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Interference Calculus

A General Framework for Interference Management and Network Utility Optimization

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Outline

1. Introduction and History
2. Axiomatic Framework
3. QoS Sets
4. User-Centric ("Fair") Strategies based on Monotonicity
5. Concave and Convex Interference Functions
6. Log-Convex Interference Functions
7. Conclusions

Design Principles for Wireless Communication Systems

- Conventional approach: interference-free (orthogonal) point-to-point links, (carriers, slots, beams, ...)
  - Pro: easy to handle, enables separate optimization of link level and system level
  - Con: ignores the interdependencies between links (interference, limited power/resources)
- Future wireless networks: high user density, high rates, mixture of macro cells with unplanned pico/femto cells, relays, MIMO, ...
- Challenge: Optimization of coupled multi-user systems
  - Flexible, system-wide resource allocation (put resources where they are needed and don't use more than necessary)
  - Interference avoidance/management (many deployment scenarios are interference-limited)

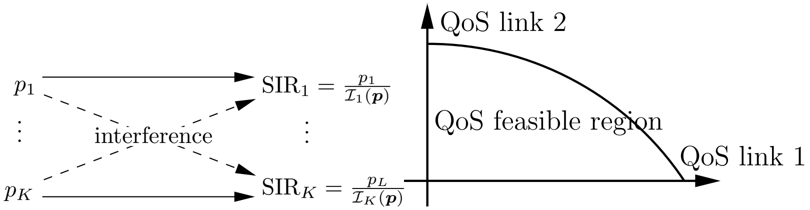
Coupled Multiuser Systems

- System abstraction: Quality-of-service (QoS) of link k is strongly monotone in the SINR, which depends on transmission powers $\mathbf{p} = [p_1, \dots, p_K]^T$

$$\mathrm{QoS}_k(\mathbf{p}) = \phi_k(\mathrm{SINR}_k(\mathbf{p})) = \phi_k\left(\frac{p_k}{\mathcal{I}(\mathbf{p})}\right)$$

Examples: BER, data rate, MMSE, ...

- Tradeoff due to power/resource constraints and interference



Motivation

- The behavior of any coupled multiuser system depends on the properties of the underlying interference functions $\mathcal{I}(\mathbf{p})$
- Situation in the literature: Many variations of interference-coupled system. Sometimes only minor differences in the problem fomulations, slight modification of assumptions, similar algorithms.
- Are there common patterns? What are the key properties that enable efficient algorithmic solutions? Can we find a high-level theory that explains previous results as special cases?
- We will show examples showing that interference is often

homogeneous: $\mathcal{I}(\alpha\mathbf{p}) = \alpha\mathcal{I}(\mathbf{p})$ for all $\alpha > 0$

monotone: $\mathcal{I}(\mathbf{p}) \geq \mathcal{I}(\mathbf{p}')$ if $\mathbf{p} \geq \mathbf{p}'$

Linear Interference Functions (since 1970s)

- given a non-negative coupling matrix $\mathbf{V}$, the interference experienced by link l is

$$\mathcal{I}_l(\mathbf{p}) = [\mathbf{V}\mathbf{p}]_l$$

- linear interference function with noise:

$$\begin{aligned}\mathcal{I}_l(\underline{\mathbf{p}}) &= [\mathbf{V}\mathbf{p} + \mathbf{n} \cdot \sigma_n^2]_l \qquad \text{where } \underline{\mathbf{p}} = [p_1, \ldots, p_L, \sigma_n^2]^T \\ &= [\mathbf{V}'\underline{\mathbf{p}}]_l \quad \text{with } \mathbf{V}' = [\mathbf{V}|\mathbf{n}]\end{aligned}$$

- Framework for analyzing and optimizing such systems: Perron-Frobenius theory of non-negative matrices

Joint Beamforming and Power Control (since 1990s)

- Interference at the beamformer output

$$\mathcal{I}_k(\underline{\mathbf{p}}) = \frac{p_k}{\max_{\|\mathbf{w}_k\|=1} \mathrm{SINR}_k(\underline{\mathbf{p}}, \mathbf{w}_k)} = \min_{\|\mathbf{w}_k\|=1} \frac{p_k}{\mathrm{SINR}_k(\underline{\mathbf{p}}, \mathbf{w}_k)}$$

$$= \min_{\|\mathbf{w}_k\|_2=1} \frac{\sum_{l\neq k} p_l \mathbf{w}_k^H \mathbf{R}_l \mathbf{w}_k + \|\mathbf{w}_k\|^2 \sigma_n^2}{\mathbf{w}_k^H \mathbf{R}_k \mathbf{w}_k}$$

Defining interference coupling coefficients

$$[\mathbf{v}_{\mathbf{w}_k}]_l = \begin{cases} \frac{\mathbf{w}_k^H \mathbf{R}_l \mathbf{w}_k}{\mathbf{w}_k^H \mathbf{R}_k \mathbf{w}_k} & 1 \leq j \leq K,\ j \neq l \\ \frac{\|\mathbf{w}_k\|^2}{\mathbf{w}_k^H \mathbf{R}_k \mathbf{w}_k} & j = K+1, \\ 0 & j = l\ . \end{cases}$$

➠ interference $\mathcal{I}_k(\underline{\mathbf{p}}) = \min_{\|\mathbf{w}_k\|_2=1} \underline{\mathbf{p}}^T \mathbf{v}_{\mathbf{w}_k}$

Beamforming with Deterministic Channel Vectors $\mathbf{h}_k$

- if $\mathbf{R}_k = \mathbf{h}_k \mathbf{h}_k^H$, then the maxSINR solution has a closed form

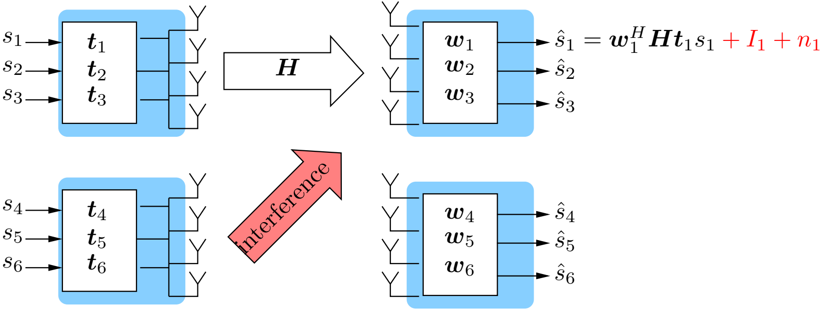
$$\mathbf{w}_k^* = \left(\sigma_n^2 \mathbf{I} + \sum_{j \neq l} p_l \mathbf{R}_l\right)^{-1} \mathbf{h}_k$$

where $\mathbf{R}_{l \neq k} = \mathrm{E}[\mathbf{h}_l \mathbf{h}_l^H]$ (stochastic CSI from interferers)

➠ interference function

$$\mathcal{I}_k(\underline{\mathbf{p}}) = \frac{1}{\mathbf{h}_k^H \left(\sigma_n^2 \mathbf{I} + \sum_{j \neq l} p_l \mathbf{R}_l\right)^{-1} \mathbf{h}_k}$$

Interference in a Multiuser MIMO Channel



- For fixed transmit filters, the interference can be modeled by functions

$$\mathcal{I}_k(\underline{\mathbf{p}}) = \min_{\|\mathbf{w}_k\|_2=1} \underline{\mathbf{p}}^T \mathbf{v}_{\mathbf{w}_k}$$

- Same for fixed receive filters (via uplink/downlink duality)

Base Station Assignment

- Consider the problem of combined beamforming and base station assignment [Yates and Ching-Yao, 1995; Rashid-Farrokhi et al., 1998; Hanly, 1995; Bengtsson, 2001].
- From a set of base stations $\mathcal{B}_k$, choose the one that maximizes the SINR.

$$\mathcal{I}_k(\underline{\mathbf{p}}) = \min_{b_k \in \mathcal{B}_k} \left(\min_{\mathbf{w}_k : \|\mathbf{w}_k\|=1} \frac{\mathbf{w}_k^H \left(\sum_{l \neq k} p_l \mathbf{R}_l^{(b_k)} + \sigma_n^2 \mathbf{I} \right) \mathbf{w}_k}{\mathbf{w}_k^H \mathbf{R}_k^{(b_k)} \mathbf{w}_k} \right) .$$

Robust Signal Processing

- Channel estimation errors or system imperfections are modeled by an uncertainty region $\mathcal{C}_k$
- The system is optimized with respect to the worst-case interference (e.g. [Biguesh et al., 2004; Payaró et al., 2007])

$$\mathcal{I}_k(\mathbf{p}) = \max_{c_k \in \mathcal{C}_k} \mathbf{p}^T \mathbf{v}(c_k), \quad \forall k \; ,$$

Power Control with Standard Interference Functions

Definition 1 (Yates, 1995)

A function $\mathcal{J}(\mathbf{p})$ is called a standard interference function if the following axioms are fulfilled.

Y1	(positivity)	$\mathcal{J}(\mathbf{p}) > 0$ for all $\mathbf{p} \in \mathbb{R}_+^K$
Y2	(scalability)	$\alpha\mathcal{J}(\mathbf{p}) > \mathcal{J}(\alpha\mathbf{p})$ for all $\alpha > 1$
Y3	(monotonicity)	$\mathcal{J}(\mathbf{p}) \geq \mathcal{J}(\mathbf{p}')$ if $\mathbf{p} \geq \mathbf{p}'$

There always exist a homogeneous monotone $\mathcal{I}$ such that

$$\mathcal{I}\left(\begin{bmatrix}\mathbf{p}\\1\end{bmatrix}\right) = \mathcal{J}(\mathbf{p}) \qquad \text{[Boche and Schubert, 2010]}$$

Examples:

- linear: $\mathcal{I}_k(\underline{\mathbf{p}}) = \mathcal{J}_k(\mathbf{p}) = [\mathbf{V}\mathbf{p} + \mathbf{n}]_k$
- non-linear: $\mathcal{I}(\underline{\mathbf{p}}) = \mathcal{J}_k(\mathbf{p}) = \min_z[\mathbf{V}(z)\mathbf{p} + \mathbf{n}(z)]_k$
- ...

Link Coupling by Limited Powers/Resources

Transmission powers of all links are chosen from a set

$$\mathcal{P} = \{\mathbf{p} \geq 0 : \|\mathbf{p}\| \leq P_{max}\}$$

Examples for the norm $\| \cdot \|$:

- Total power contraint $\|\mathbf{p}\| = \sum_{k=1}^{K} p_k$
- Per-user power constraint: $\|\mathbf{p}\| = \max_k p_k$
- Per-base station constraints $\|\mathbf{p}\| = \max_{b \in \mathcal{B}} \sum_{k \in \mathcal{A}_b} p_k$
  where $\mathcal{A}_b$ is the set of all users assigned to base station b
- Observation: $\| \cdot \|$ behaves like interference (monotone homogeneous function)

Admission Control

- SI(N)R (resp. QoS) values $\boldsymbol{\gamma} = [\gamma_1, \ldots, \gamma_K]$ are feasible iff for any $\epsilon > 0$

$$\mathrm{SINR}_k(\mathbf{p}) = \frac{p_k}{\mathcal{I}_k(\mathbf{p})} \geq \gamma_k + \epsilon \quad \text{for all links } k = 1, \ldots, K$$

- Indicator for feasibility:

$$C(\boldsymbol{\gamma}) \leq 1 \qquad \text{where} \quad C(\boldsymbol{\gamma}) = \inf_{\mathbf{p}>0} \Big(\max_k \frac{\gamma_k \mathcal{I}_k(\mathbf{p})}{p_k} \Big)$$

- $C(\boldsymbol{\gamma})$ also behaves like "interference" (monotone homogeneous function)

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Axiomatic Framework of General Interference Functions

- Monotone homogeneous functions are ubiquitous.
  They model the dependency on underlying resources $\mathbf{r} \geq 0$.
- Is there a comon pattern? Are the previous examples special cases of a general framework?
- The following set of axioms provides a common basis.

Definition 2 (general interference function $\mathcal{I}$ defined on $\mathbb{R}^L_+$)

A1	(positivity)	There exists an $\mathbf{r} > 0$ such that $\mathcal{I}(\mathbf{r}) > 0$
A2	(scale invariance)	$\mathcal{I}(\alpha\mathbf{r}) = \alpha\mathcal{I}(\mathbf{r})$ for all $\alpha > 0$
A3	(monotonicity)	$\mathcal{I}(\mathbf{r}) \geq \mathcal{I}(\mathbf{r}')$ if $\mathbf{r} \geq \mathbf{r}'$

Adding Structure

- A1,A2,A3 is useful as a common basis for analysis and optimization of coupled systems
- But in most cases we want to consider additional properties:
  - strict monotonicity (dependency between links)
  - convexity
  - logarithmic convexity
  - ...
- This additional structure can be exploited for the design of algorithms (as shown later)

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The SIR Feasible Set is Comprehensive

- The indicator function

$$C(\gamma) = \inf_{\mathbf{p}>0} \left(\max_{1 \leq k \leq K} \frac{\gamma_k \mathcal{I}_k(\mathbf{p})}{p_k} \right) \quad \text{s.t.} \quad \|\mathbf{p}\| = P_{max}$$

is an interference function (thus monotone).

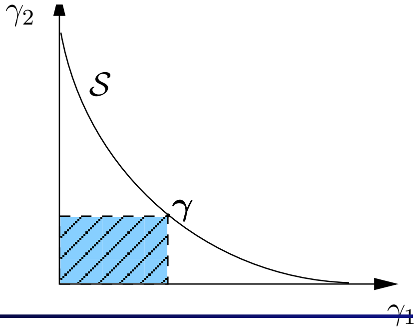
➠ The SIR feasible region

$$\mathcal{S} = \{\gamma : C(\gamma) \leq 1\}$$

is downward-comprehensive, i.e.,

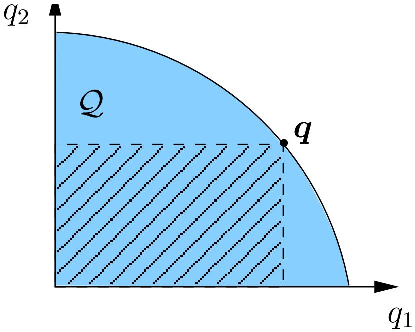
for all $\gamma \in \mathcal{S}$ and $\gamma' \in \mathbb{R}^K_{++}$

$\gamma' \leq \gamma \implies \gamma' \in \mathcal{S}$



The QoS Region is Comprehensive

$$\mathrm{QoS}_k = \phi_k\left(\frac{p_k}{\mathcal{I}_k(\underline{\mathbf{p}})}\right)$$



- Properties of the QoS region depend on underlying interference functions $\mathcal{I}$ and the monotonic "utility function" ϕ
- Let $\gamma(\mathbf{q})$ be the SINR required for achieving some QoS values $\mathbf{q}$
  - The QoS region

$$\mathcal{Q} = \{\mathbf{q} : C\big(\gamma(\mathbf{q})\big) \leq 1\}$$ is downward-comprehensive.

Representation of General Interference Functions

Theorem 3 ([Boche and Schubert, 2008c])

Let $\mathcal{I}$ be an arbitrary interference function, then

$$\mathcal{I}(\mathbf{p}) = \min_{\hat{\mathbf{p}} \in \underline{L}(\mathcal{I})} \max_k \frac{p_k}{\hat{p}_k} = \max_{\hat{\mathbf{p}} \in \overline{L}(\mathcal{I})} \min_k \frac{p_k}{\hat{p}_k}$$

- $\mathcal{I}(\mathbf{p})$ can always be represented as the optimum of a weighted max-min (or min-max) optimization problem

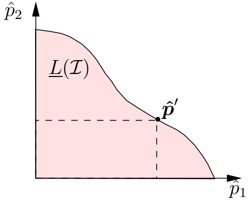
The weights $\hat{\mathbf{p}}$ are elements of convex/concave level sets

$$\underline{L}(\mathcal{I}) = \{\hat{\mathbf{p}} > 0 : \mathcal{I}(\hat{\mathbf{p}}) \leq 1\}$$
$$\overline{L}(\mathcal{I}) = \{\hat{\mathbf{p}} > 0 : \mathcal{I}(\hat{\mathbf{p}}) \geq 1\}$$

Interference Functions and Utility/Cost Regions

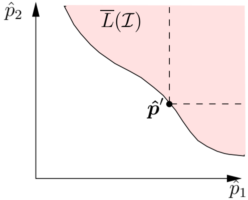
- the set $\underline{L}(\mathcal{I})$ is closed bounded and downward-comprehensive

$$\hat{\mathbf{p}} \leq \hat{\mathbf{p}}' \,, \quad \hat{\mathbf{p}}' \in \underline{L}(\mathcal{I}) \quad \Longrightarrow \quad \hat{\mathbf{p}} \in \underline{L}(\mathcal{I})$$



- the set $\overline{L}(\mathcal{I})$ is closed and upward-comprehensive

$$\hat{\mathbf{p}} \geq \hat{\mathbf{p}}' \,, \quad \hat{\mathbf{p}}' \in \overline{L}(\mathcal{I}) \quad \Longrightarrow \quad \hat{\mathbf{p}} \in \overline{L}(\mathcal{I})$$



➠ Every interference function can be interpreted as an optimum of a utility/cost optimization problem

Comprehensive Power Sets

- Set of transmission powers:

$$\mathcal{P} = \{\mathbf{p} \geq 0 : \|\mathbf{p}\| \leq P_{max}\}$$

- If the norm $\|\cdot\|$ is monotone, then $\mathcal{P}$ is comprehensive ("free disposability of powers")
- If $\|\cdot\|$ is convex, then $\mathcal{P}$ is convex

➠ Useful properties for the design of algorithms that optimize over the set $\mathcal{P}$

Interference Functions and Comprehensive Sets

Theorem 4 ([Boche and Schubert, 2008b])

Every compact comprehensive utility set from $\mathbb{R}^K_{++}$ can be expressed as a sub-level set of an interference function $C(\mathbf{u})$.

$$\mathcal{U} = \{\mathbf{u} \in \mathbb{R}^K_{++} : C(\mathbf{u}) \leq 1\}$$

The sub-level set $\mathcal{U}$ is convex if and only if $C(\mathbf{u})$ is a convex

- Interference functions and comprehensive sets are closely connected.
- Analyzing interference functions helps to better understand the structure of utility/cost sets. Applications in resource allocation, game theory, algorithm design, etc.
- Example: Computation of the comprehensive/convex hull of a given non-comprehensive set [Schubert and Boche, 2012].

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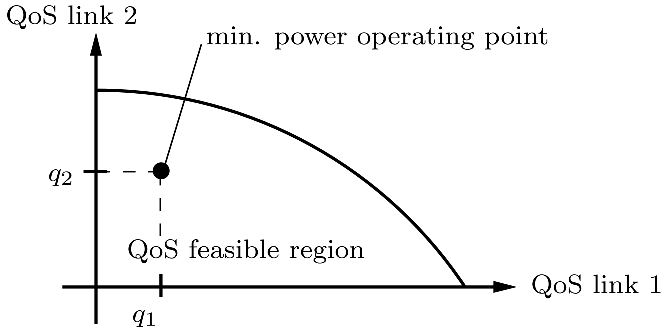
QoS-Constrained Power Minimization

- Frequently used problem formulation: Minimize the total power subject to QoS targets $q_1, \dots, q_K$

$$\min_{\mathbf{p} \in \mathcal{P}} \sum_{l \in \mathcal{K}} p_l \quad \text{s.t.} \quad \phi_k\big(\mathrm{SINR}_k(\mathbf{p})\big) \geq q_k \text{ for all } k \, .$$



QoS-Constrained Power Minimization (cont.)

- Problem only meaningful with standard interference functions

$$\mathcal{J}(\mathbf{p}) = \mathcal{I}\left(\begin{bmatrix}\mathbf{p}\\ \sigma_n^2\end{bmatrix}\right)$$

  (otherwise no solution exists)

- QoS is a strongly monotone function of the SINR. ➠ Replace QoS constraints by SINR constraints $\boldsymbol{\gamma} = [\gamma_1, \dots, \gamma_K]$.

$$\min_{\mathbf{p}\geq 0} \sum_{k=1}^{K} p_k \quad \text{s.t. } \frac{p_k}{\mathcal{J}_k(\mathbf{p})} \geq \gamma_k,\ \forall k\ ,$$

Fixed Point Iteration

For standard interference functions it was shown [Yates, 1995]

If target SINR $\boldsymbol{\gamma} = [\gamma_1, \ldots, \gamma_K]$ are feasible then for any initialization $\mathbf{p}^{(0)} \geq 0$, the iteration

$$p_k^{(n+1)} = \gamma_k \cdot \mathcal{J}_k(\mathbf{p}^{(n)}), \quad k = 1, 2, \ldots, K$$

converges to the optimum of the power minimization problem

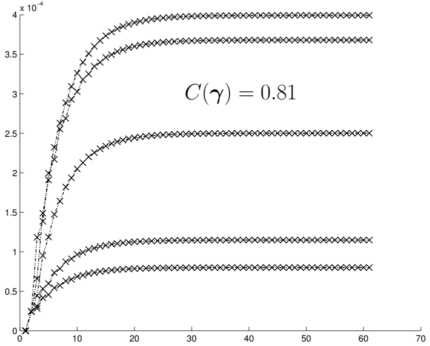
$$\min_{\mathbf{p} \geq 0} \sum_{k=1}^{K} p_k \quad \text{s.t.} \ \frac{p_k}{\mathcal{J}_k(\mathbf{p})} \geq \gamma_k, \ \forall k \ ,$$

Properties of the Fixed Point Iteration

The fixed-point iteration has the following properties:

- component-wise monotonicity
- optimum achieved iff $p_k^{(n+1)} = \gamma_k \mathcal{J}_k(\mathbf{p}^{(n)})$, $\forall k$
- optimizer $\lim_{n\to\infty} \mathbf{p}^{(n)}$ is unique

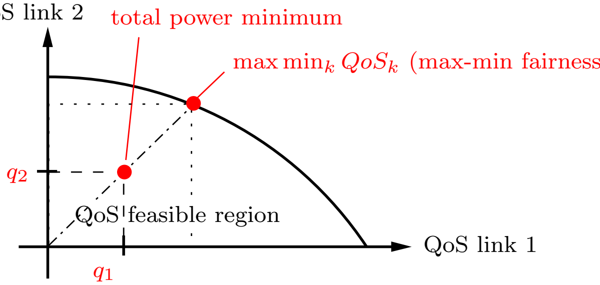
convergence to the optimal power levels



QoS Balancing

- Another possible strategy: maximize the worst-case QoS:

$$\max_{\mathbf{p}\in\mathcal{P}}\left(\min_{l\in\mathcal{L}} \frac{p_l}{\gamma_l \mathcal{I}_l(\mathbf{p})}\right)$$



- For "strongly coupled" interference functions [Vucic and Schubert, 2011], the problem is solved by the iteration

$$\mathbf{p}^{(n+1)} = \frac{1}{\|\boldsymbol{\mathcal{I}}\|} \cdot \boldsymbol{\Gamma}\boldsymbol{\mathcal{I}}(\mathbf{p}^{(n)})$$

where $\boldsymbol{\mathcal{I}} = [\mathcal{I}_1, \dots, \mathcal{I}_K]$ and $\boldsymbol{\Gamma} = \mathrm{diag}\{[\gamma_1, \dots, \gamma_K]\}$

Observation: if $\boldsymbol{\mathcal{I}} = \mathbf{V}\mathbf{p}$, then this is the well-known Power Method

Discussion

- Convexity is commonly considered as the dividing line between "easy" and "difficult" problems
- Interference functions have a special structure that enables globally optimal solutions, even without convexity
- Monotonicity is a key property
- If the interference functions are additionally convex or concave, then more efficient solutions are possible (➠ following slides)

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Power Minimization with Convex/Concave $\mathcal{J}_k(\mathbf{p})$

- The problem can be written as

$$\min_{\mathbf{p}\geq 0} \sum_{1\leq l\leq K} p_l \quad \text{s.t.} \quad \gamma_k \mathcal{J}_k(\mathbf{p}) - p_k \leq 0 \,, \quad \forall k \,. \tag{1}$$

- If $\mathcal{J}_k$ are convex (e.g. robust optimization), then (1) is convex
- If $\mathcal{J}_k$ are concave (e.g. beamforming), then (1) is non-convex in general
- Monotonicity enables an equivalent convex reformulation

$$\max_{\mathbf{p}\geq 0} \sum_{l\in\mathcal{K}} p_l \quad \text{s.t.} \quad p_k - \gamma_k \mathcal{J}_k(\mathbf{p}) \leq 0 \quad \text{for all } k \,.$$

Special Case: Multiuser Downlink Beamforming

$$\min_{\mathbf{w}_1,\ldots,\mathbf{w}_K \in \mathbb{C}^M} \sum_{k=1}^{K} p_k \quad \text{s.t.} \quad \mathrm{SINR}_k(\mathbf{w}_1, \ldots, \mathbf{w}_K) \geq \gamma_k \ , \quad \forall k$$

- This problem was studied for more than a decade. Different solutions exist based on uplink/downlink duality [Rashid-Farrokhi et al., 1998; Schubert and Boche, 2004], semidefinite ralaxation [Bengtsson and Ottersten, 2001] and conic optimization [Wiesel et al., 2006]
- Interference calculus
  - helps to better understand the underlying structure of the problem (equivalent convex reformulation)
  - generalizes the results to arbitrary concave or convex interference functions

Representation of Concave Interference Functions

Theorem 5 ([Boche and Schubert, 2008b])

Let $\mathcal{I}(\mathbf{p})$ be an arbitrary concave interference function, then

$$\mathcal{I}(\mathbf{p}) = \min_{\mathbf{w} \in \mathcal{N}_0(\mathcal{I})} \sum_{k=1}^{K} w_k p_k \ , \quad \textit{for all } \mathbf{p} > 0.$$

where

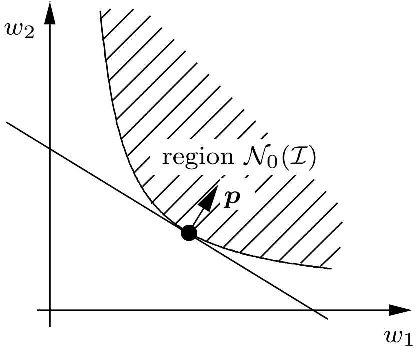
$$\mathcal{N}_0(\mathcal{I}) = \{\mathbf{w} \in \mathbb{R}_+^K : \underline{\mathcal{I}}^*(\mathbf{w}) = 0\}$$

and $\underline{\mathcal{I}}^*(\mathbf{w}) = \inf_{\mathbf{p}>0} \Big(\sum_{l=1}^{K} w_l p_l - \mathcal{I}(\mathbf{p}) \Big)$ *is the conjugate of* $\mathcal{I}$.

Interpretation of Concave Interference Functions

$$\mathcal{I}(\mathbf{p}) = \min_{\mathbf{w} \in \mathcal{N}_0(\mathcal{I})} \sum_{k=1}^{K} w_k p_k$$

- the set $\mathcal{N}_0(\mathcal{I})$ is closed, convex, and upward-comprehensive
- any concave interference function can be interpreted as the solution of a loss/cost minimization problem



Structure of Concave Standard Interference Functions

Theorem 6 ([Boche and Schubert, 2010])

$\mathcal{J}_l(\mathbf{p})$ is a concave standard interference function iff there exists a non-empty, closed, convex, comprehensive set $\mathcal{V}_l \subset \mathbb{R}_+^{L+1}$ such that

$$\mathcal{J}_l(\mathbf{p}) = \min_{\mathbf{v} \in \mathcal{V}_l} \Big(\sum_{j=1}^{L} p_j v_j + v_{L+1} \Big), \qquad \textit{where } v_{L+1}^l > 0$$

- Interpretation: interference resulting from adaptive receive strategies z_k:

$$\mathcal{I}_k(\mathbf{p}) = \min_{z_k \in \mathcal{Z}_k} \Big(\underbrace{\mathbf{p}^T \mathbf{v}(z_k)}_{\textit{Interference}} + \underbrace{n_k(z_k)}_{\textit{Noise}} \Big) , \quad k = 1, 2, \ldots, K$$

Exploiting the Structure of Concave Interference Functions

- Structure of concave interference functions can be exploited for the development of (sub-)gradient algorithms
- Example: power minimization problem

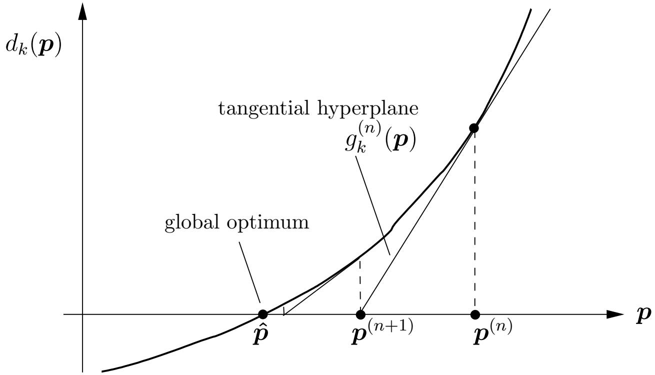
$$\min_{\mathbf{p}\in\mathcal{P}} \sum_{k\in\mathcal{K}} p_k \quad \text{s.t.} \quad \frac{p_k}{\mathcal{I}_k(\underline{\mathbf{p}})} \geq \gamma_k \text{ for all } k \, .$$

- The constraints can be rewritten as

$$\mathbf{d}(\mathbf{p}) = \mathbf{p} - \mathbf{\Gamma}\mathcal{I}(\mathbf{p})$$

- Due to strict monotonicity, the unique global optimum is completely characterized by $\mathbf{d}(\mathbf{p}) = 0$ (fixed point)

Newton-Type Iteration



- Newton-type iteration [Boche and Schubert, 2008d]
    - Jacobian: coupling matrix $\mathbf{V}(z) = [\mathbf{v}_1(z_1), \ldots, \mathbf{v}_K(z_K)]^T$
    - No assumptions on smoothness

Example: Adaptive Receive Strategy

Alternating optimization of receive strategies $z^{(n)}$ and power allocation $\mathbf{p}^{(n)}$

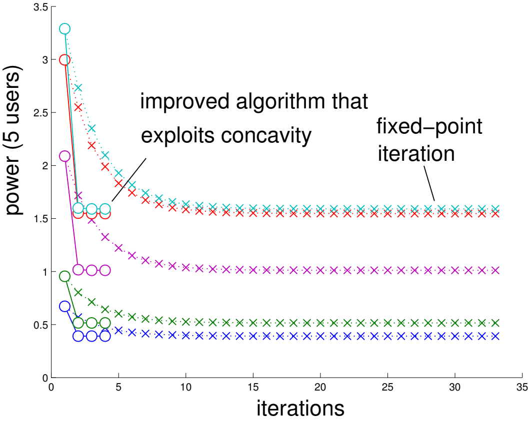
1. $z_k^{(n)} = \arg\min_{z_k \in \mathcal{Z}_k} \Big[\mathbf{V}(z)\mathbf{p}^{(n)} + \mathbf{n}(z) \Big]_k , \quad k \in \{1, 2, \ldots, K\}$
2. $\mathbf{p}^{(n+1)} = (\mathbf{I} - \mathbf{\Gamma}\mathbf{V}(z^{(n)}))^{-1} \cdot \mathbf{\Gamma} N(z^{(n)})$

- This algorithm can be applied whenever the underlying interference functions are strongly monotone (standard) and concave
- Corresponding results can be shown for convex interference functions (robust optimization) [Schubert and Boche, 2012]

Convergence Analysis

The sequence $\mathbf{p}^{(n)}$ has super-linear convergence [Boche and Schubert, 2008d].

$$\lim_{n\to\infty} \frac{\|\mathbf{p}^{(n+1)} - \mathbf{p}^*\|_1}{\|\mathbf{p}^{(n)} - \mathbf{p}^*\|_1} = 0$$



Convex Interference Functions

- Similar results can be shown for convex interference functions
- Example: Robust Optimizaztion. Worst-case interference

$$\mathcal{I}_k(\mathbf{p}) = \max_{c_k \in \mathcal{C}_k} \mathbf{p}^T \mathbf{v}(c_k), \quad \forall k \ ,$$

  where the parameter c_k models an 'uncertainty' (e.g. caused by channel estimation errors or system imperfections).
- the optimization is over a compact uncertainty region $\mathcal{C}_k$

Representation of Convex Interference Functions

Theorem 7 ([Boche and Schubert, 2008b])

Let $\mathcal{I}(\mathbf{p})$ be an arbitrary convex interference function, then

$$\mathcal{I}(\mathbf{p}) = \max_{\mathbf{w} \in \mathcal{W}_0(\mathcal{I})} \sum_{k=1}^{K} w_k \cdot p_k \ , \quad \textit{for all } \mathbf{p} > 0.$$

where

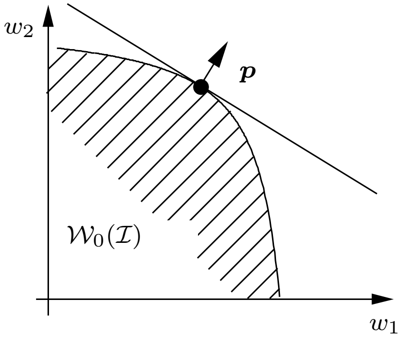
$$\mathcal{W}_0(\mathcal{I}) = \{\mathbf{w} \in \mathbb{R}_+^K : \bar{\mathcal{I}}^*(\mathbf{w}) = 0\}$$

and $\bar{\mathcal{I}}^*(\mathbf{w}) = \sup_{\mathbf{p}>0}\Big(\sum_{l=1}^{K} w_l p_l - \mathcal{I}(\mathbf{p})\Big)$ *is the conjugate of* $\mathcal{I}$.

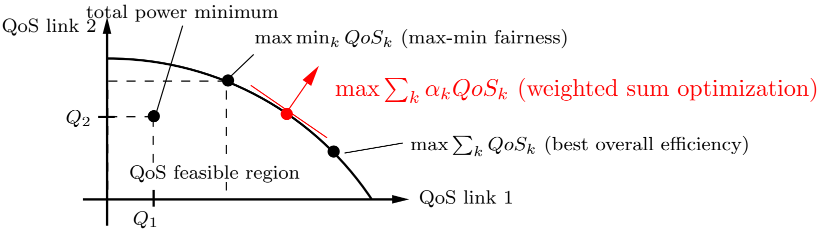
Interpretation of Convex Interference Functions

$$\mathcal{I}(\mathbf{p}) = \max_{\mathbf{w} \in \mathcal{W}_0(\mathcal{I})} \sum_{k=1}^{K} w_k \cdot p_k$$

- the set $\mathcal{W}_0(\mathcal{I})$ is closed, convex, and downward-comprehensive
- any convex interference function can be interpreted as the solution of a utility maximization problem



Weighted Sum Rate Maximization



$$R(\boldsymbol{\alpha}) = \max_{\mathbf{p}\in\mathcal{P}} \sum_{l=1}^{L} \alpha_l \log\left(1 + \frac{p_l}{\mathcal{I}_l(\mathbf{p})}\right)$$

NP hard [Hayashi and Luo, 2009]. For global optimization it is important to exploit concavity and monotonicity

Exploiting Concavity and Monotonicity

- Problem can be rewritten as

$$R(\boldsymbol{\alpha}) = \max_{\mathbf{p}\in\mathcal{P}} \sum_{l=1}^{L} \alpha_l \log\left(1 + \frac{p_l}{\mathcal{I}_l(\mathbf{p})}\right)$$

$$= \max_{\mathbf{p}\in\mathcal{P}} \left(\sum_{l=1}^{L} \alpha_l \log\big(p_l + \mathcal{I}_l(\mathbf{p})\big) - \sum_{l=1}^{L} \alpha_l \log\big(\mathcal{I}_l(\mathbf{p})\big) \right)$$

- Difference of monotone functions ➠ polyblock strategies
- Difference of convex functions ➠ DC programming

➠ Efficient algorithms available, e.g. [Eriksson et al., 2010] (but still exponential complexity)

Outline

1. Introduction and History
2. Axiomatic Framework
3. QoS Sets
4. User-Centric ("Fair") Strategies based on Monotonicity
5. Concave and Convex Interference Functions
6. **Log-Convex Interference Functions**
7. Conclusions

Motivation: “Hidden Convexity”

- In the context of linear interfererence functions, it was observed ([Sung, 2002; Stanczak et al., 2008; Tan et al., 2007]) that certain power control problems can be convexified by a change of variable

$$\mathbf{p} = \exp \mathbf{s} \qquad \text{(component-wise exponential)}$$

- A more general approach is provided by the framework of log-convex interference functions

Two main aspects:

- convexification of the QoS region
- convexification/concavification of the target function

Log-Convex Interference Functions

Definition 8

We say that $\mathcal{I} : \mathbb{R}_+^K \mapsto \mathbb{R}_+$ is a log-convex interference function if it fulfills the axioms:

A1	(non-negativeness)	$\mathcal{I}(\mathbf{p}) \geq 0$	
A2	(scale invariance)	$\mathcal{I}(\alpha\mathbf{p}) = \alpha\mathcal{I}(\mathbf{p})$	$\forall \alpha \in \mathbb{R}_+$
A3	(monotonicity)	$\mathcal{I}(\mathbf{p}) \geq \mathcal{I}(\mathbf{p}')$	if $\mathbf{p} \geq \mathbf{p}'$
C3	(log-convexity)	$\mathcal{I}_k(\mathrm{e}^{\mathbf{s}})$ is log-convex on $\mathbb{R}^K$	

Examples of Log-Convex Interference Functions

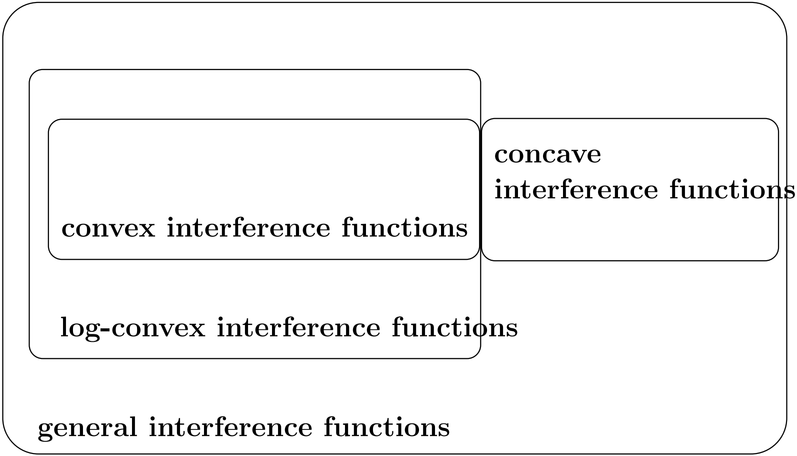
- Any convex interference function is a log-convex interference function (notice the change of variable!)
- this includes the class of linear interference functions
- This also includes the class of worst-case designs

$$\mathcal{I}_k(\mathbf{p}) = \max_{c_k \in \mathcal{C}_k} \mathbf{p}^T \mathbf{v}(c_k), \quad \forall k \,,$$

  where the parameter c_k models an 'uncertainty' (e.g. caused by channel estimation errors or system imperfections).
- Another example: multiplicative utility function

$$\mathcal{I}(\mathbf{r}) = \prod_{l=1}^{K} (r_l)^{w_l} \qquad \text{where} \sum_k w_k = 1$$

Categories of Interference Functions



Representation of Log-Convex Interference Functions

Theorem 9 ([Boche and Schubert, 2008a])

Every log-convex interference function $\mathcal{I}(\mathbf{p})$*, with* $\mathbf{p} > 0$*, can be represented as*

$$\mathcal{I}(\mathbf{p}) = \max_{\mathbf{w} \in \mathcal{L}(\mathcal{I})} \Big(f_{\mathcal{I}}(\mathbf{w}) \cdot \prod_{l=1}^{K} (p_l)^{w_l} \Big) .$$

where $f_{\mathcal{I}}(\mathbf{w}) = \inf_{\mathbf{p} > 0} \frac{\mathcal{I}(\mathbf{p})}{\prod_{l=1}^{K} (p_l)^{w_l}} , \quad \mathbf{w} \in \mathbb{R}_+^K, \quad \sum_k w_k = 1$

$$\mathcal{L}(\mathcal{I}) = \big\{ \mathbf{w} \in \mathbb{R}_+^K : f_{\mathcal{I}}(\mathbf{w}) > 0 \big\}$$

Application Example: Weighted Sum QoS

$$\inf_{\mathbf{s}\in\mathbb{R}^K} \sum_k \alpha_k \, g\big(\mathcal{I}_k(\mathrm{e}^{\mathbf{s}})/\mathrm{e}^{s_k}\big) \quad \text{s.t.} \;\; \|\mathrm{e}^{\mathbf{s}}\|_1 \leq P_{\max} \;,$$

Theorem 10 ([Boche and Schubert, 2008a])

Suppose that $\mathcal{I}_k(\mathrm{e}^{\mathbf{s}})$ is log-convex for all and g is monotone increasing. Then the problem is convex if and only if $g(\mathrm{e}^{x})$ is convex on $\mathbb{R}$.

Application example:

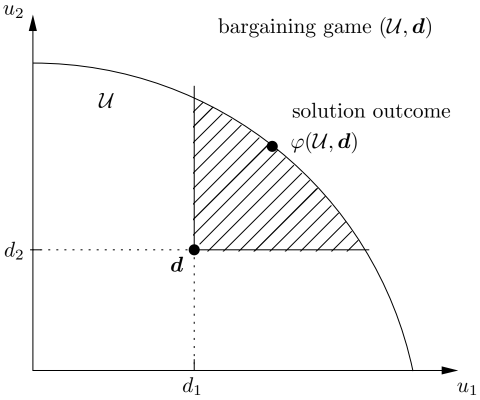
$$\max_{\mathbf{p}\geq 0} \sum_k \log(1 + \mathsf{SINR}(\mathbf{p})) \quad \text{(Sum Rate Maximization)}$$

can be convexified under the approximation
$\log(1 + \mathsf{SINR}(\mathbf{p})) \approx \log(\mathsf{SINR}(\mathbf{p}))$

Application Example: Cooperative Nash Bargaining

K players try to reach an unanimous agreement on utilities $\mathbf{u} = [u_1, \ldots, u_k]$

- the utility region $\mathcal{U} \subset \mathbb{R}^K_{++}$ is convex, comprehensive, closed, bounded
- Depending on the chosen strategy, the solution outcome φ results
- If the bargaining fails, the disagreement outcome $\mathbf{d}$ results



Axiomatic Framework for Symmetric Nash Bargaining

- WPO **Weak Pareto Optimality**. The players should not be able to collectively improve upon the solution outcome.
- IIA **Independence of Irrelevant Alternatives.** If the feasible set shrinks but the solution outcome remains feasible, then the outcome is also the solution of the smaller set.
- SYM **Symmetry.** If the region is symmetric, then the outcome does not depend on the identities of the users.
- STC **Scale Transformation Covariance.** The outcome is component-wise scale-invariant.

The Nash Product

For convex comprehensive sets the unique Nash bargaining solution fulfilling the axioms WPO, IIA, SYM, STC is obtained as the solution of

$$\max_{\{\mathbf{u}\in\mathcal{U}:\mathbf{u}>\mathbf{d}\}} \prod_{k=1}^{K} (u_k - d_k)$$

Often, the choice of the zero of the utility scales does not matter, so we can choose $\mathbf{d} = \mathbf{0}$

$$\max_{\mathbf{u}\in\mathcal{U}} \prod_{k=1}^{K} u_k$$

Nash Bargaining and Proportional Fairness

- the product optimization approach is equivalent to proportional fairness [Kelly et al., 1998]

$$\hat{\mathbf{u}} = \arg\max_{\mathbf{u}\in\mathcal{U}} \prod_{k=1}^{K} u_k = \arg\max_{\mathbf{u}\in\mathcal{U}} \log \prod_{k=1}^{K} u_k = \arg\max_{\mathbf{u}\in\mathcal{U}} \sum_{k=1}^{K} \log u_k$$

- if the region $\mathcal{U}$ is convex closed comprehensive and bounded, then symmetric Nash bargaining and proportional fairness are equivalent

Bargaining over SIR Feasible Sets

- for wireless systems, an important performance measure is the signal-to-interference ratio

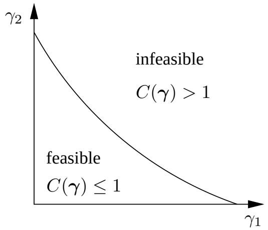
$$\mathsf{SIR}_k(\mathbf{p}) = \frac{p_k}{\mathcal{I}_k(\mathbf{p})} \qquad \begin{array}{l} \leftarrow \text{useful power} \\ \leftarrow \text{interference (+noise) power} \end{array}$$

- indicator of feasibility: $C(\gamma) = \inf_{\mathbf{p}>0}\Big(\max_k \frac{\gamma_k \mathcal{I}_k(\mathbf{p})}{p_k}\Big)$

- the SIR region

$$\mathcal{S} = \{\gamma \in \mathbb{R}_+^K : C(\gamma) \leq 1\}$$

is generally not convex, so results from classical bargaining theory cannot be applied directly



The Log-Convex Case

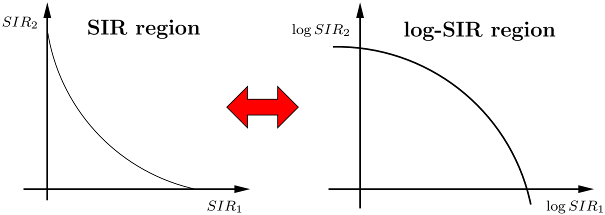
- Let $\mathcal{I}_1, \ldots, \mathcal{I}_K$ be log-convex interference functions, then

$$C(\gamma) = \inf_{\mathbf{p}>0} \Big(\max_{1 \leq k \leq K} \frac{\gamma_k \mathcal{I}_k(\mathbf{p})}{p_k} \Big)$$

  is a log-convex interference function, i.e., $C(\exp \mathbf{q})$ is a log-convex (thus convex) function.
- the SIR feasible set $\mathcal{S} = \{\gamma : C(\gamma) \leq 1\}$ is convex on a logarithmic scale
- this "hidden convexity" can be exploited for designing resource allocation algorithms

Logarithmically Convex Regions, “Hidden Convexity”

- If the underlying interference functions are log-convex, then the SIR region is log-convex

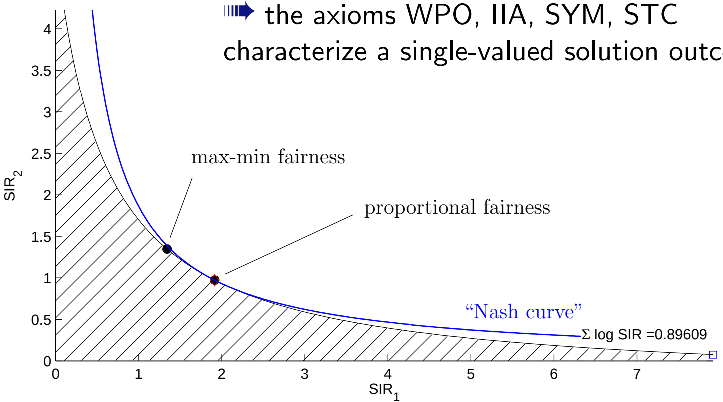


- SIR region has special properties which can be exploited for bargaining (closed, comprehensive, log-convex)

Extension of the Classical Nash Bargaining Framework

- The classical Nash bargaining framework extends to utility sets that are strictly convex after a log-transformation [Boche and Schubert, 2009]

➠ the axioms WPO, IIA, SYM, STC characterize a single-valued solution outcome



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Conclusions

- Coupled multiuser systems are often difficult to handle. Adaptive strategies for interference mitigation/avoidance (MIMO, scheduling, etc) offer new degrees of freedom, but they also complicate the task of resource allocation.
- A thorough understanding of the interference structure is the key to the development of efficient algorithmic solutions
- Interference calculus offers
  - ➠ abstract model, focus on core properties
  - ➠ rigorous, allows to handle problems analytically
  - ➠ provides intuition and roadmap for design of algorithms
- Applications beyond wireless communications. Coupled systems also play a central role in other disciplines.

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