Irreducible decompositions of binomial ideals

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Joint with Thomas Kahle and Ezra Miller

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Question

Fact

Every ideal $I \subset \mathbb{k}[x_1, \ldots, x_n]$ can be written as a finite intersection of irreducible ideals (an irreducible decomposition).

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An ideal $I \subset \mathbb{k}[x_1, \ldots, x_n]$ is a *binomial ideal* if it is generated by polynomials with at most two terms. Example: $I = \langle x - 2y, x^2 \rangle$.

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Question (Eisenbud-Sturmfels, 1996)

Do binomial ideals have *binomial* irreducible decompositions?

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Question (Eisenbud-Sturmfels, 1996)

Do binomial ideals have *binomial* irreducible decompositions?

Answer (Kahle-Miller-O., 2014)

No.

Monomial ideals

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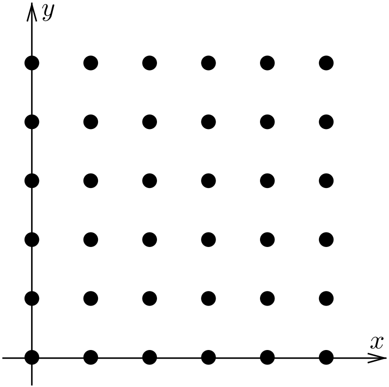

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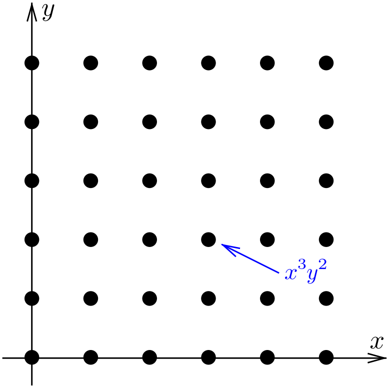

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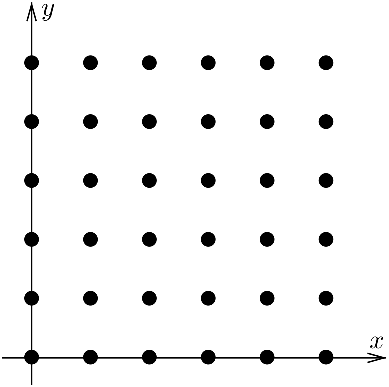

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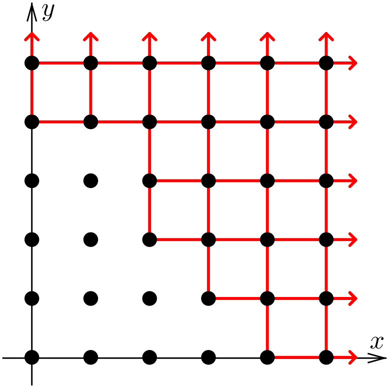

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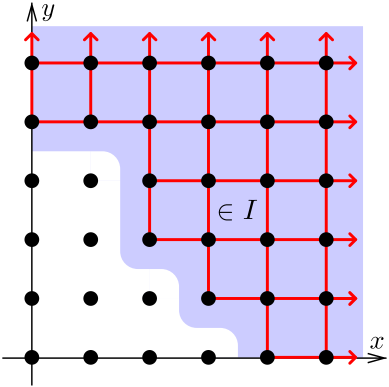

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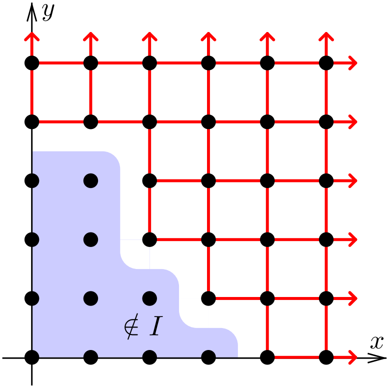

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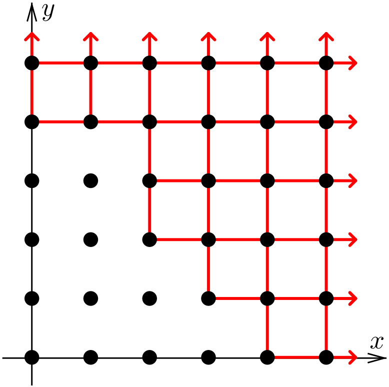

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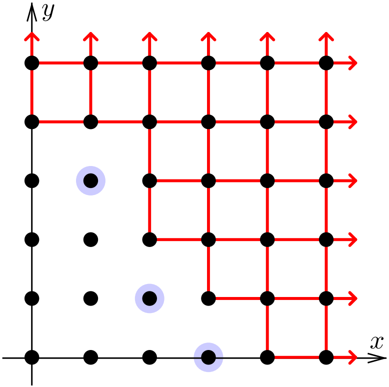

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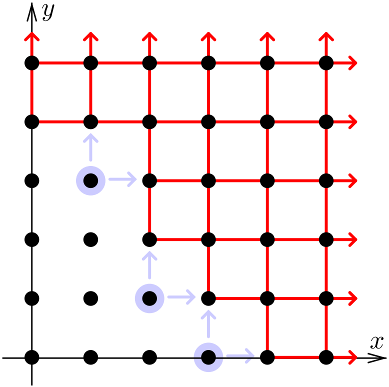

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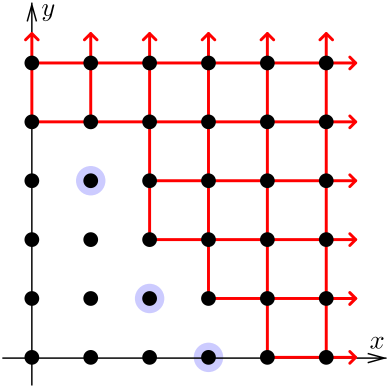

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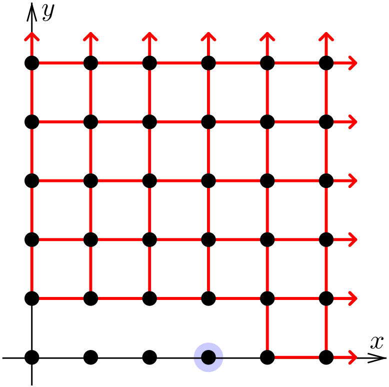

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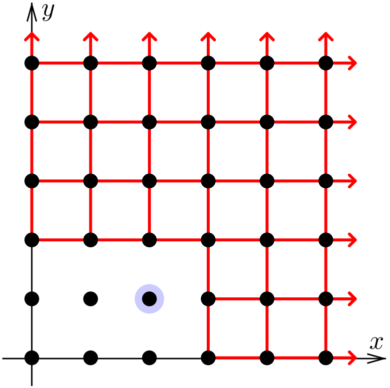

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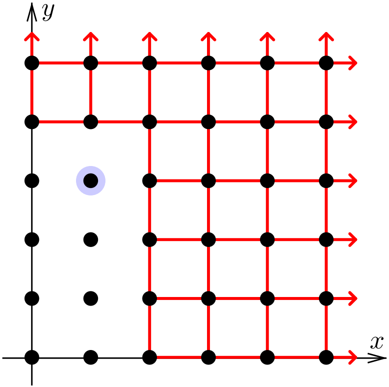

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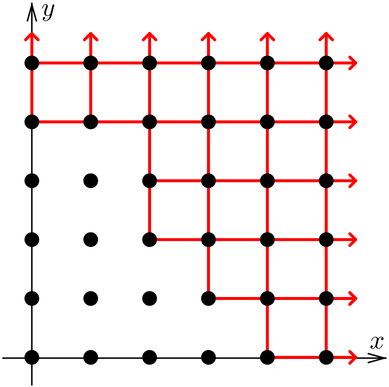

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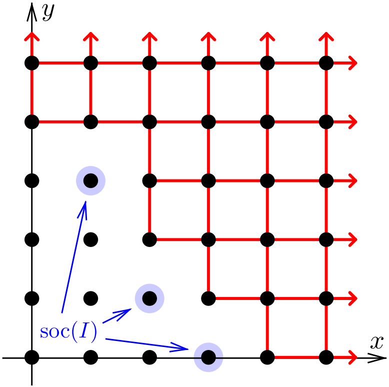

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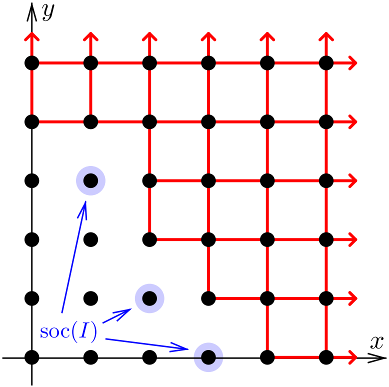
irreducible $\Leftrightarrow$ "simple socle"



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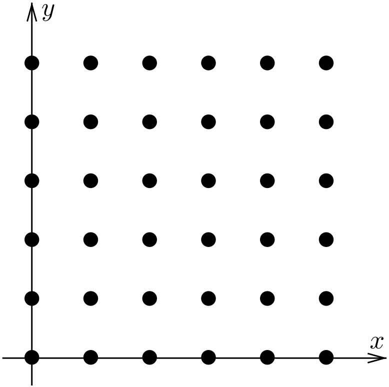

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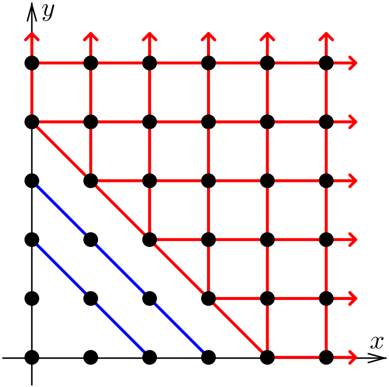

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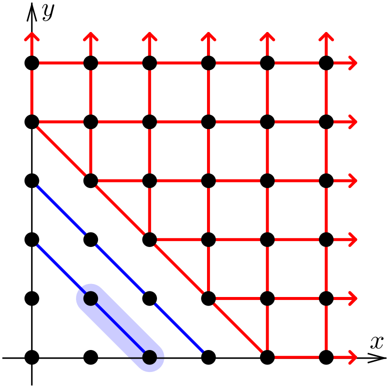
$$x^2 = xy \text{ in } \Bbbk[x, y]/I$$



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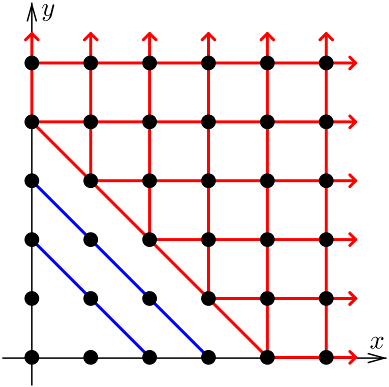

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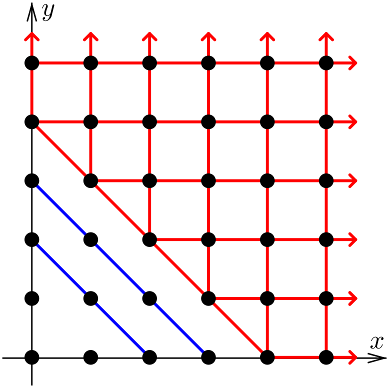
"witnesses" = monomials that merge in all directions



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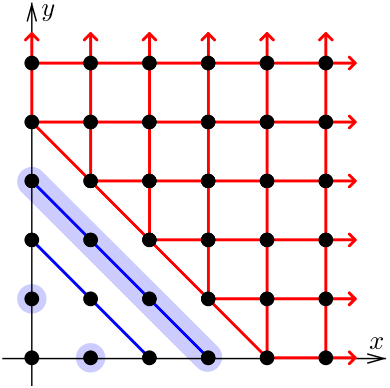

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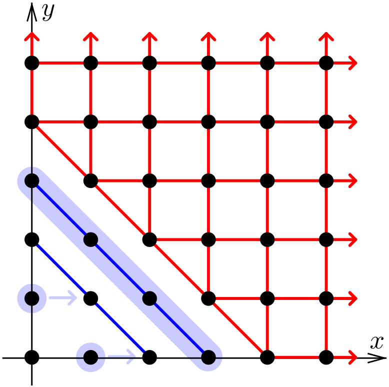

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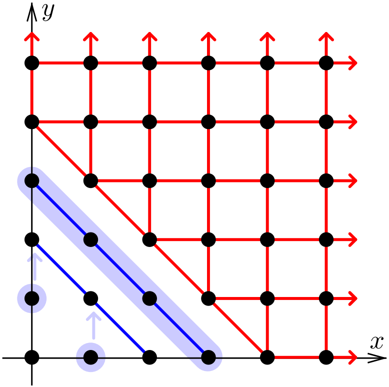

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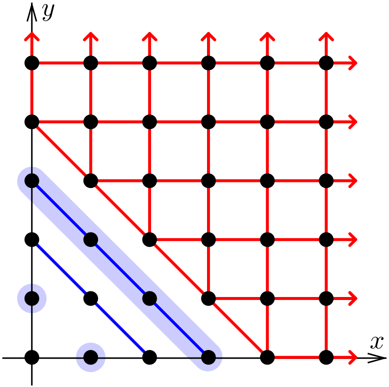
To decompose I: force each witness to be a simple socle



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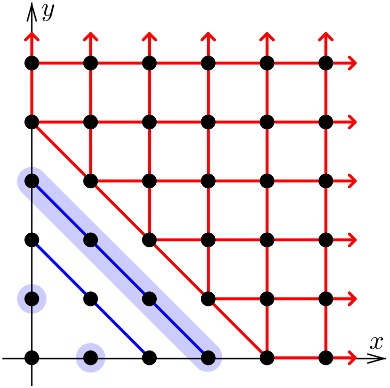

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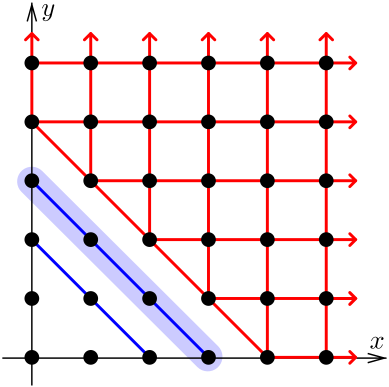

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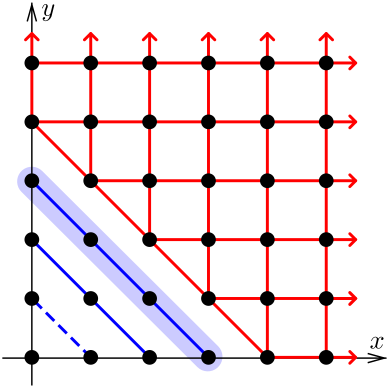
$$=$$

$$x - y \in \text{socle}$$



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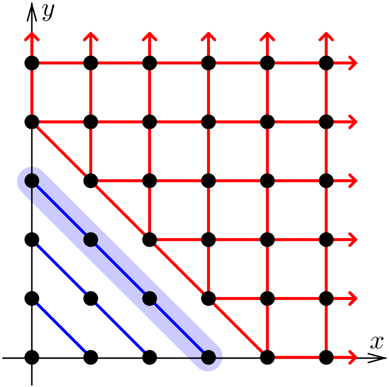

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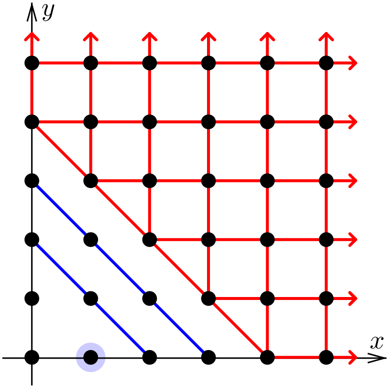

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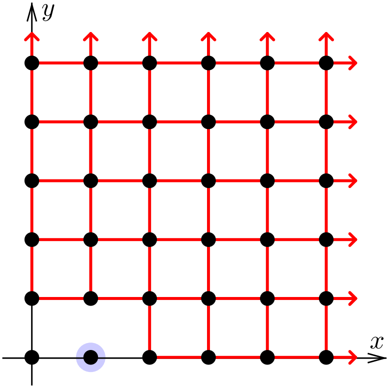

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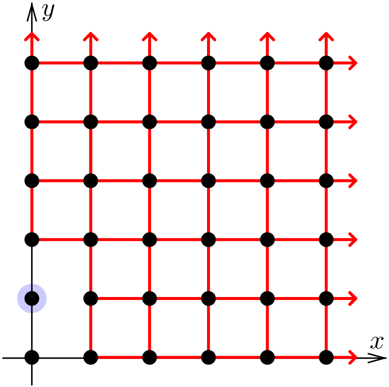

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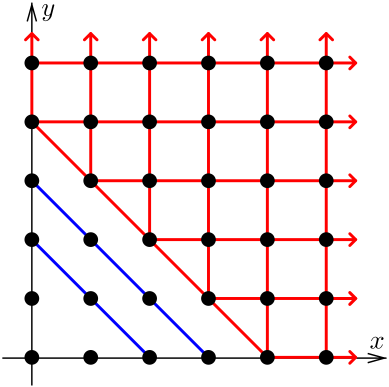

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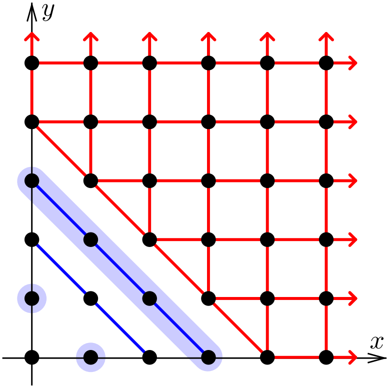

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Counterexample revisited

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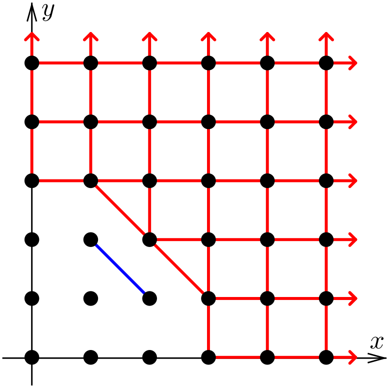

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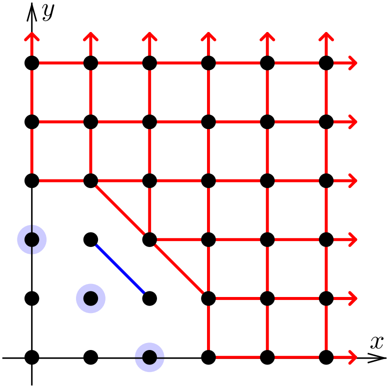

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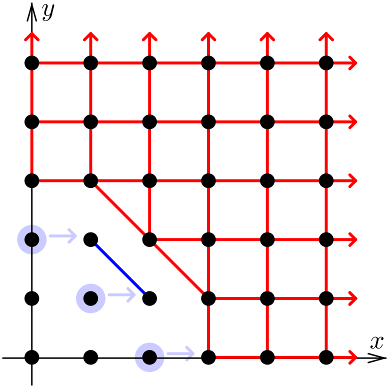

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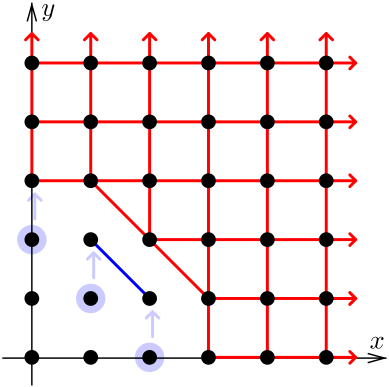

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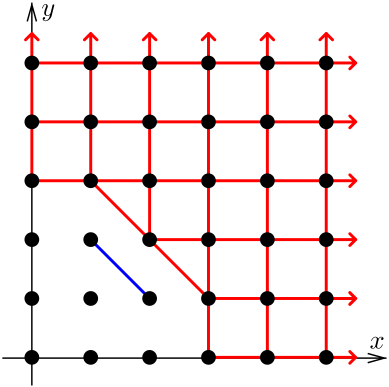

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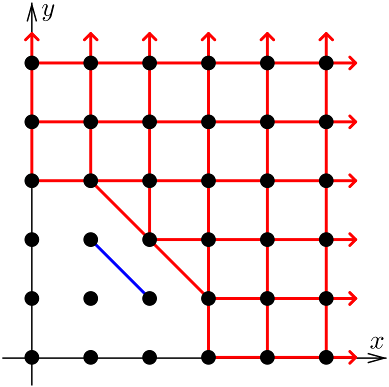
I admits no binomial irreducible decompositions



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Thomas Kahle, Ezra Miller (2013)
Decompositions of commutative monoid congruences and binomial ideals.
arXiv:1107.4699 [math].

Thomas Kahle, Ezra Miller, Christopher O'Neill (2014)
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