Localization Error Analysis in Wireless Sensor Networks

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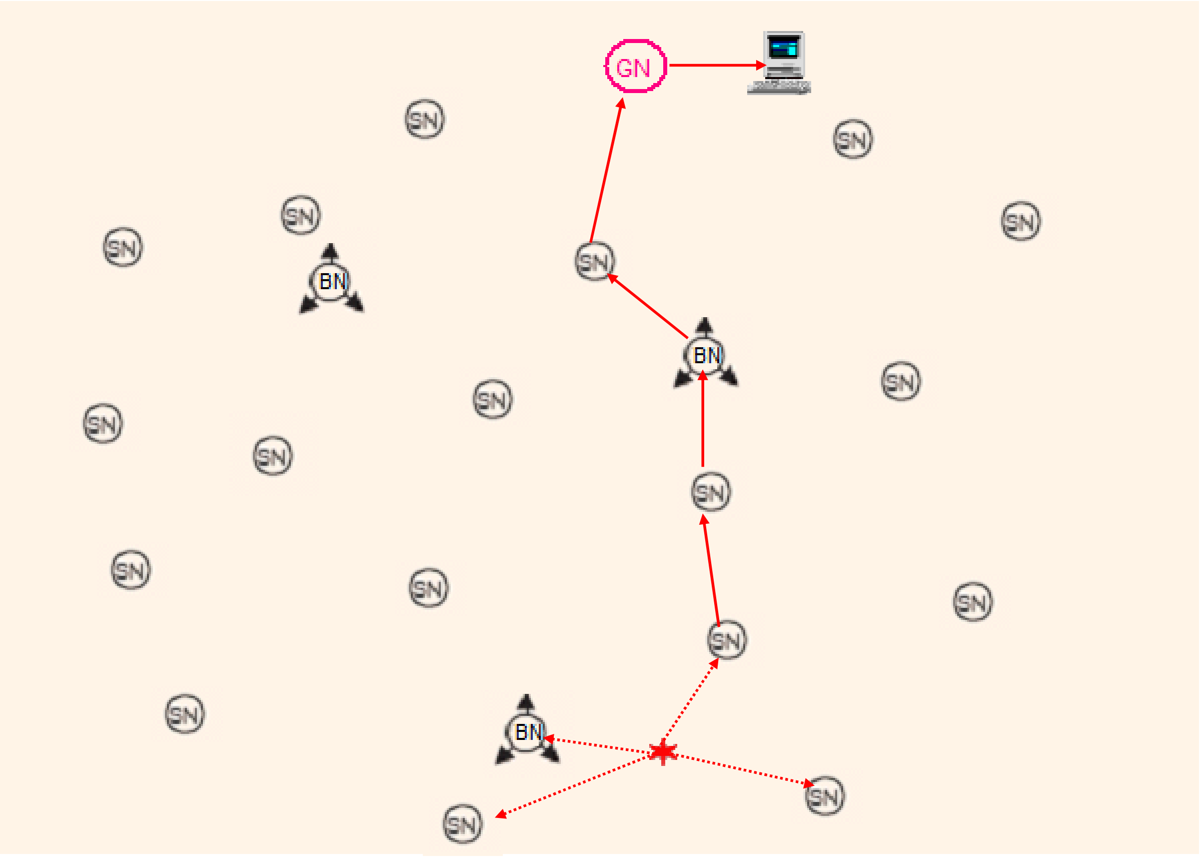
What is a Wireless Sensor Network ?

- Entities
  - Sensor Nodes
  - Beacon Nodes
  - Gateway Nodes
- Function
  - Sample
  - Process
  - Communicate



Wireless Sensor Networks

- Why WSNs ?

  Small size, Low cost, High Reliability and Accessibility
- Unique challenges

  Power, Random Deployment, Unreliable Communication
- Applications

  Habitat monitoring, Battle fields, Surveillance, Nuclear power plants, etc.
- Functions

  Parameter measurement, Target localization and tracking.

Localization

- Node Localization
- Source Localization
- Differences
  - Cooperative/Non-cooperative
  - Node Density
  - Computational Complexity

Localization – A generic definition

- Given a set of entities (nodes) with known locations and a source entity, the problem is to estimate the location of the source.
- Location aware --- Sensor node
- Location unaware --- Source

Localization

- Measurements Modalities
  - Received Signal Strength (RSS)
  - Time of Flight (TOF)
  - Time of Arrival (TOA)
  - Time Difference of Arrival (TDOA)
  - Direction of Arrival (DOA)
- Measurements
  - Range
  - Range Difference

Localization Algorithms

- Mostly non iterative
- Range Difference
  - Locus is a hyperbola
  - At least four nodes are required
  - Least Square Estimation
- Range
  - Locus is a circle
  - At least three nodes are required
  - Also Least Square Estimation

Localization Error

- Network Parameters
  - Node density
  - Available energy resources
  - Circuit noise
  - Location errors
- Environmental Parameters
  - Sensing modality and its propagation model
  - Terrain’s geographical topology
  - Ambient noise levels

Problem Formulation

- *To characterize the localization error with respect to the network and environmental parameters in an algorithm independent manner*

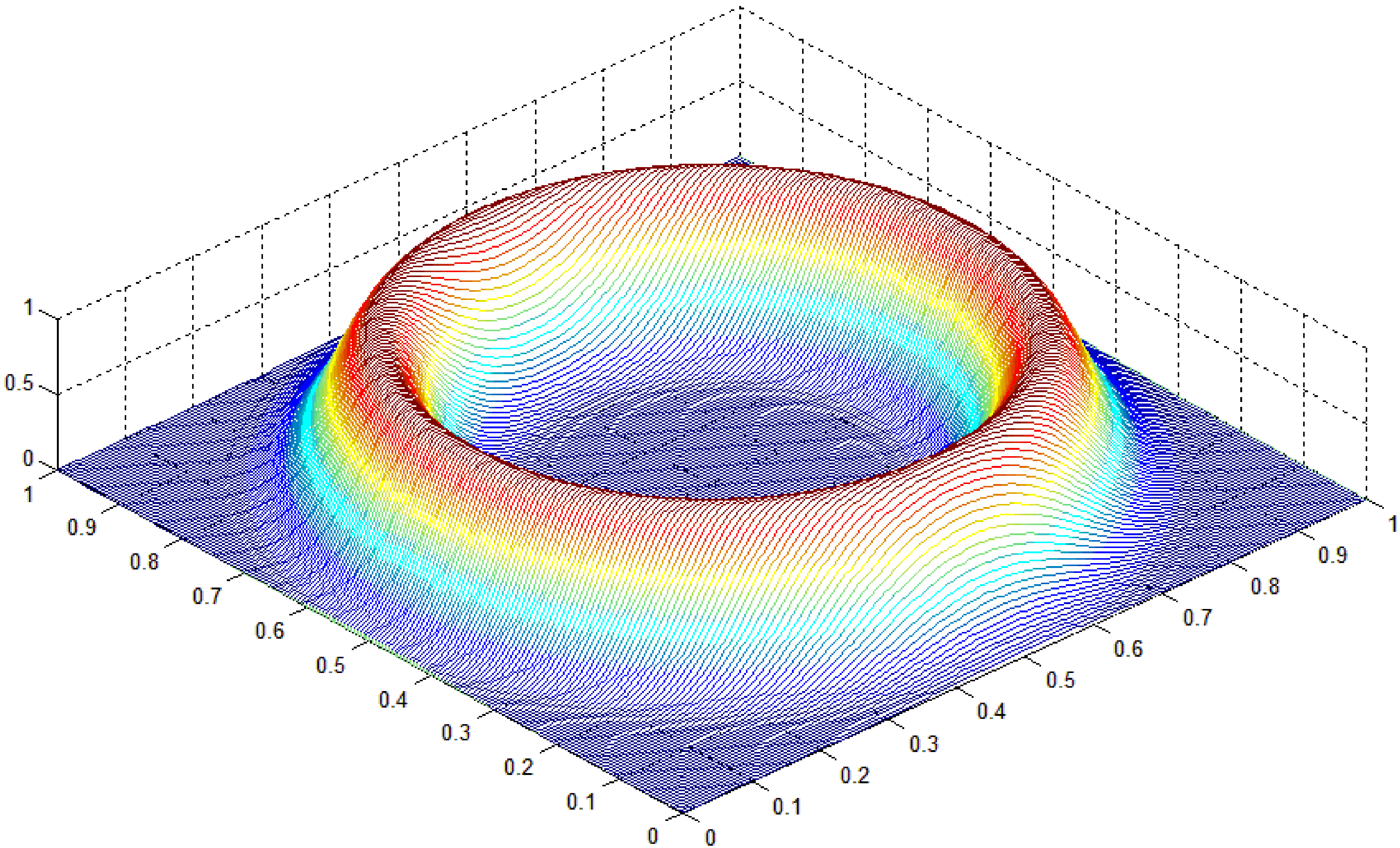
Notation

- (x_s , y_s) --- source location
- $(\hat{x}_s, \hat{y}_s)$ --- estimated source location
- (x_i , y_i) --- i_{th} sensor node
- r_i --- distance between the source and i_{th} node
- m_i --- range measurement at the i_{th} sensor node.
- m_{ij} --- range-difference measurement between i_{th} and the j_{th} sensor nodes

Error models

- Range Measurements
  - Gaussian error [illegible]

$$f\left(m \mid (x,y);\ (x_i,y_i)\right) = \frac{1}{\sqrt{2\pi\sigma}} \exp\left(\frac{\left(m - \sqrt{(x-x_i)^2 + (y-y_i)^2} \right)^2}{2\sigma^2} \right)$$



1
0.5
0
1
0.9
0.8
0.7
0.6
0.5
0.4
0.3
0.2
0.1
0
0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
1

Error models

- Range Difference Measurements
  - Joint Gaussian error

$$f\left(m_{21}, m_{31}, \dots, m_{N1} \mid (x_1, y_1), (x_2, y_2), \dots, (x_N, y_N); (x_s, y_s)\right) =$$

$$\frac{1}{\left(\sqrt{2\pi}\right)^{N} \det(C)} \exp\left(-\frac{1}{2}\left[\bar{m}-\bar{r}\right]^{T} C^{-1}\left[\bar{m}-\bar{r}\right]\right)$$

$$\bar{m} = \left[m_{21} \dots m_{N1}\right] \qquad \bar{r} = \left[r_{21} \dots r_{N1}\right]$$

Error models

- Range Difference Measurements
  - Derived Gaussian error

$$f(m_{21}, m_{31}, \ldots, m_{N1} | (x_1, y_1), \ldots, (x_N, y_N); (x_s, y_s))$$

$$= c\ exp\left\{ \left(\frac{(r_1 - (m_2 - r_2) - (m_3 - r_3) - \ldots - (m_N - r_N))^2}{2N\sigma^2} \right) - \left(\frac{{r_1}^2 + (m_2 - r_2)^2 + (m_3 - r_3)^2 + \ldots + (m_N - r_N)^2}{2\sigma^2} \right) \right\}$$

Data Collection Techniques

- Closest N Activation Model (CNAM)
- Fixed Radius Activation Model (FRAM)

Post-deployment and a priori error performance

- Given sensor network
- Random network (Poisson points)
  - CNAM $f((x_1, y_1), \ldots, (x_N, y_N)) = (\lambda)^N e^{-\lambda\pi\left((x_s - x_N)^2 + (y_s - y_N)^2\right)}$
  - FRAM $f((x_1, y_1), (x_2, y_2), \ldots, (x_N, y_N) | N) = \frac{1}{(\pi R^2)^N}$

Cramer-Rao Lower Bound

[illegible]

$$J_{ij} = E\left[\frac{\partial^2 \log(f)}{\partial\theta_i \partial\theta_j}\right]$$

Cramer-Rao Lower Bound

$$\begin{bmatrix} \mathrm{var}(x) & \mathrm{cov}(x,y) \\ \mathrm{cov}(x,y) & \mathrm{var}(y) \end{bmatrix} \geq \frac{1}{J_{11}J_{22} - J_{12}J_{21}} \begin{bmatrix} J_{22} & -J_{12} \\ -J_{21} & J_{11} \end{bmatrix}$$

$$J_{11} = -E\left[\frac{\partial^2 \log f}{\partial x^2}\right] \qquad J_{22} = -E\left[\frac{\partial^2 \log f}{\partial y^2}\right]$$

$$J_{12} = -E\left[\frac{\partial^2 \log f}{\partial x \partial y}\right] = J_{21}$$

Range Measurements – Post-deployment CRLB

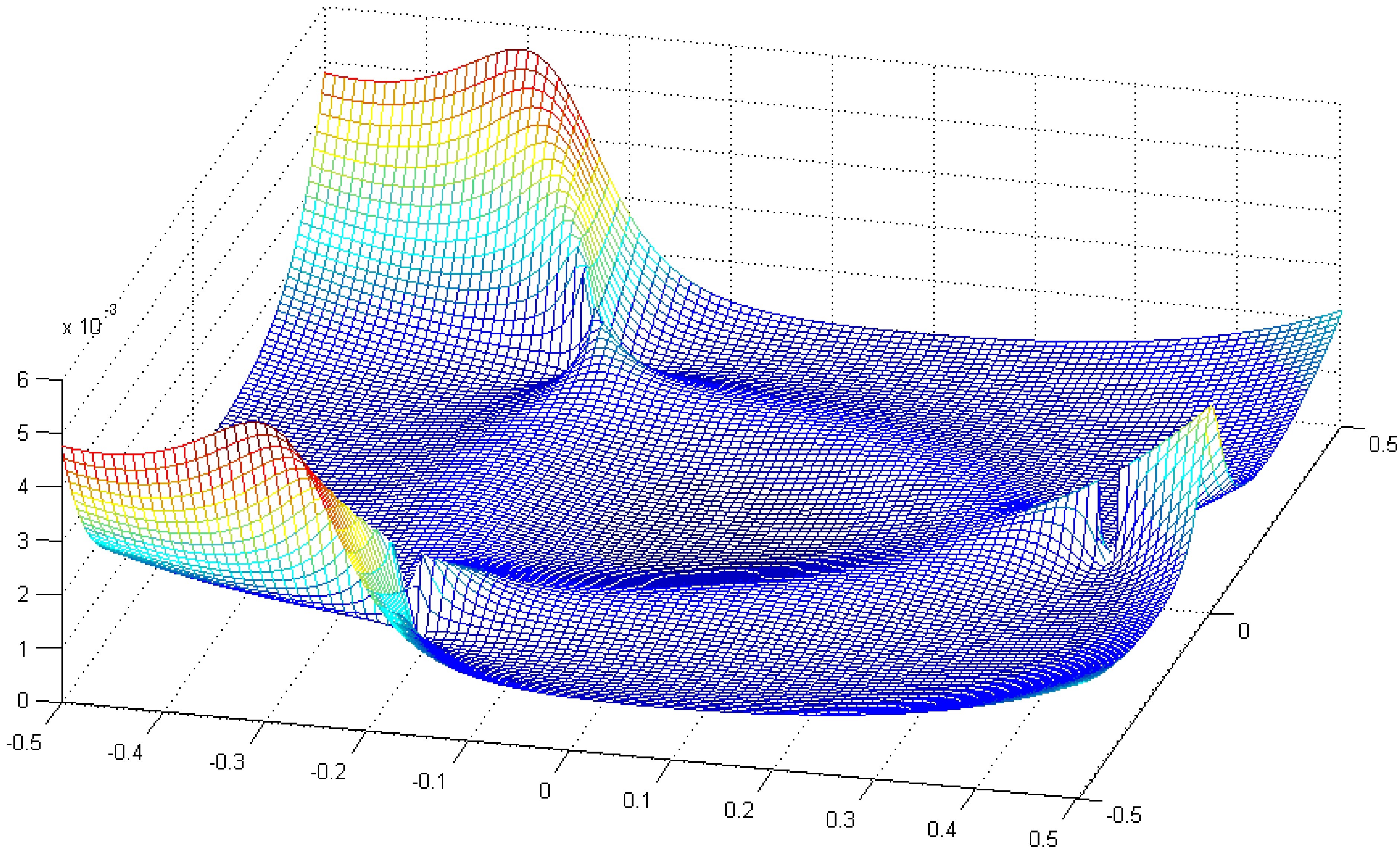
$$J_{11} = \sum_{i=1}^{N} \frac{(1+2\alpha^2)(x_s - x_i)^2}{\alpha^2 r_i^2 (K + r_i)^2}$$

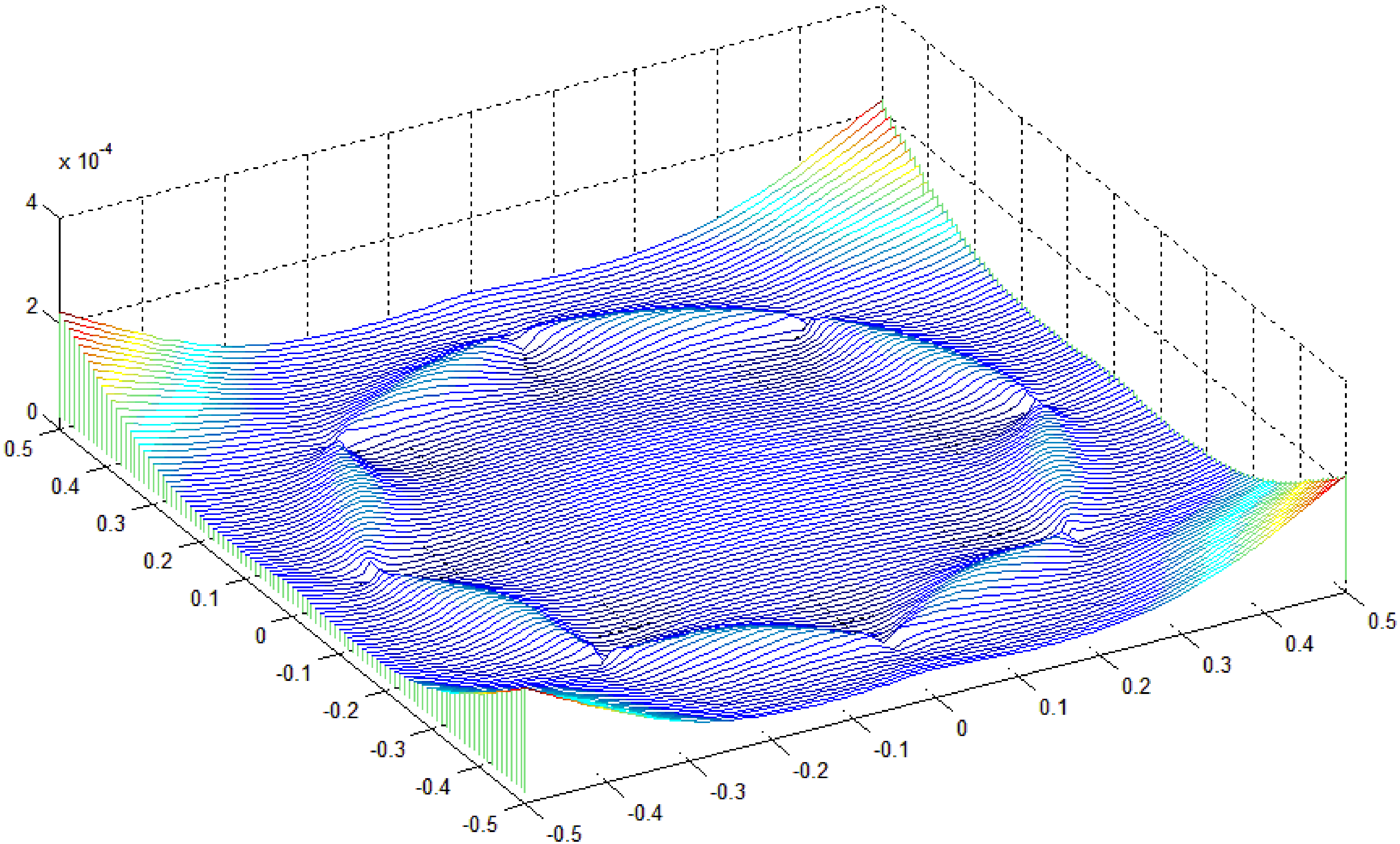
$$J_{12} = \sum_{i=1}^{N} \frac{(1+2\alpha^2)(x_s - x_i)(y_s - y_i)}{\alpha^2 r_i^2 (K + r_i)^2}$$

$$J_{22} = \sum_{i=1}^{N} \frac{(1+2\alpha^2)(y_s - y_i)^2}{\alpha^2 r_i^2 (K + r_i)^2}$$

Re-deployment stratagies

- Constrained optimization


x 10^-3
6
5
4
3
2
1
0
-0.5
-0.4
-0.3
-0.2
-0.1
0
0.1
0.2
0.3
0.4
0.5
-0.5
0
0.5



x 10^-4
4
2
0
0.5
0.4
0.3
0.2
0.1
0
-0.1
-0.2
-0.3
-0.4
-0.5
-0.5
-0.4
-0.3
-0.2
-0.1
0
0.1
0.2
0.3
0.4
0.5

Range Measurements – A priori CRLB

$$J_{i,j} = -E_{M,L}\left(\frac{\partial^2 \ln(f(M,L))}{\partial X_i \partial X_j}\right)$$

$$J_{i,j} = JL_{i,j} + JM_{i,j} = -E_L\left(\frac{\partial^2 \ln(f(L))}{\partial X_i \partial X_j}\right) - E_L\left(E_{M|L}\left(\frac{\partial^2 \ln(f(M \mid L))}{\partial X_i \partial X_j}\right)\right)$$

CNAM

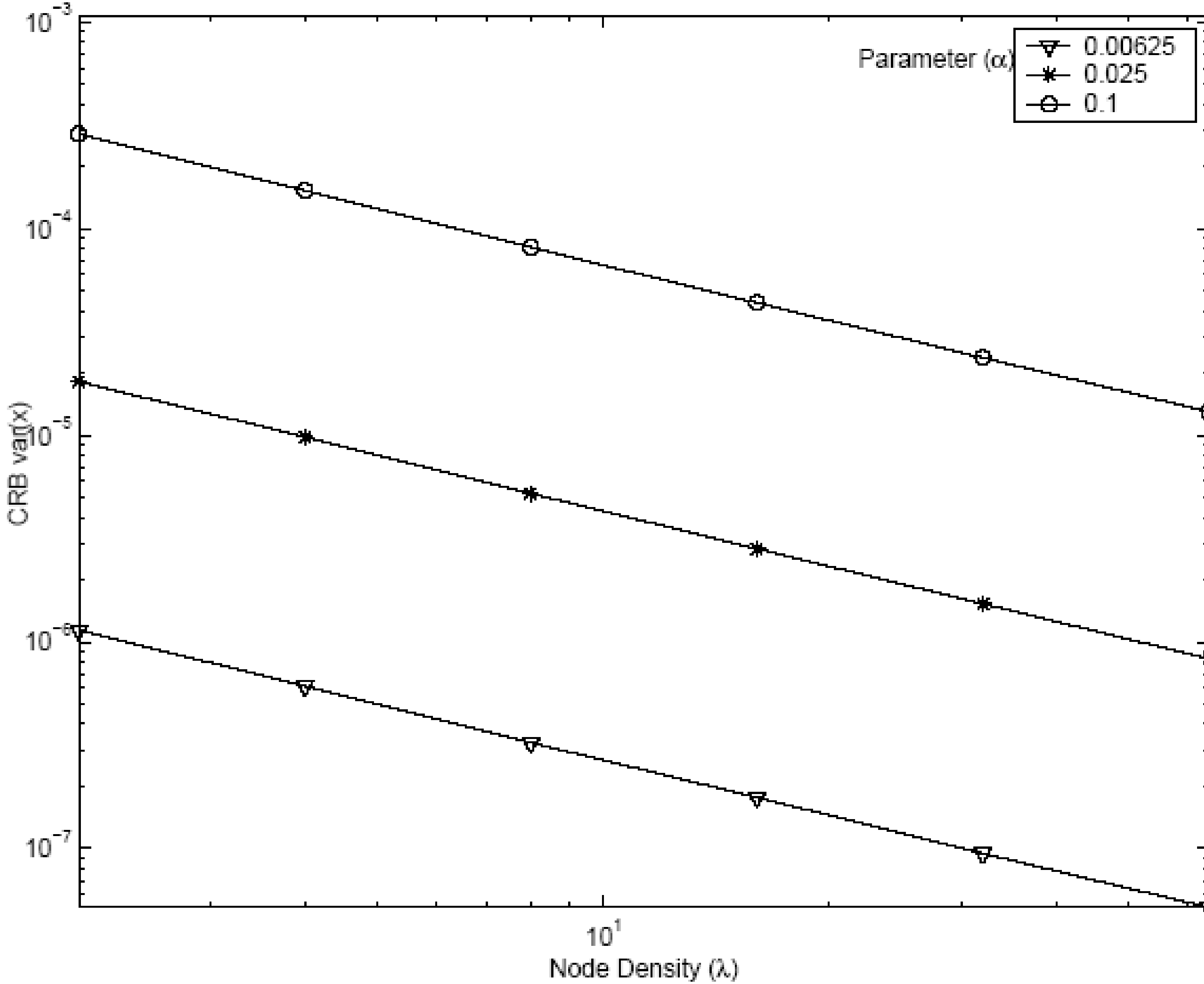
$$JL_{i,j} = \begin{cases} 2\lambda\pi & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases}$$

$$JM_{11} = JM_{22} = E_{\{r_k\}}\left[\sum_{k=1}^{\mathrm{N}} \left(\frac{(2\pi)^N(1+2\alpha^2)}{2\alpha^2(K+r_k)^2}\right)\right]$$

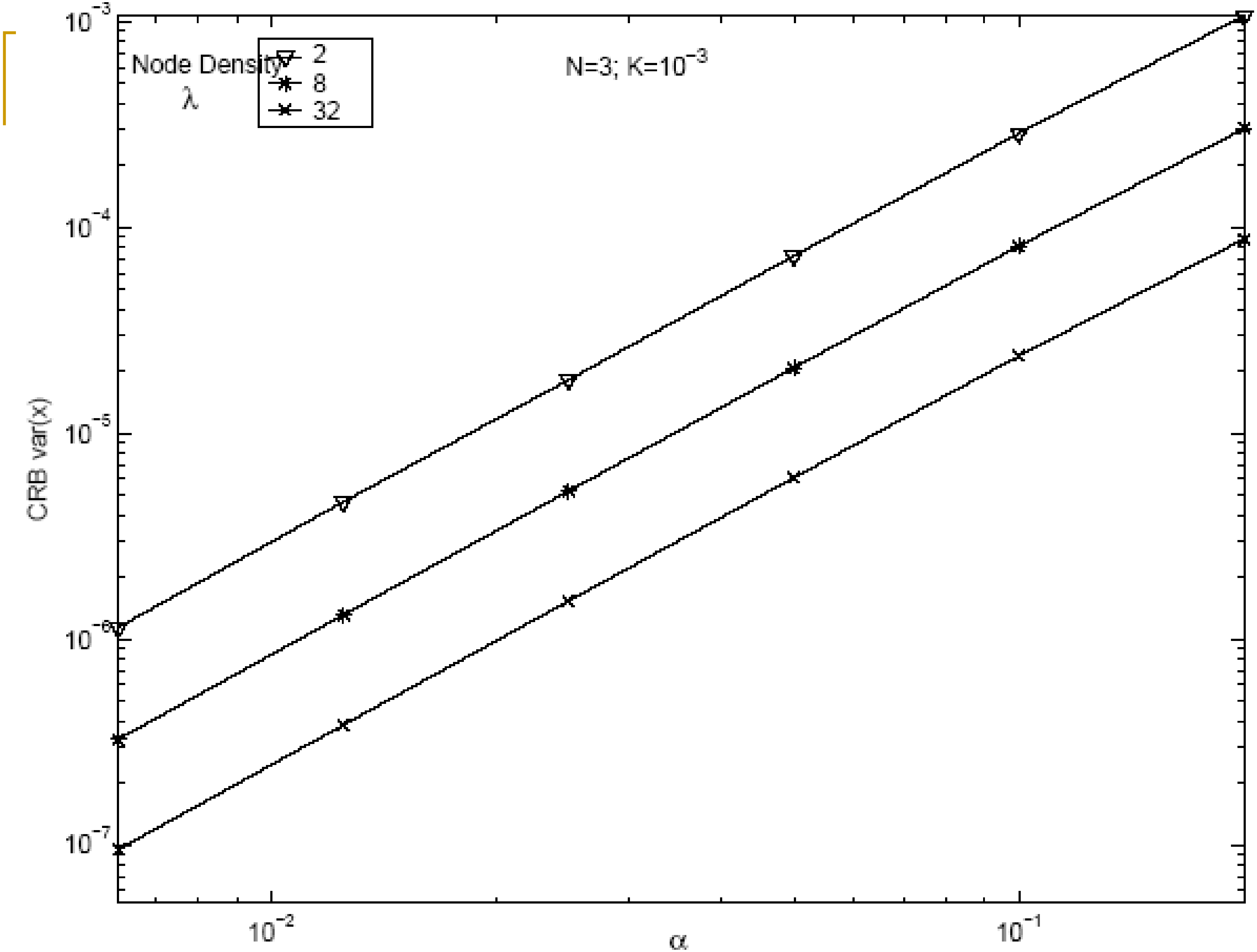
$$J_{ij} = \begin{cases} 2\lambda\pi + E_{\{r_k\}}\left(\sum_{k=1}^{\mathrm{N}}\left(\frac{(2\pi)^N(1+2\alpha^2)}{2\alpha^2(K+r_k)^2}\right)\right) & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases}$$

CNAM : K = 0

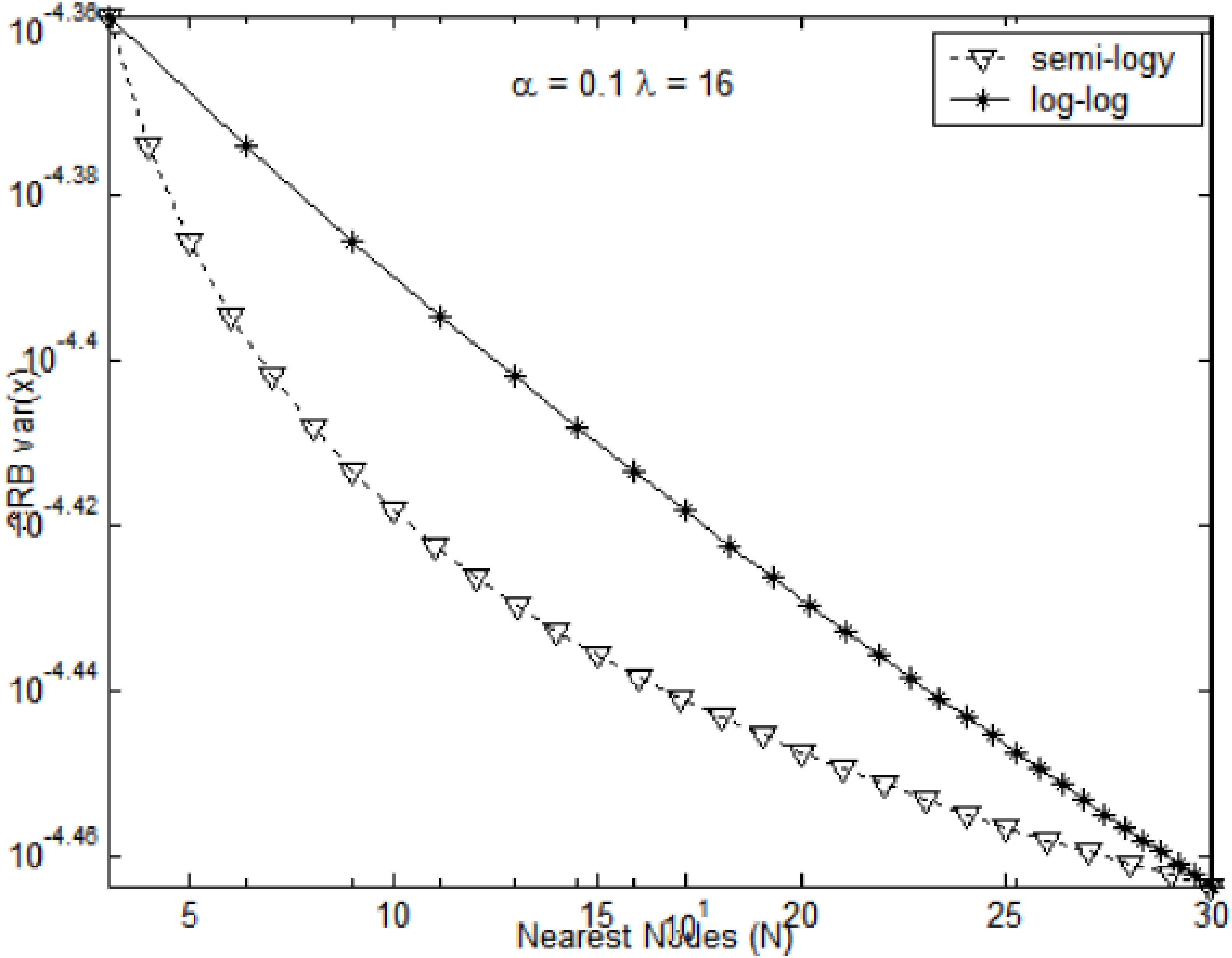
$$\overline{J}_{11} = \overline{J}_{22} = 2\lambda\pi + \frac{\lambda\pi(2\alpha^2 + 1)(\sum_{i=2}^{i=N} \frac{1}{i-1})}{2\alpha^2}$$



Plot of CRLB vs λ with α as the parameter

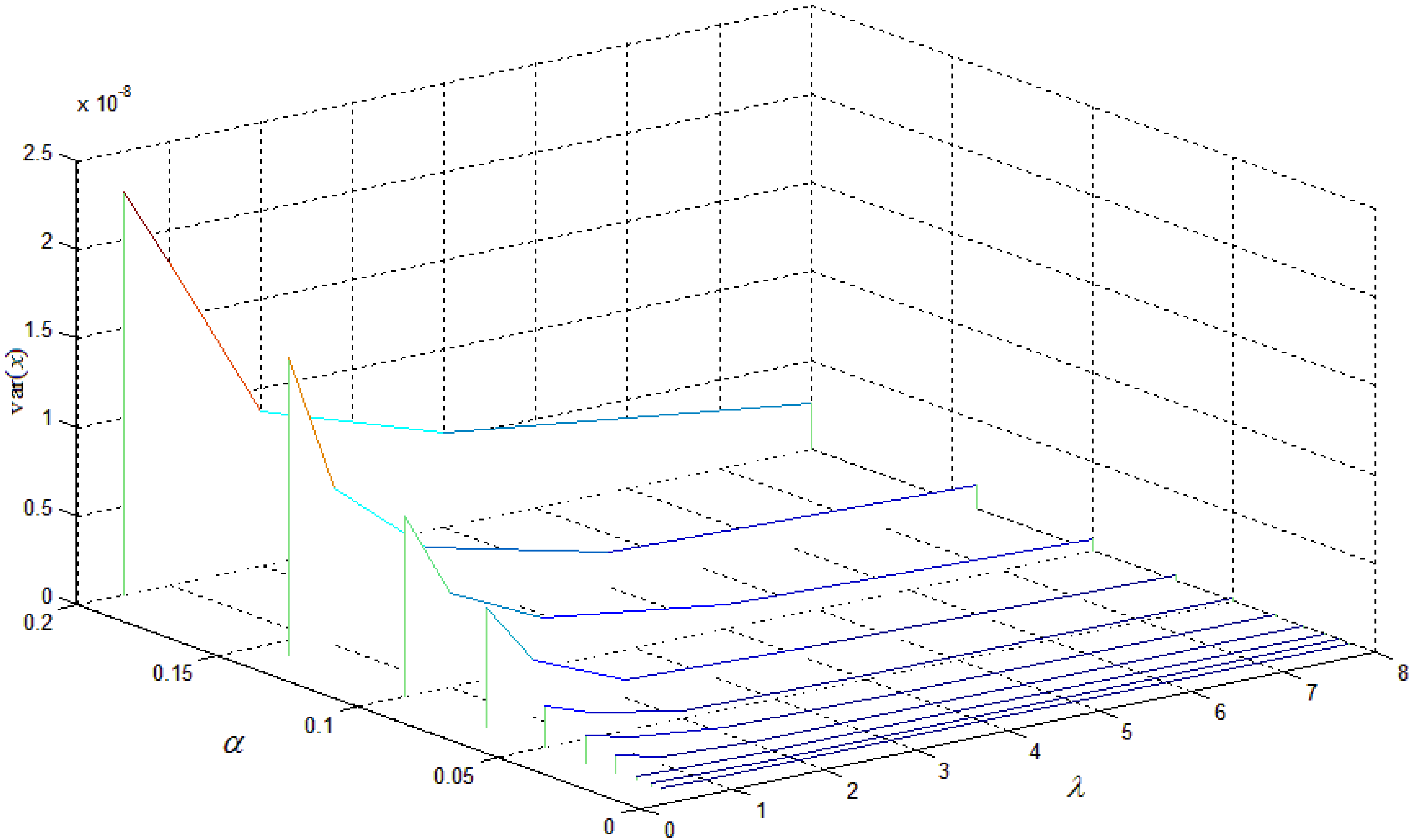


Plot of CRLB vs α with λ as the parameter



Plot of CRLB vs number of sensors used in localization (N)

CRLB 3D plot for the nearest 6 nodes as a function of α and λ



FRAM

$$
\begin{aligned}
J_{i,j} &= -E_N\left(E_{M,L}\left[\frac{\partial^2 \ln(f(M,L,N))}{\partial X_i \partial X_j}\right]\right)\\
&= -E_N\left(E_{M,L}\left(\frac{\partial^2 \ln(f(N))}{\partial X_i \partial X_j} + \frac{\partial^2 \ln(f(L \mid N))}{\partial X_i \partial X_j} + \frac{\partial^2 \ln(f(M \mid, L, N))}{\partial X_i \partial X_j}\right)\right)\\
&= E_N\left(JN_{ij} + JL_{ij} + JM_{ij}\right)
\end{aligned}
$$

$$
J_{ij} = \begin{cases} \frac{\lambda\pi(1+2\alpha^2)}{\alpha^2}\left[\log\left(\frac{K+R}{R}\right) - \frac{R}{K+R}\right] & \text{for } i = j \\ 0 & \text{for } i \neq j \end{cases}
$$

$$
CRLB_x = CRLB_y = \frac{\alpha^2}{\lambda\pi(1+2\alpha^2)}\left[\log\left(\frac{K+R}{R}\right) - \frac{R}{K+R}\right]^{-1}
$$

Error Comparisons

$$(x_s - x_i)^2 + (y_s - y_i)^2 = m_i^2$$

$$2x_s(x_1 - x_i) + 2y_s(y_1 - y_i) = (m_i^2 - m_1^2) - ((x_i^2 + y_i^2) - (x_1^2 + y_1^2))$$

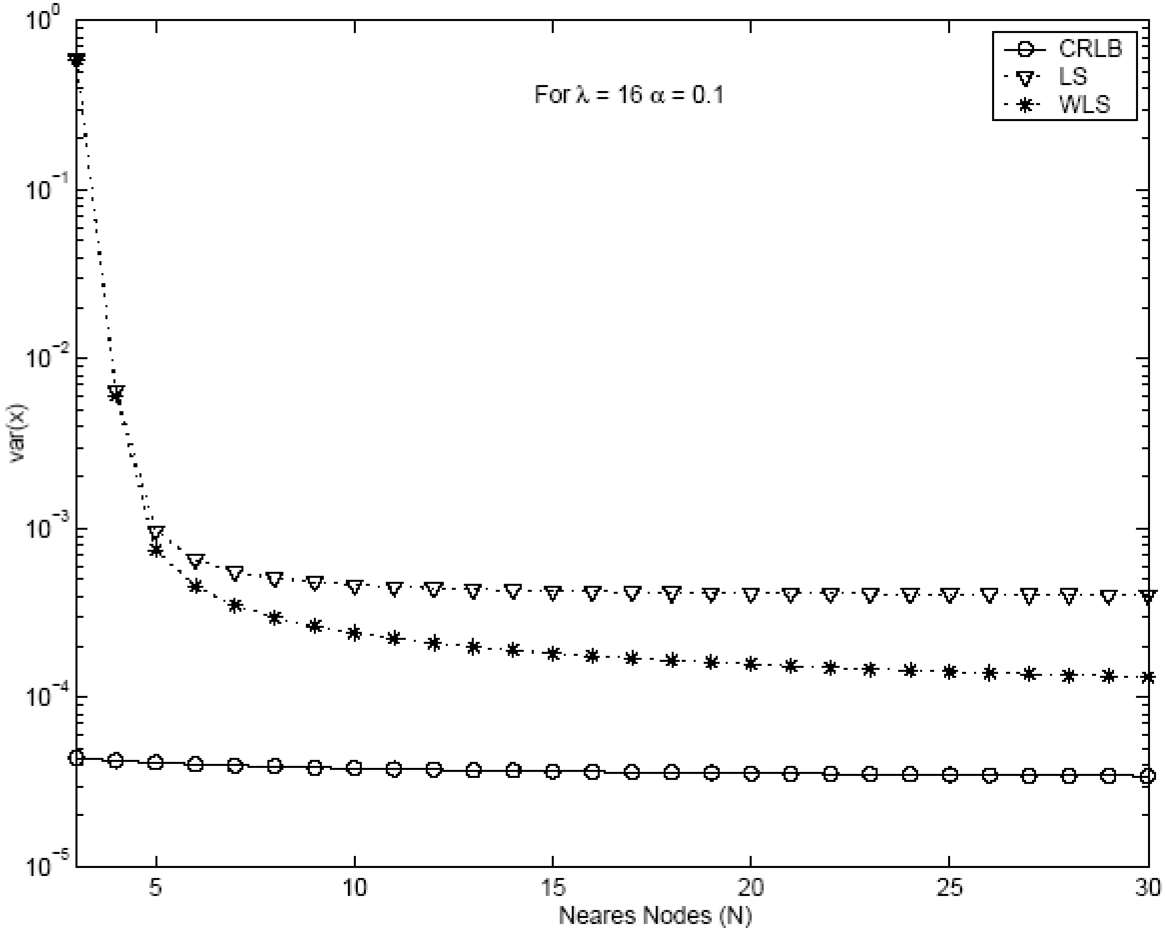
$$AX = B + \epsilon$$

$$\hat{X} = (A^T A)^{-1} A^T B$$ LSE

$$\hat{X} = (A^T W^{-1} A)^{-1} A^T W^{-1} B$$ WLSE

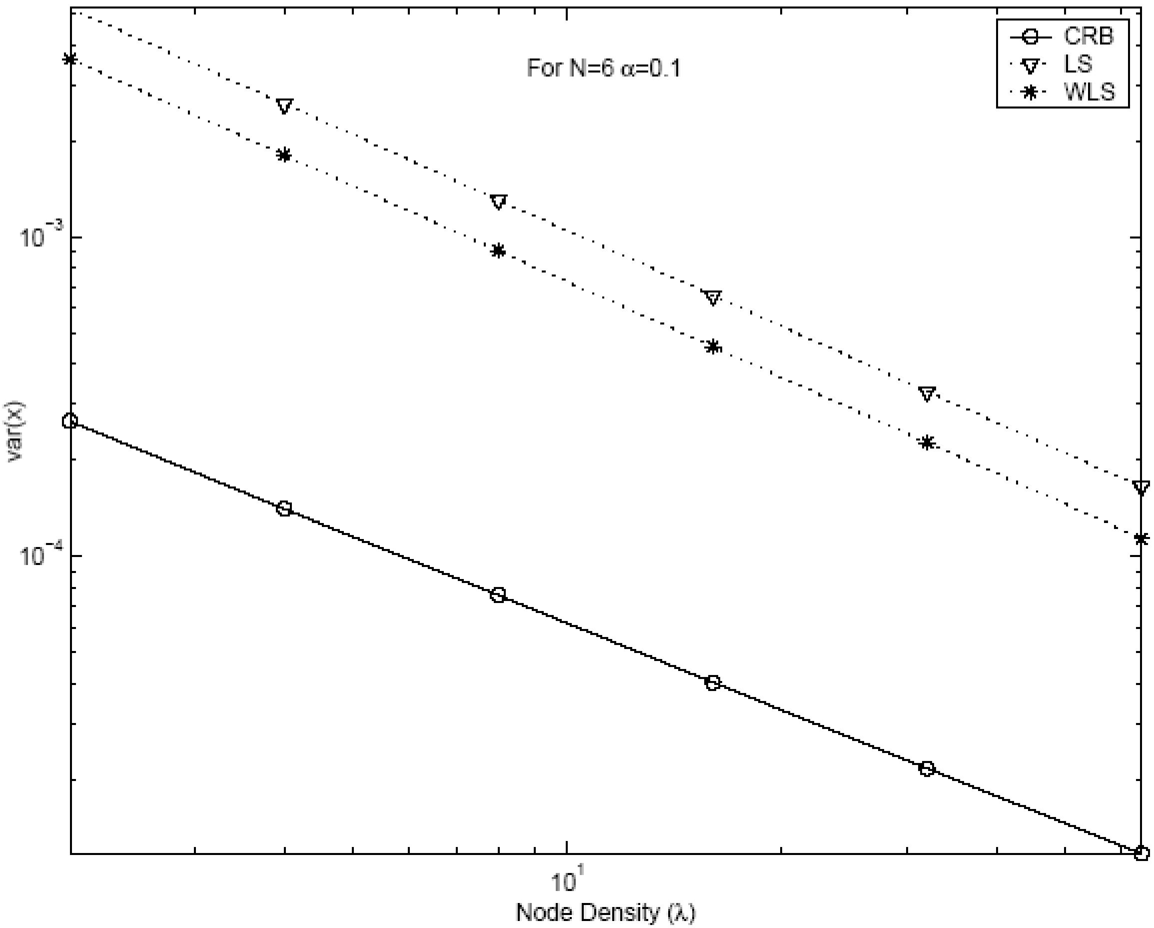
$$W = Cov(\epsilon\epsilon^T)$$

Error Comparisons



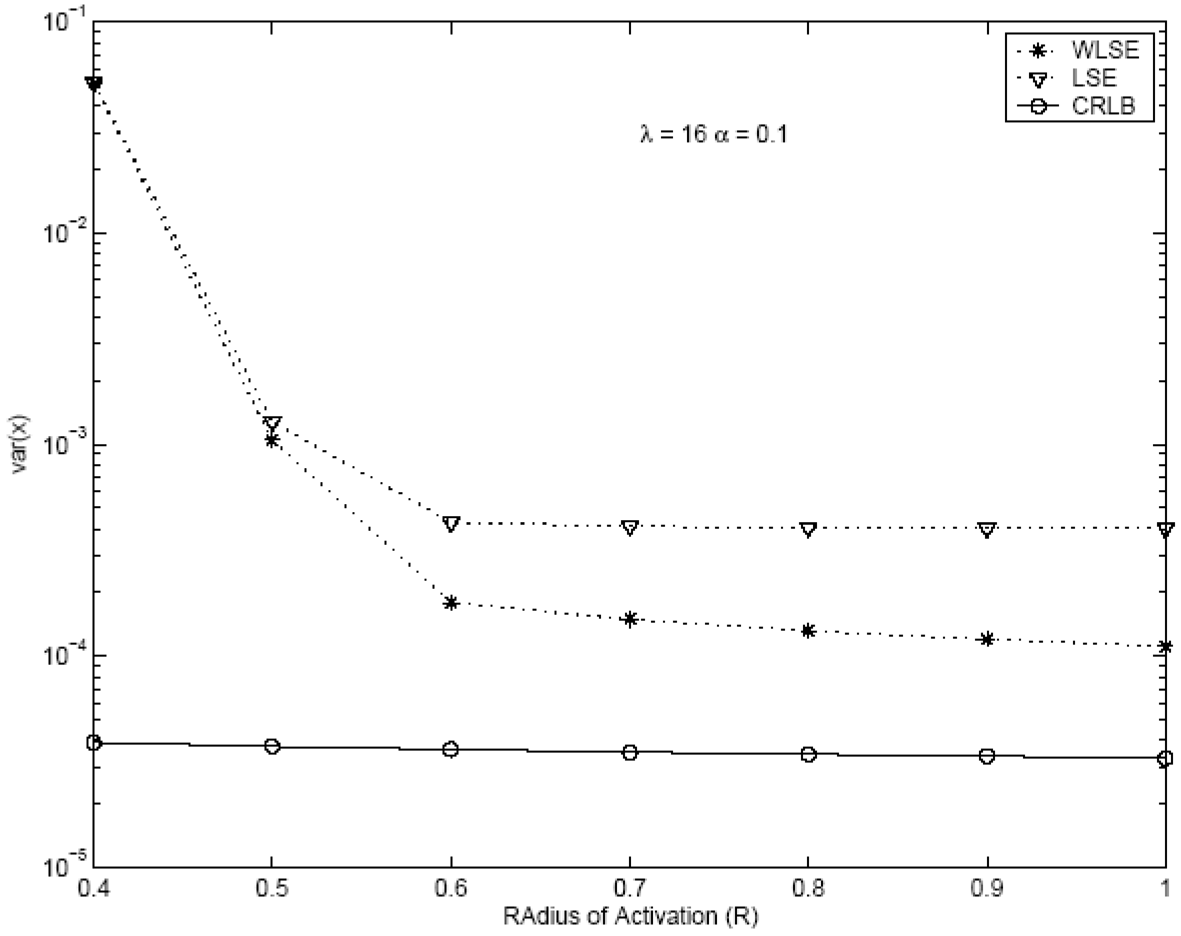
CNAM : Comparison of LS and WLS estimators with CRLB

Error Comparisons



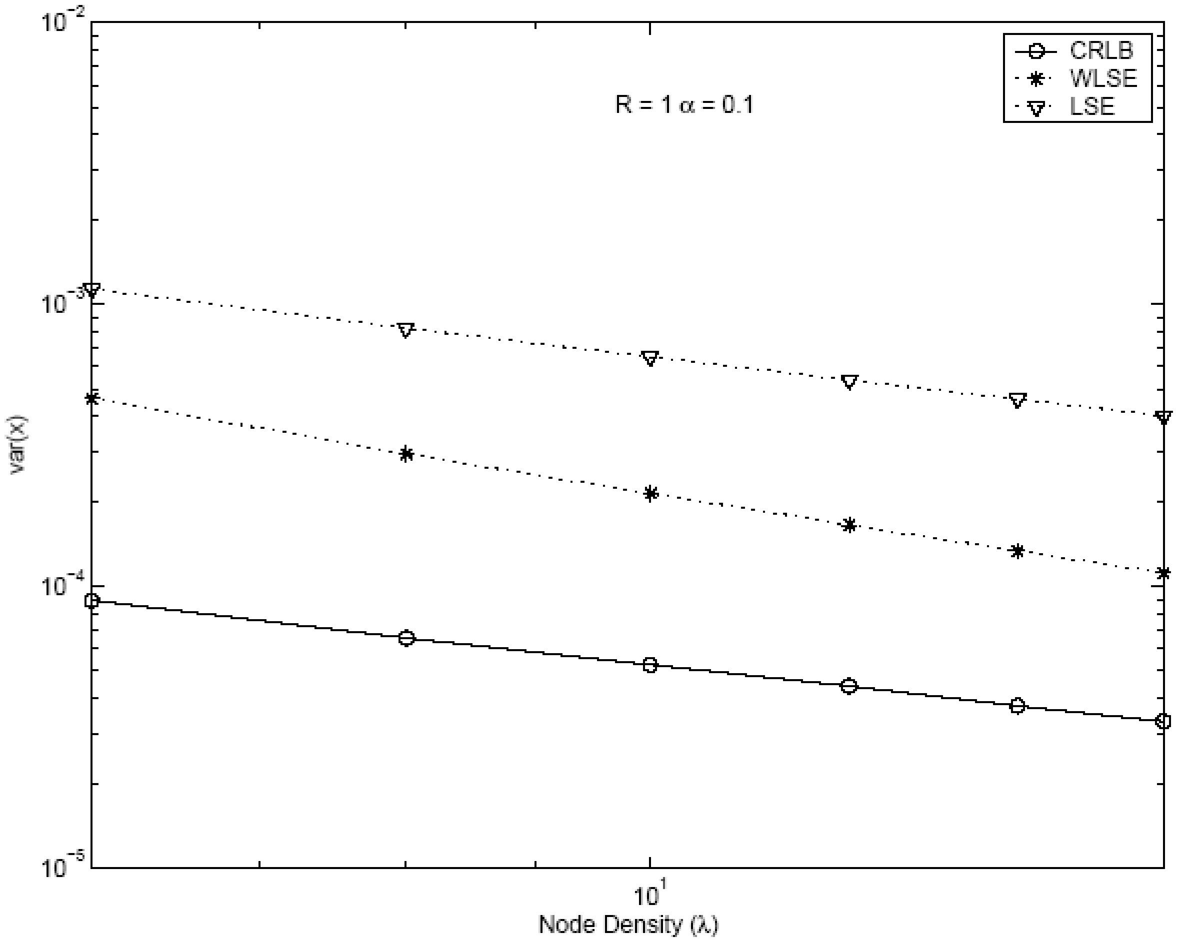
CNAM : Comparison of LS and WLS estimators with CRLB for varying λ

Error Comparisons



FRAM : Comparison of LS and WLS estimators with CRLB as localizing radius R varies

Error Comparisons



FRAM : Comparison of LS and WLS estimators with CRLB for varying λ

Range Difference Measurements

- Joint Gaussian model

$$f(m_{21}, m_{31}, \dots, m_{N1} | (x_1, y_1), \dots, (x_N, y_N); (x_s, y_s)) = \frac{1}{\left(\sqrt{2\pi}\right)^N \det(C)} e^{\left(-\frac{1}{2}[\bar{m}-\bar{r}]^T C^{-1} [\bar{m}-\bar{r}]\right)}$$

.

$$\mathbf{C} = \frac{\sigma^2}{2} \begin{pmatrix} 2 & 1 & 1 & \dots \\ 1 & 2 & 1 & \dots \\ 1 & 1 & 2 & \dots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}_{(N-1)\times(N-1)}$$

$$P_{ij} = \begin{cases} \frac{2(N-1)}{N\sigma^2} & \text{if } i = j \\ \frac{2(-1)}{N\sigma^2} & \text{if } i \neq j \end{cases}$$

Joint Gaussian model

- CNAM

$$CRLB_x = CRLB_y = \frac{1}{2\lambda\pi + \frac{2(\lambda A - 1)}{\sigma^2}}$$

- FRAM

$$CRLB_x = CRLB_y = \frac{\sigma^2}{2(\lambda A - 1)}$$

Derived Gaussian model

- CNAM

$$CRLB_x = CRLB_y = \frac{1}{2\lambda\pi + \frac{(\lambda A - 1)}{2\sigma^2}}$$

- FRAM

$$CRLB_x = CRLB_y = \frac{2\sigma^2}{(\lambda A - 1)}$$

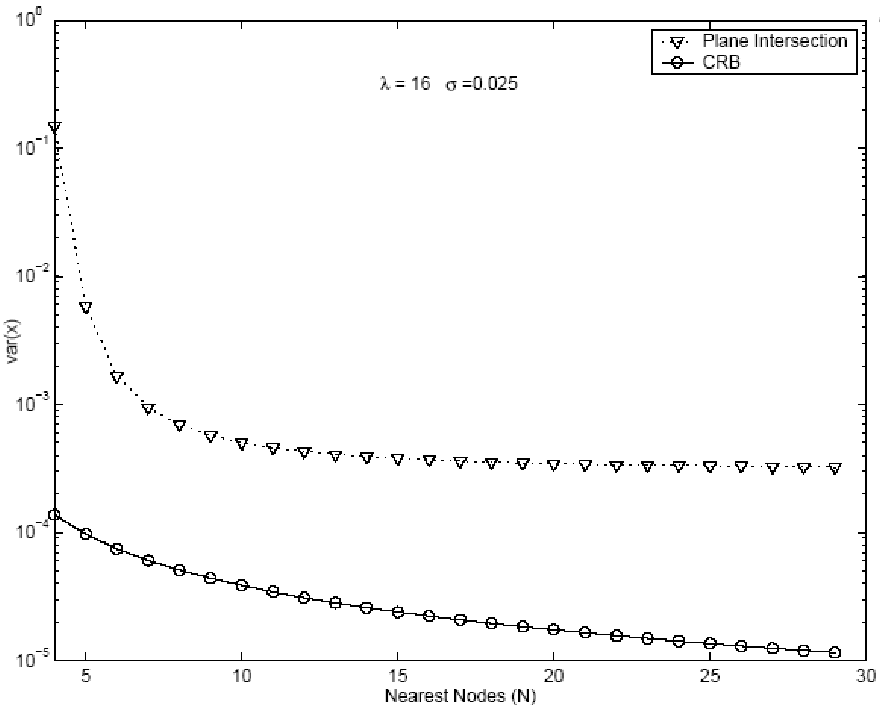
Error Comparisons

$$AX = B + \epsilon$$

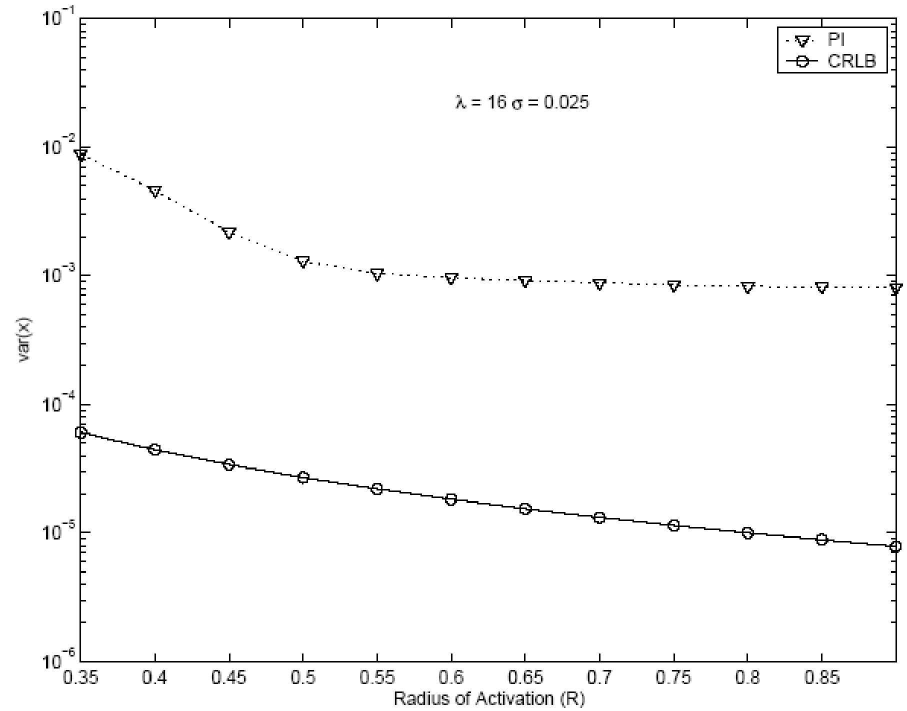
$$\begin{aligned}
A(i,1) &= x_1 m_{2i} + x_2 m_{i1} + x_i m_{12} \\
A(i,2) &= y_1 m_{2i} + y_2 m_{i1} + y_i m_{12} \\
B(i,1) &= \frac{1}{2}\left(m_{12} m_{2i} m_{i1} + m_{2i}(x_1^2 + y_1^2) + m_{i1}(x_2^2 + y_2^2) + m_{12}(x_i^2 + y_i^2)\right)
\end{aligned}$$

$$\hat{X} = (A^T A)^{-1} A^T B$$ LSE

Error Comparisons – Joint Gaussian Model

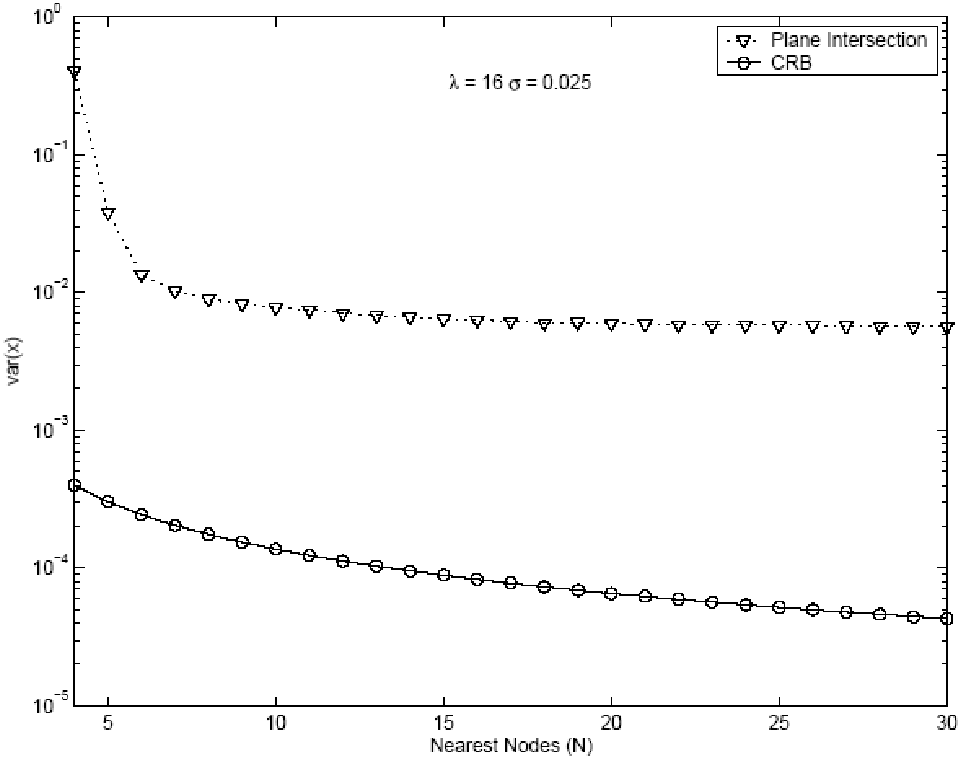


CNAM : Comparison of PI estimator with CRLB as localizing nodes N varies for Joint Gaussian Error Model

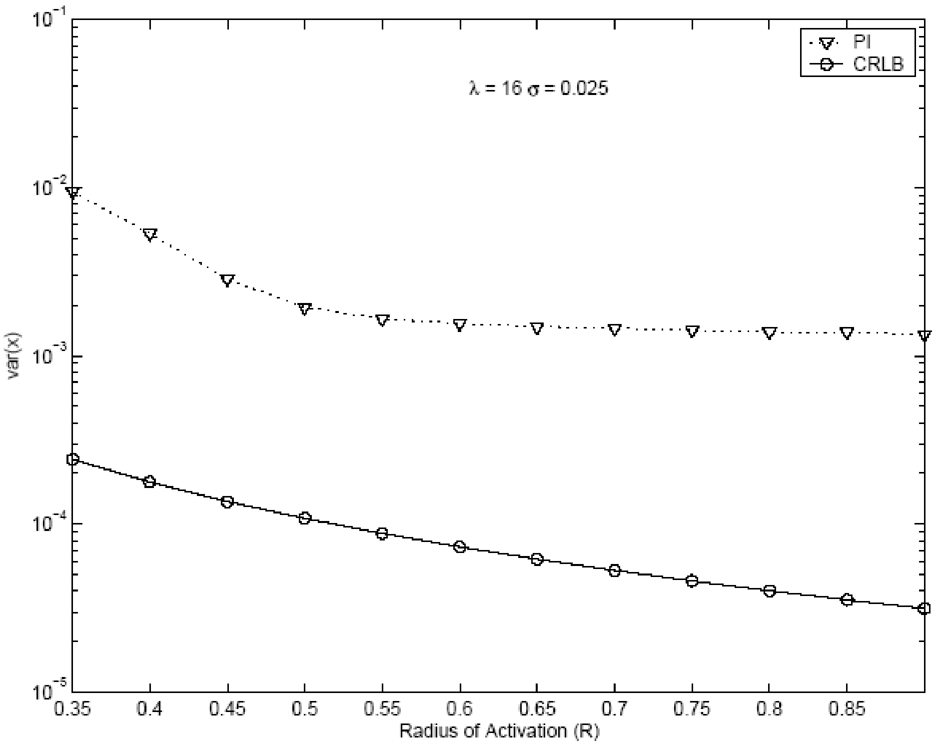


FRAM : Comparison of PI estimator with CRLB as activation radius R varies for Joint Gaussian Error Model

Error Comparisons – Derived Gaussian Model



CNAM : Comparison of PI estimator with CRLB as localizing nodes N varies for Derived Gaussian Error Model



FRAM : Comparison of PI estimator with CRLB as activation radius R varies for Derived Gaussian Error Model

Thank You