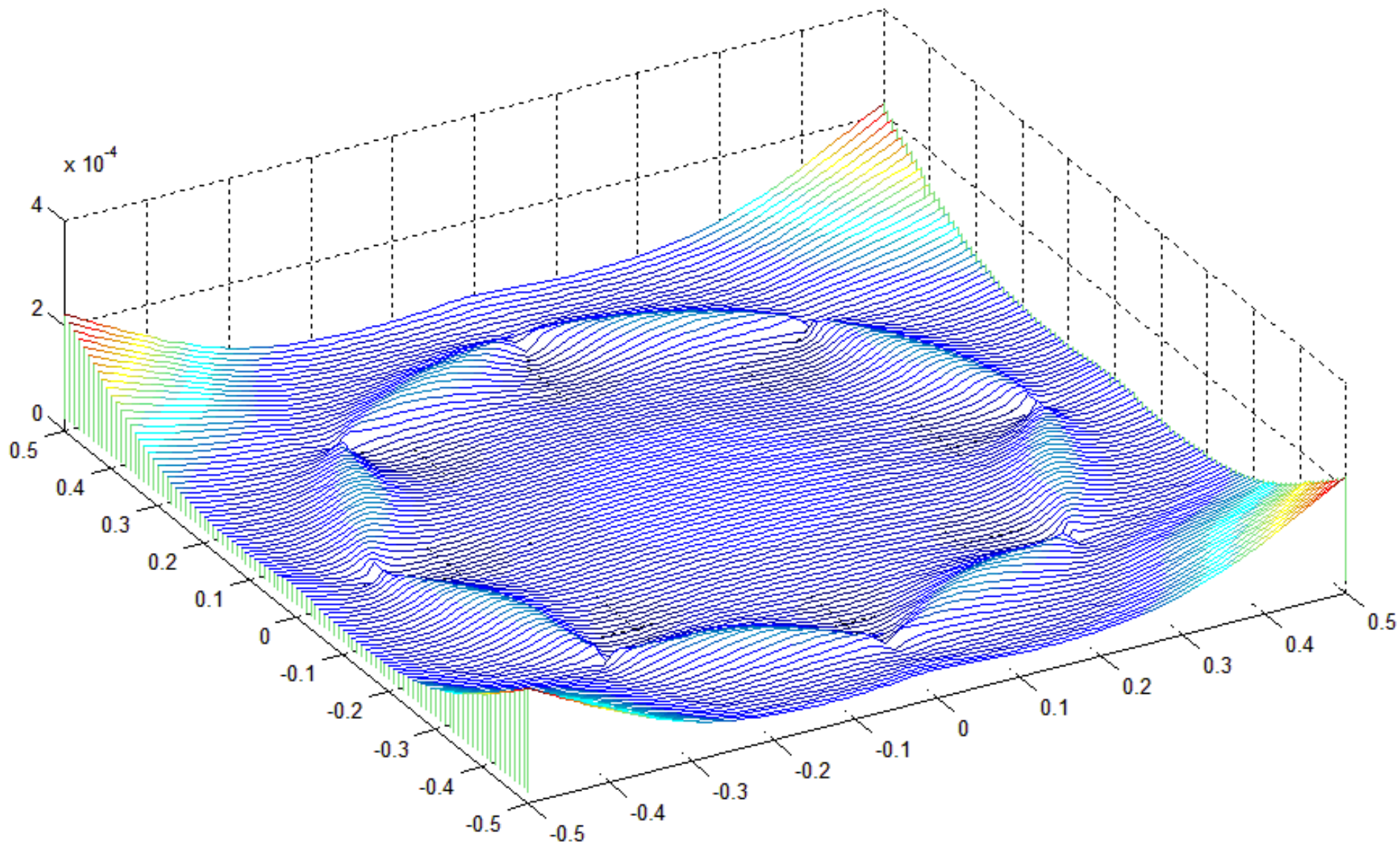
~~$$J_{IF} = \sum_{i=1}^N \frac{(A + 2\alpha) (xx)^2}{\alpha r_i^2 (K + r_i)^2}$$~~

Re-deployment strategies

- Constrained optimization







Range Measurements – A priori CRLB

$$J_{i,j} = -E_{M,L} \left(\frac{\partial^2 \ln(f(M, L))}{\partial X_i \partial X_j} \right)$$

$$J_{i,j} = J_{L_{i,j}} + J_{M_{i,j}} = -E_L \left(\frac{\partial^2 \ln(f(L))}{\partial X_i \partial X_j} \right) - E_L \left(E_{M|L} \left(\frac{\partial^2 \ln(f(M | L))}{\partial X_i \partial X_j} \right) \right)$$

CNAM

$$JL_{i,j} = \begin{cases} 2\lambda\pi & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases}$$

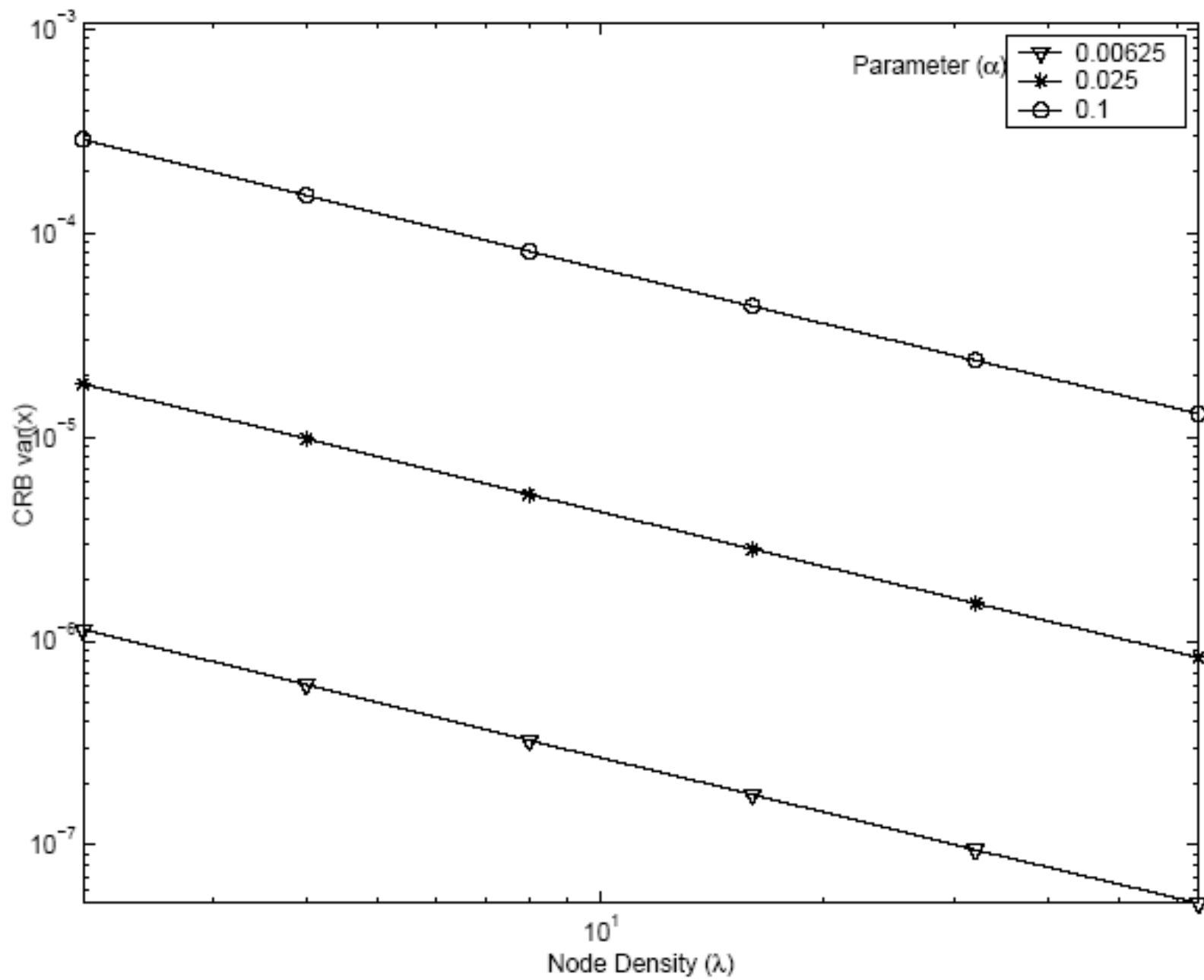
$$JM_{11} = JM_{22} = E_{\{r_k\}} \left[\sum_{k=1}^N \left(\frac{(2\pi)^N (1 + 2\alpha^2)}{2\alpha^2 (K + r_k)^2} \right) \right]$$

$$J_{ij} = \begin{cases} 2\lambda\pi + E_{\{r_k\}} \left(\sum_{k=1}^N \left(\frac{(2\pi)^N (1 + 2\alpha^2)}{2\alpha^2 (K + r_k)^2} \right) \right) & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases}$$

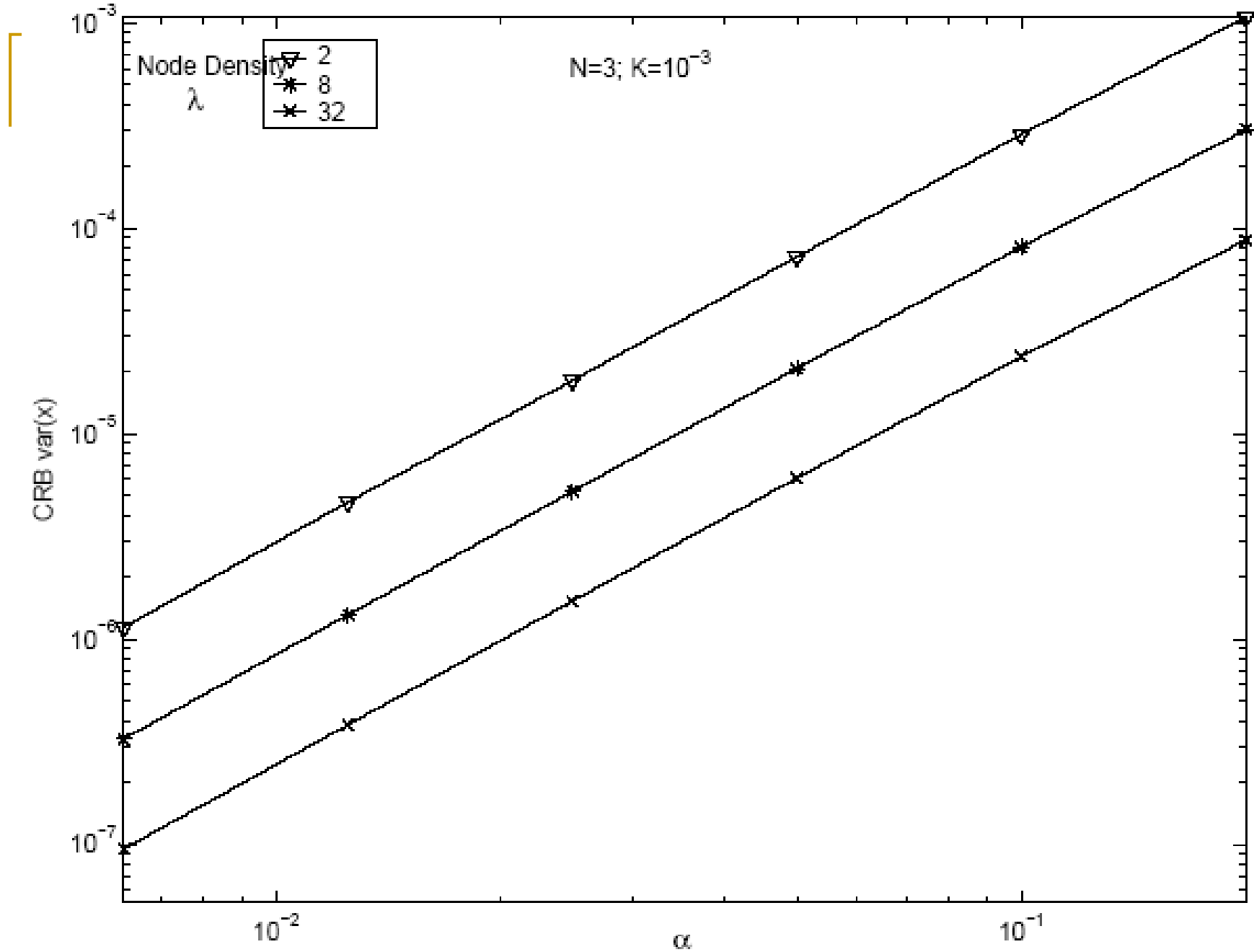
CNAM : $K = 0$

$$\bar{J}_{11} = \bar{J}_{22} = 2\lambda\pi + \frac{\lambda\pi(2\alpha^2 + 1)(\sum_{i=2}^{i=N} \frac{1}{i-1})}{2\alpha^2}$$

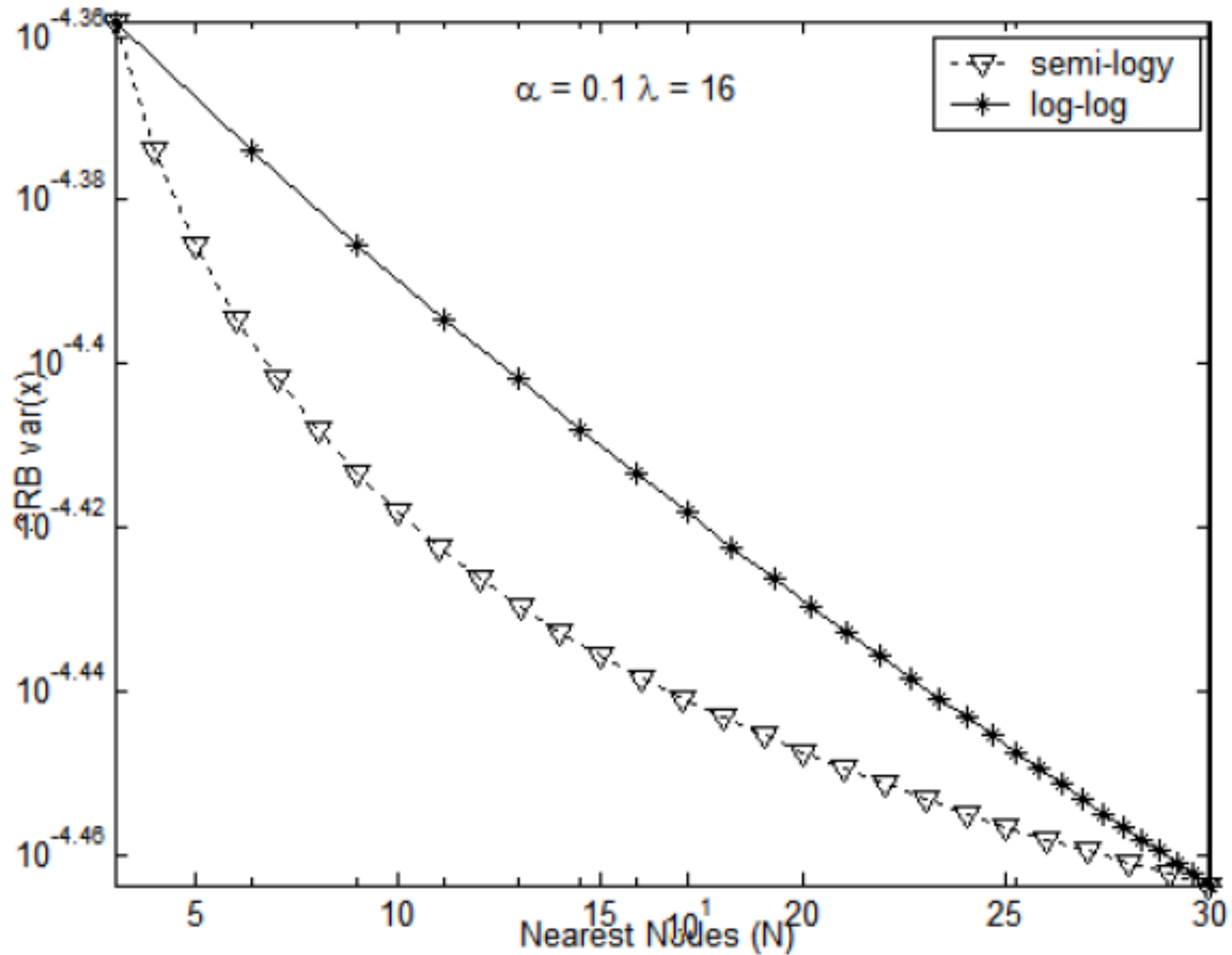




Plot of CRLB vs λ with α as the parameter

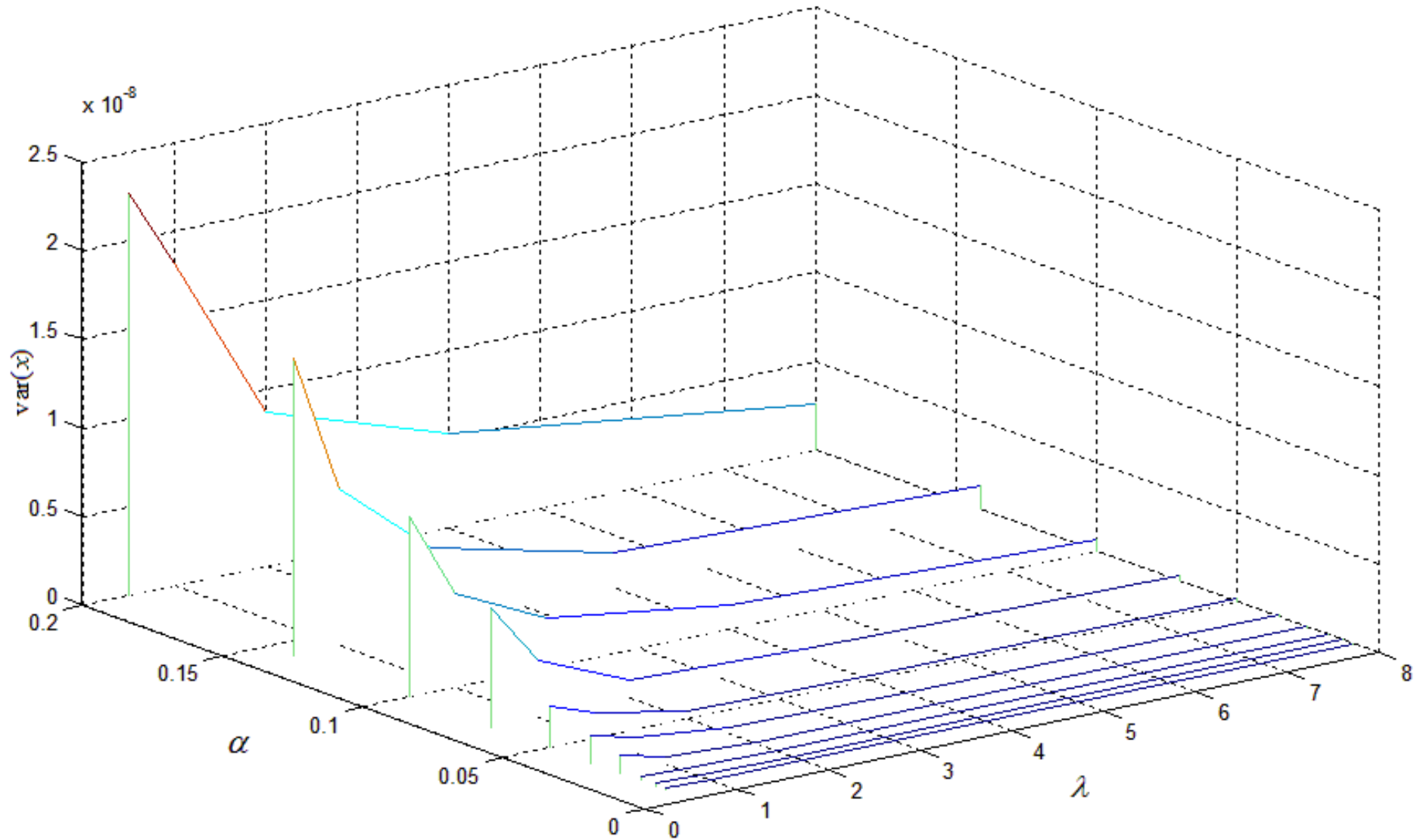


Plot of CRLB vs α with λ as the parameter



Plot of CRLB vs number of sensors used in localization (N)

CRLB 3D plot for the nearest 6 nodes as a function of α and λ



FRAM

$$\begin{aligned} J_{i,j} &= -E_N \left(E_{M,L} \left[\frac{\partial^2 \ln(f(M, L, N))}{\partial X_i \partial X_j} \right] \right) \\ &= -E_N \left(E_{M,L} \left(\frac{\partial^2 \ln(f(N))}{\partial X_i \partial X_j} + \frac{\partial^2 \ln(f(L | N))}{\partial X_i \partial X_j} + \frac{\partial^2 \ln(f(M | L, N))}{\partial X_i \partial X_j} \right) \right) \\ &= E_N (JN_{ij} + JL_{ij} + JM_{ij}) \end{aligned}$$

$$J_{ij} = \begin{cases} \frac{\lambda\pi(1+2\alpha^2)}{\alpha^2} \left[\log \left(\frac{K+R}{R} \right) - \frac{R}{K+R} \right] & \text{for } i = j \\ 0 & \text{for } i \neq j \end{cases}$$

$$CRLB_x = CRLB_y = \frac{\alpha^2}{\lambda\pi(1+2\alpha^2)} \left[\log \left(\frac{K+R}{R} \right) - \frac{R}{K+R} \right]^{-1}$$

Error Comparisons

$$(x_s - x_i)^2 + (y_s - y_i)^2 = m_i^2$$

$$2x_s(x_1 - x_i) + 2y_s(y_1 - y_i) = (m_i^2 - m_1^2) - ((x_i^2 + y_i^2) - (x_1^2 + y_1^2))$$

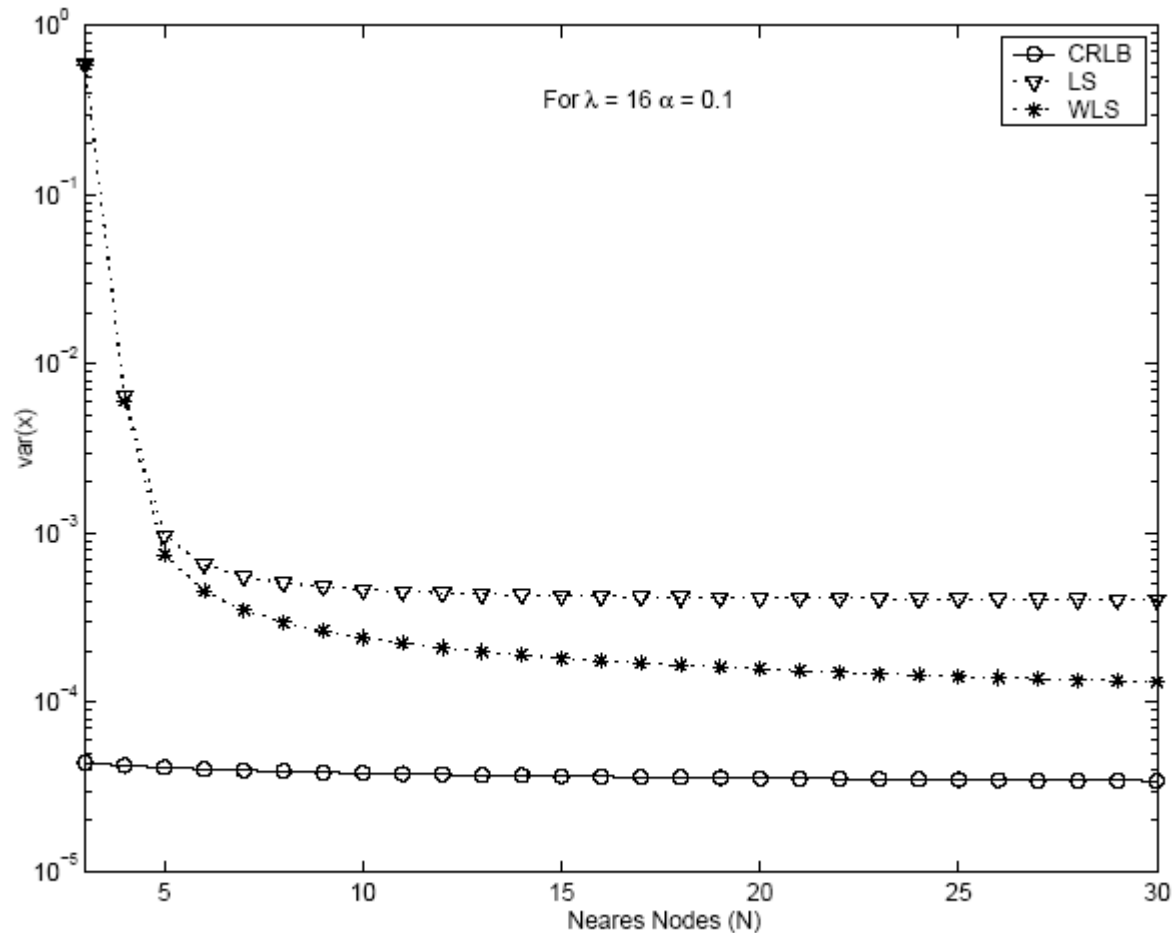
$$AX = B + \epsilon$$

$$\hat{X} = (A^T A)^{-1} A^T B \quad \text{LSE}$$

$$\hat{X} = (A^T W^{-1} A)^{-1} A^T W^{-1} B \quad \text{WLSE}$$

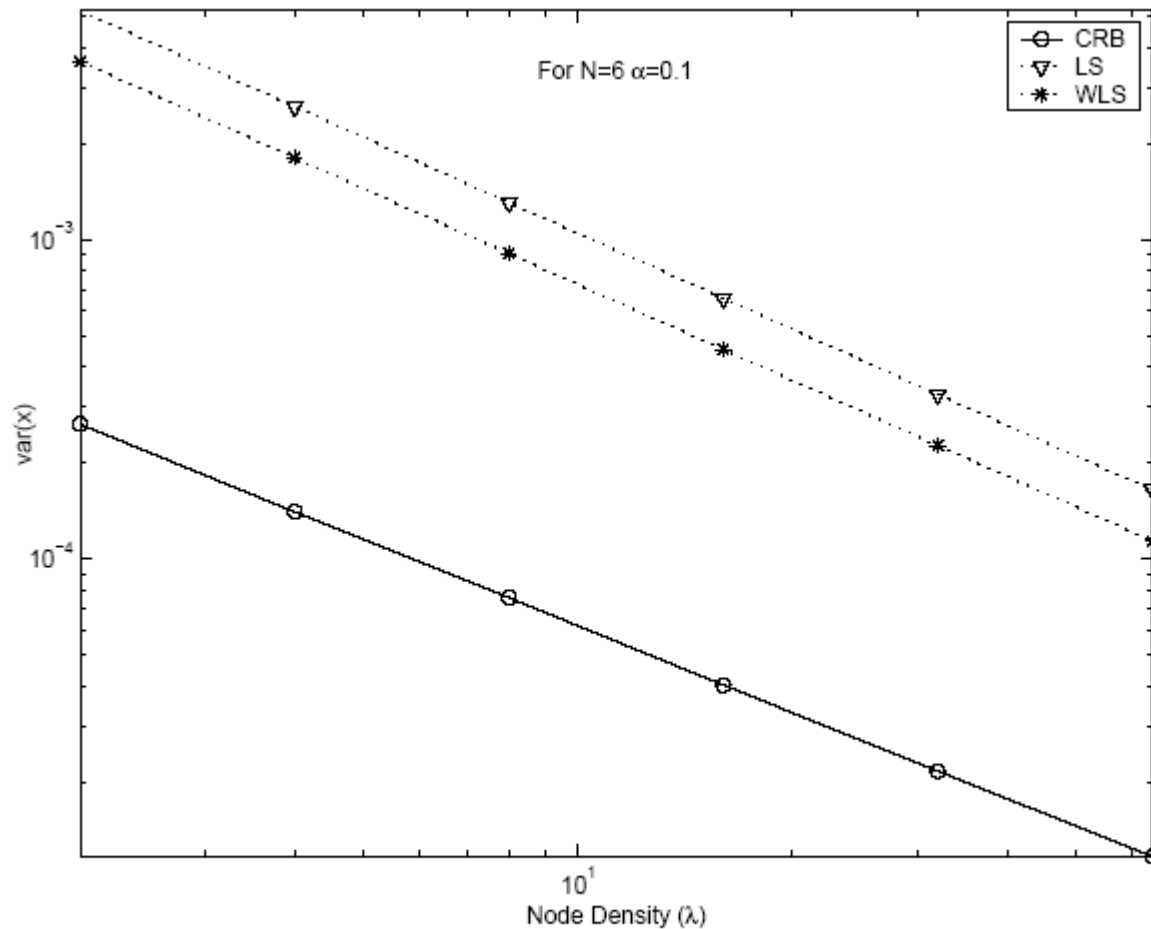
$$W = \text{Cov}(\epsilon\epsilon^T)$$

Error Comparisons



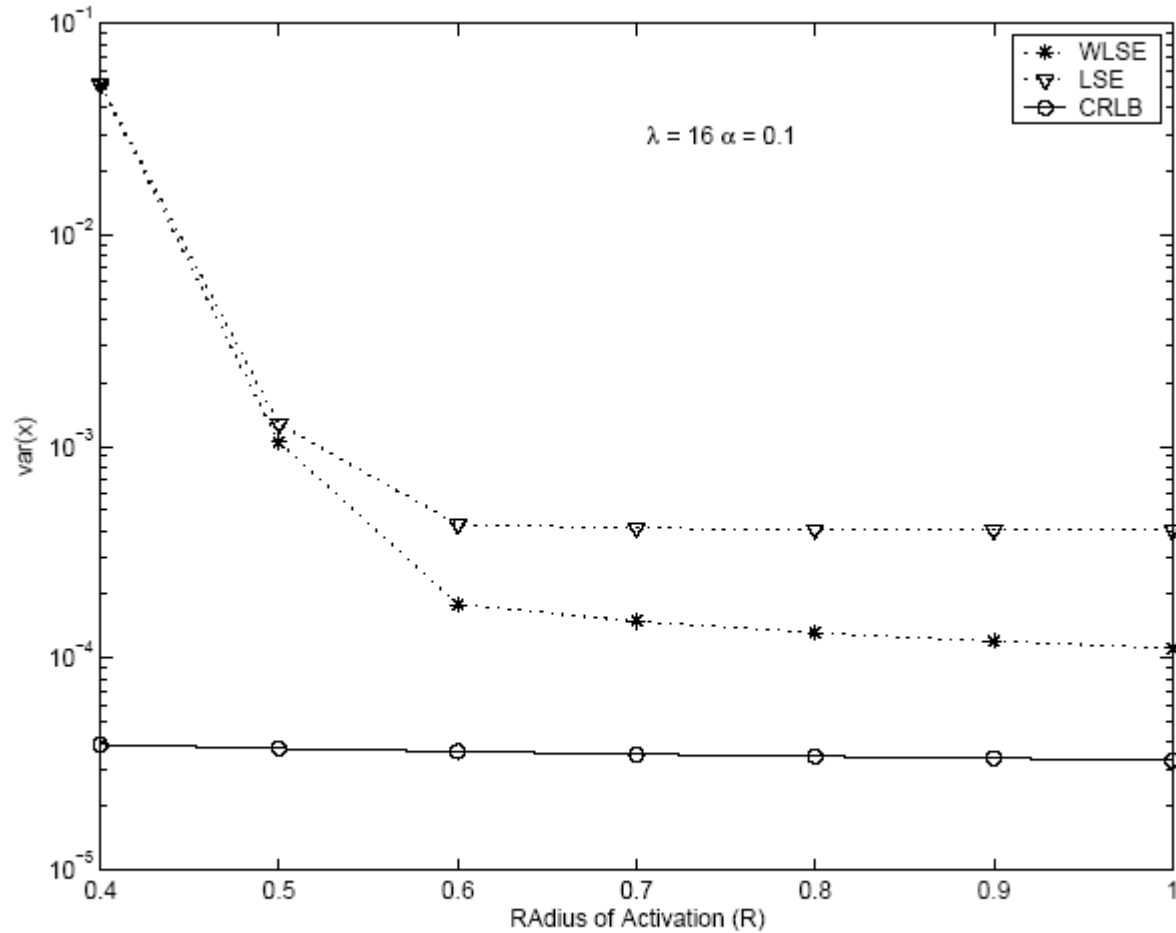
CNAM : Comparison of LS and WLS estimators with CRLB

Error Comparisons



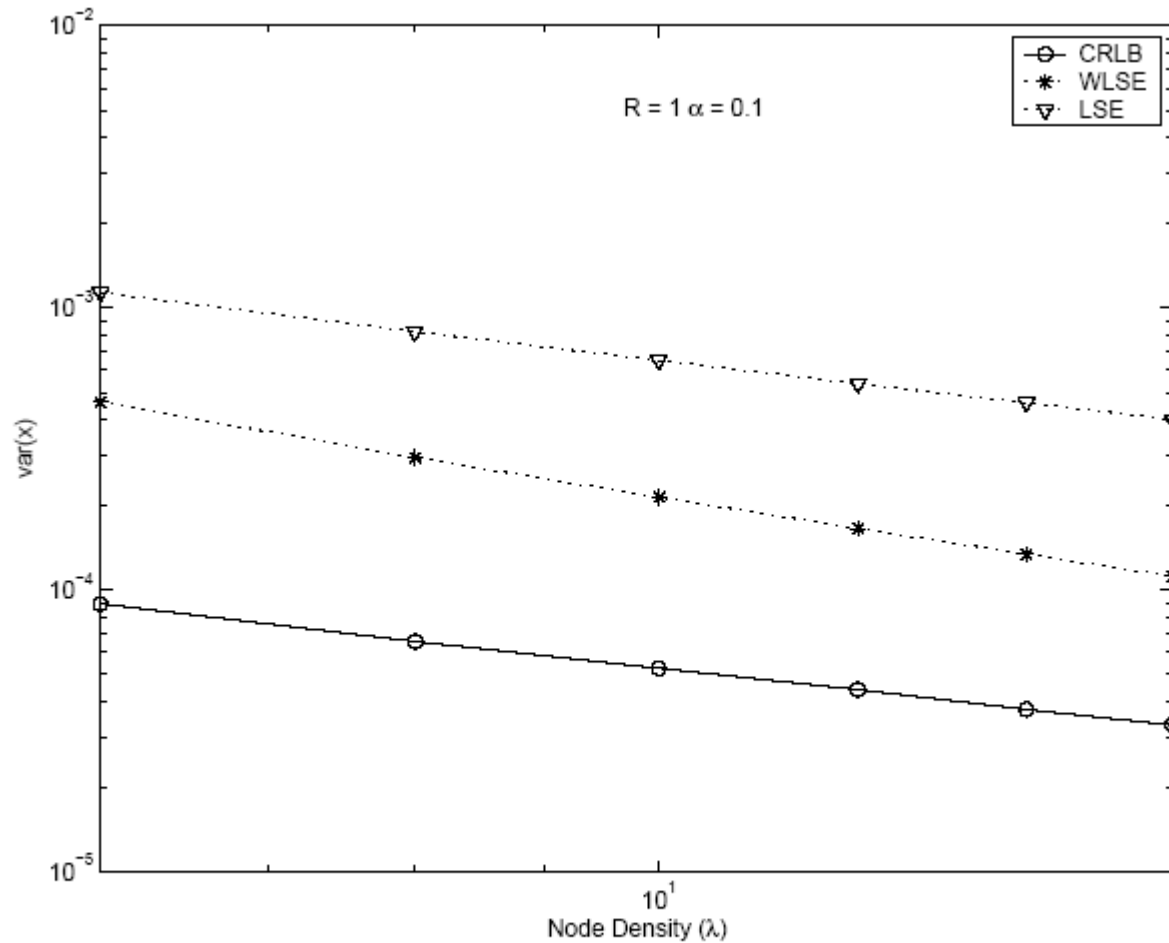
CNAM : Comparison of LS and WLS estimators with CRLB for varying λ

Error Comparisons



FRAM : Comparison of LS and WLS estimators with CRLB as localizing radius R varies

Error Comparisons



FRAM : Comparison of LS and WLS estimators with CRLB for varying λ

Range Difference Measurements

- Joint Gaussian model

$$f(m_{21}, m_{31}, \dots, m_{N1} | (x_1, y_1), \dots, (x_N, y_N); (x_s, y_s)) = \frac{1}{(\sqrt{2\pi})^N \det(C)} e^{(-\frac{1}{2}[\bar{m}-\bar{r}]^T C^{-1}[\bar{m}-\bar{r}])}$$

$$C = \frac{\sigma^2}{2} \begin{pmatrix} 2 & 1 & 1 & \dots \\ 1 & 2 & 1 & \dots \\ 1 & 1 & 2 & \dots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}_{(N-1) \times (N-1)}$$

$$P_{ij} = \begin{cases} \frac{2(N-1)}{N\sigma^2} & \text{if } i = j \\ \frac{2(-1)}{N\sigma^2} & \text{if } i \neq j \end{cases}$$

Joint Gaussian model

- CNAM

$$CRLB_x = CRLB_y = \frac{1}{2\lambda\pi + \frac{2(\lambda A - 1)}{\sigma^2}}$$

- FRAM

$$CRLB_x = CRLB_y = \frac{\sigma^2}{2(\lambda A - 1)}$$

Derived Gaussian model

- CNAM

$$CRLB_x = CRLB_y = \frac{1}{2\lambda\pi + \frac{(\lambda A - 1)}{2\sigma^2}}$$

- FRAM

$$CRLB_x = CRLB_y = \frac{2\sigma^2}{(\lambda A - 1)}$$

Error Comparisons

$$AX = B + \epsilon$$

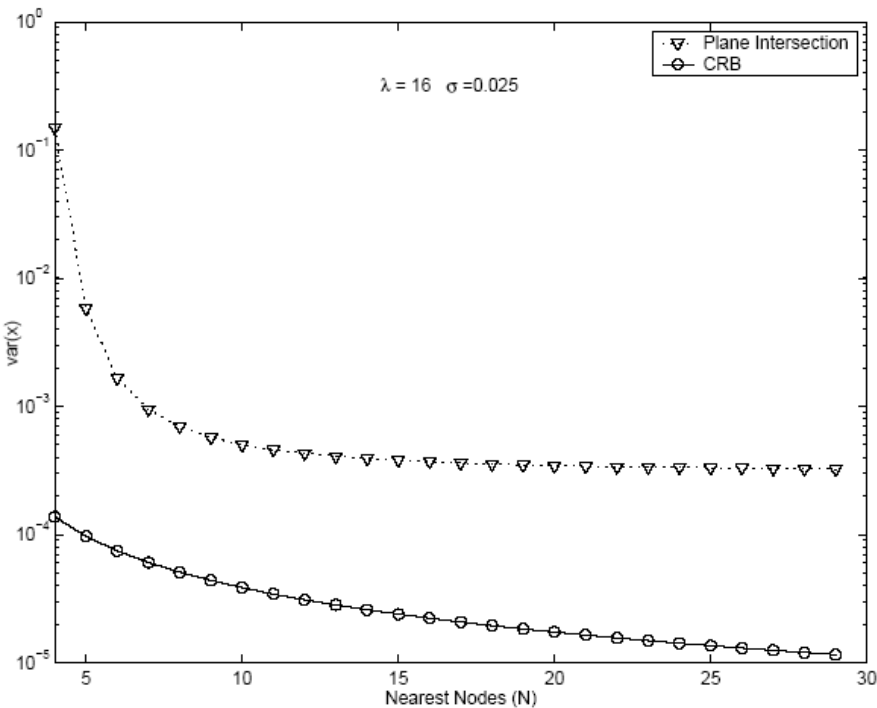
$$A(i, 1) = x_1 m_{2i} + x_2 m_{i1} + x_i m_{12}$$

$$A(i, 2) = y_1 m_{2i} + y_2 m_{i1} + y_i m_{12}$$

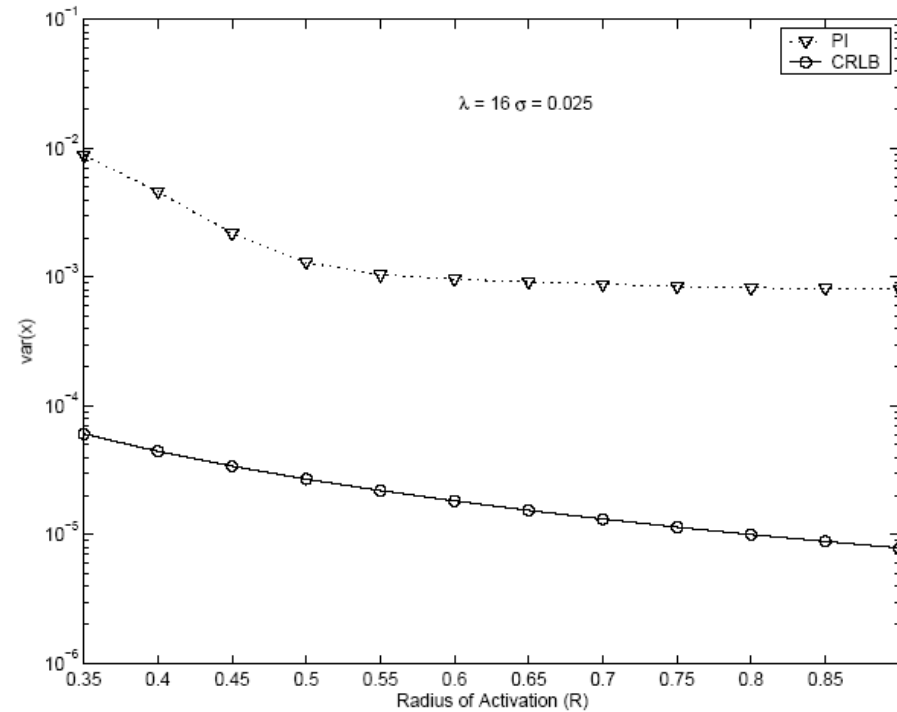
$$B(i, 1) = \frac{1}{2} (m_{12} m_{2i} m_{i1} + m_{2i} (x_1^2 + y_1^2) + m_{i1} (x_2^2 + y_2^2) + m_{12} (x_i^2 + y_i^2))$$

$$\hat{X} = (A^T A)^{-1} A^T B \quad \text{LSE}$$

Error Comparisons – Joint Gaussian Model

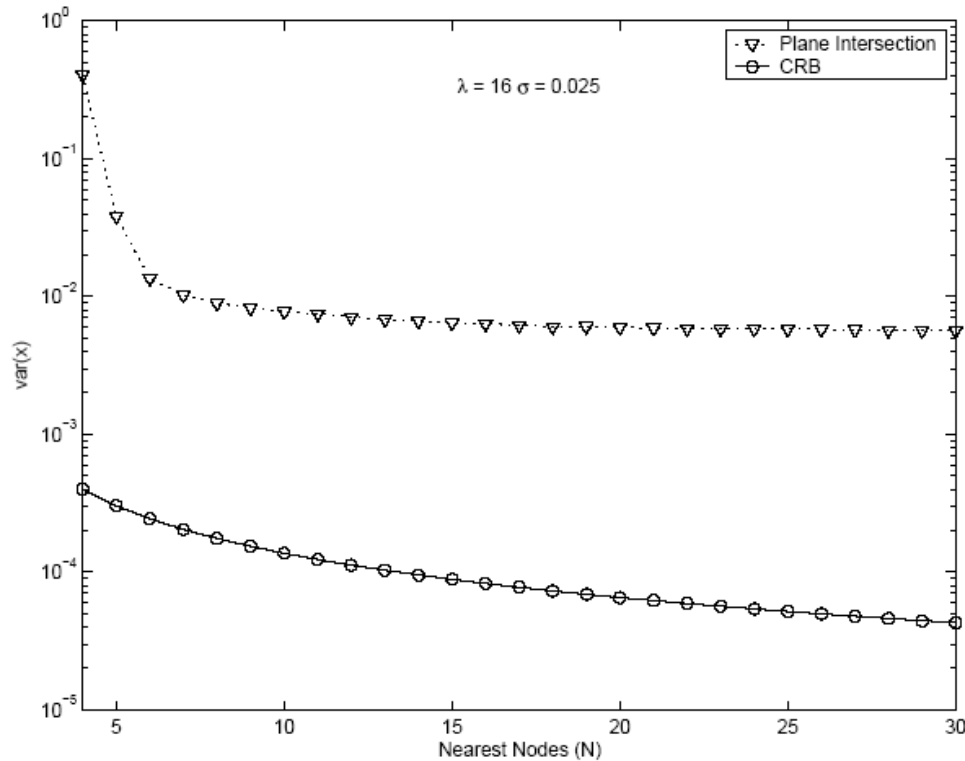


CNAM : Comparison of PI estimator with CRLB as localizing nodes N varies for Joint Gaussian Error Model

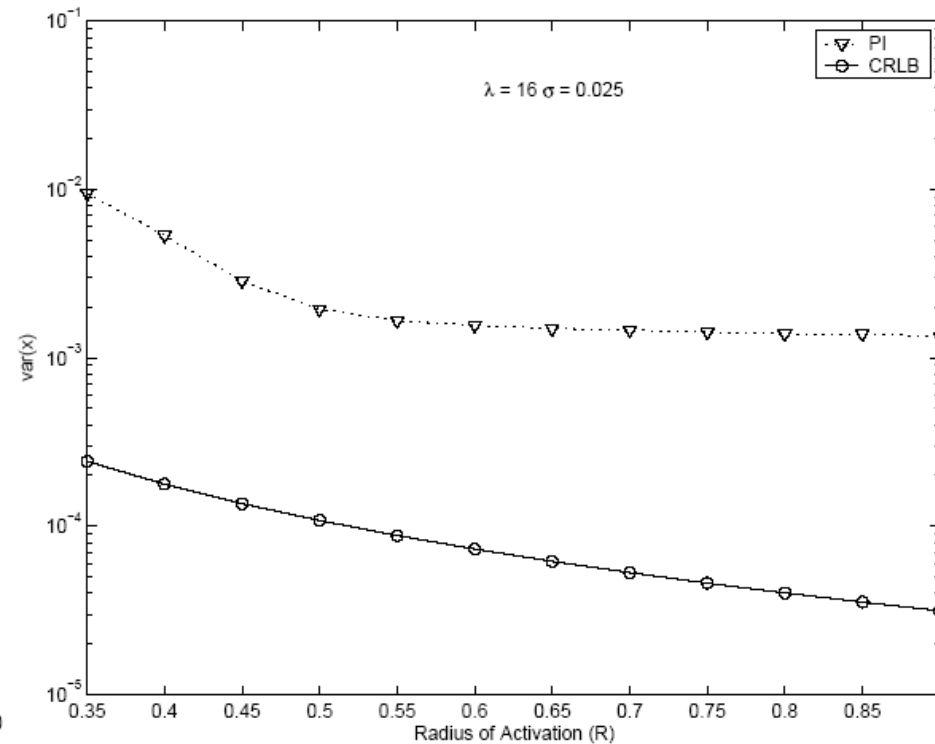


FRAM : Comparison of PI estimator with CRLB as activation radius R varies for Joint Gaussian Error Model

Error Comparisons – Derived Gaussian Model



CNAM : Comparison of PI estimator with CRLB as localizing nodes N varies for Derived Gaussian Error Model



FRAM : Comparison of PI estimator with CRLB as activation radius R varies for Derived Gaussian Error Model

Thank You
