Approximate Counting and the Lovasz Local Lemma

Ankur Moitra (MIT)

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Ex: $(\overline{x}_1 \vee x_3 \vee x_8) \wedge (x_1 \vee \overline{x}_6 \vee x_7) \wedge \ldots \wedge (\overline{x}_2 \vee \overline{x}_3 \vee x_7)$

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(2) A random assignment can be exponentially unlikely to satisfy the formula

(3) There is an efficient algorithm to find a satisfying assignment due to **[Moser, Tardos '10]**

THE CONSTRUCTIVE LOCAL LEMMA

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Moser-Tardos works under some constraints on how events are described, many improvents and generalizations:

[Haeupler, Saha, Srinivasan '11] [Harris, Srinivasan '14]

[Achlioptas, Iliopoulis '14] [Harvey, Vondrak '15] [Kolmogorov '16]

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But the distribution on solutions it finds can be far from uniform!

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Approximate Counting Local Lemma:

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Can we count the number of needles?

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Counting under the LLL conditions is not self-reducible, but nevertheless we'll solve both problems simultaneously!

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[Sly '10] showed that approximate counting is NP-hard above this

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(4) **Computational:** When does approximate counting go from easy to hard?

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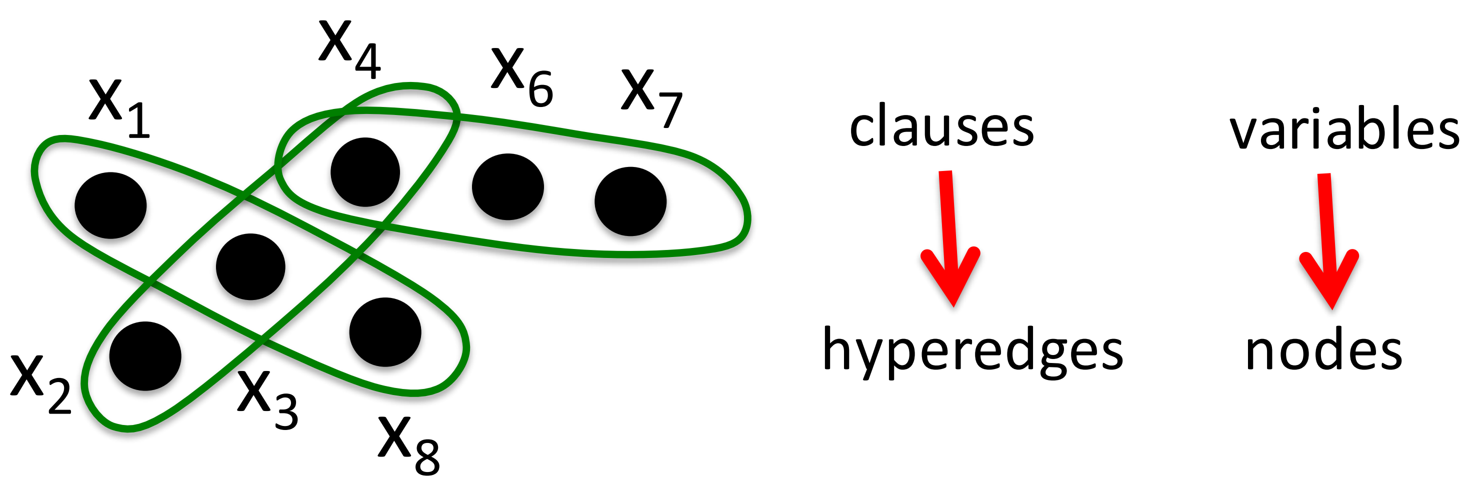
we get the **hypergraph independent set** problem:



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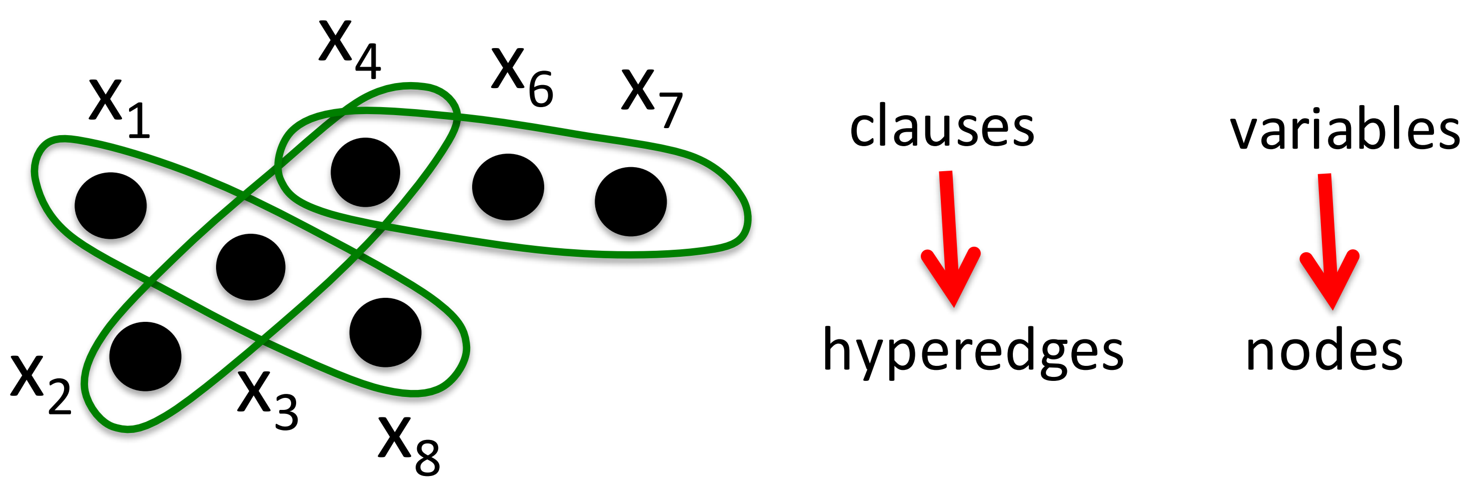

Definition: An independent set (in a hypergraph) is a set of nodes with no induced hyperedge

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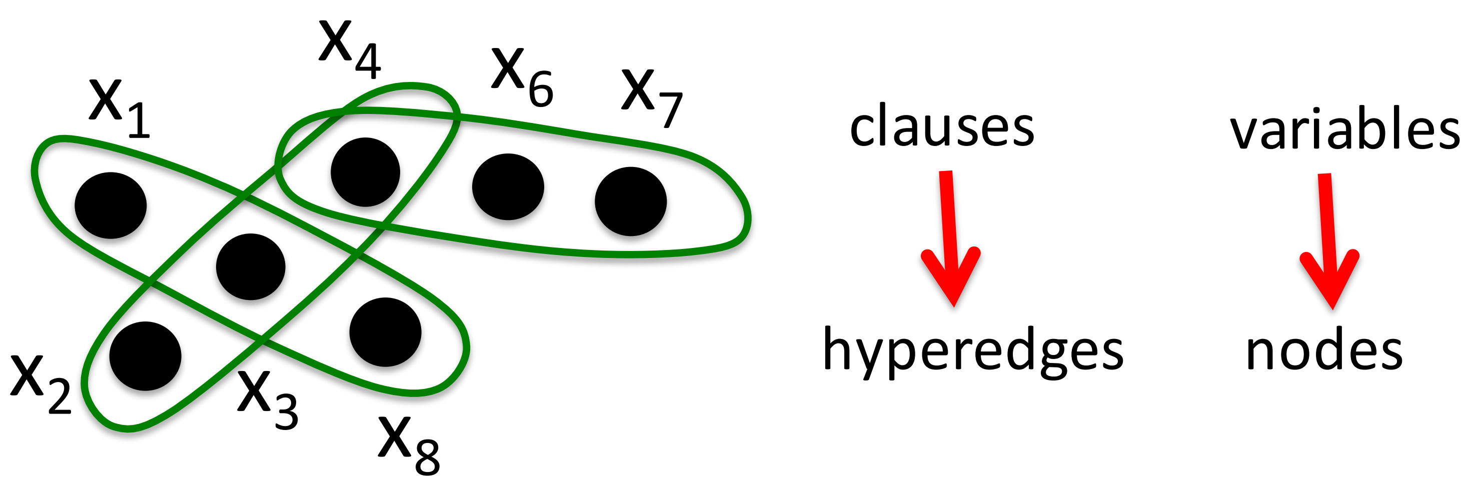

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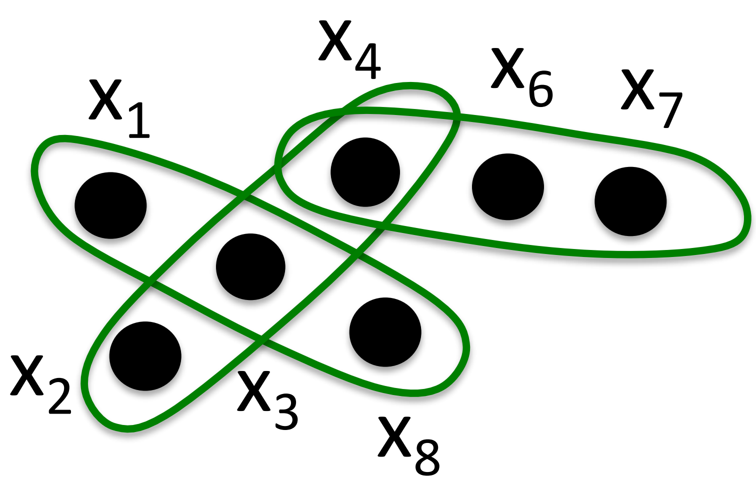

clauses → hyperedges

variables → nodes

Claim: The number of satisfying assignments is equal to the number of independent sets

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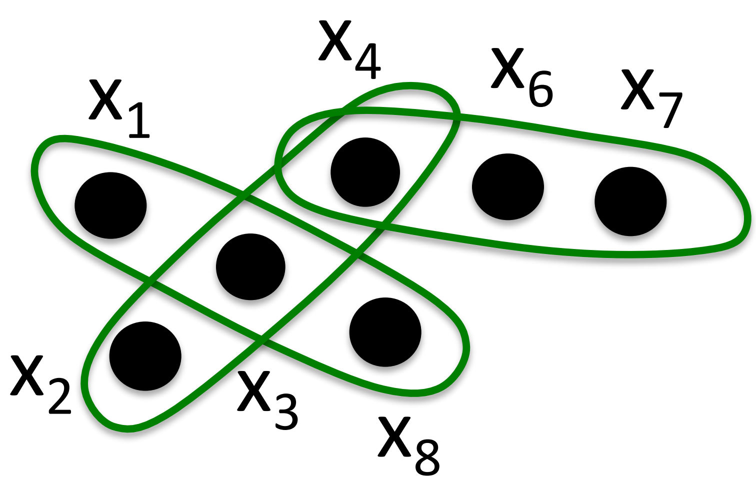

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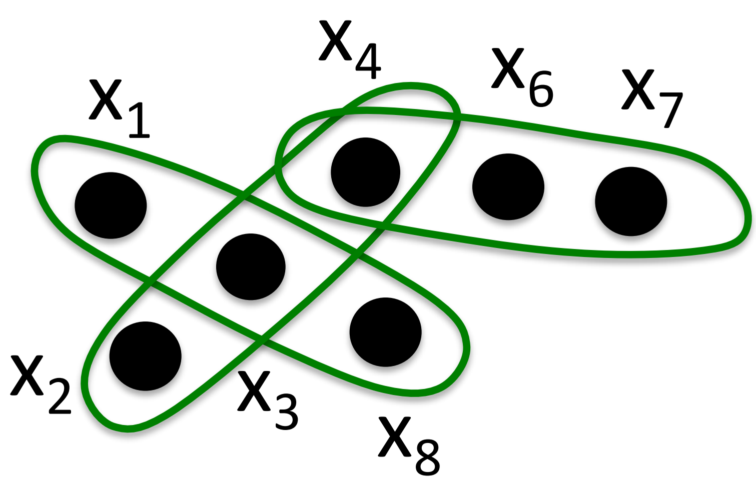

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Comment: Let d be maximum degree, then $d \leq D \leq 2kd$ if at most 2k variables per clause

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[Bezakova et al.] gave a deterministic algorithm under same conds.

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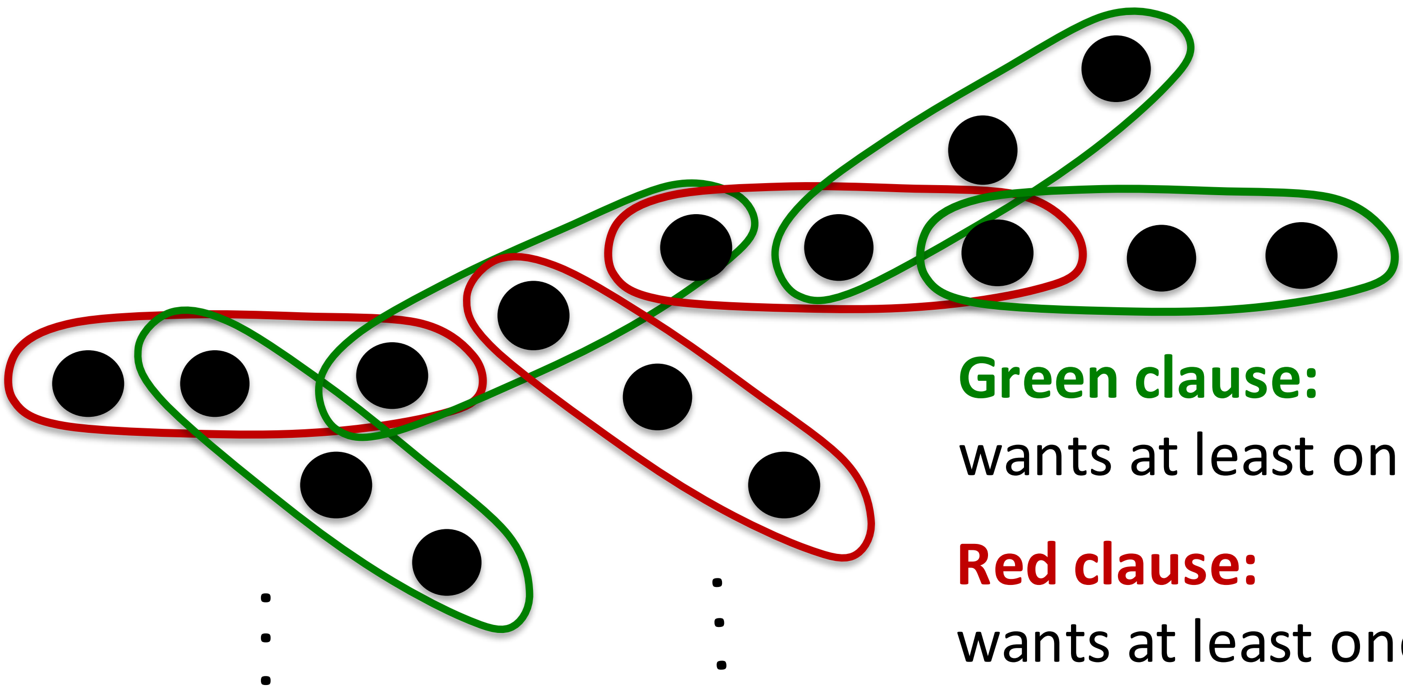

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wants at least one green

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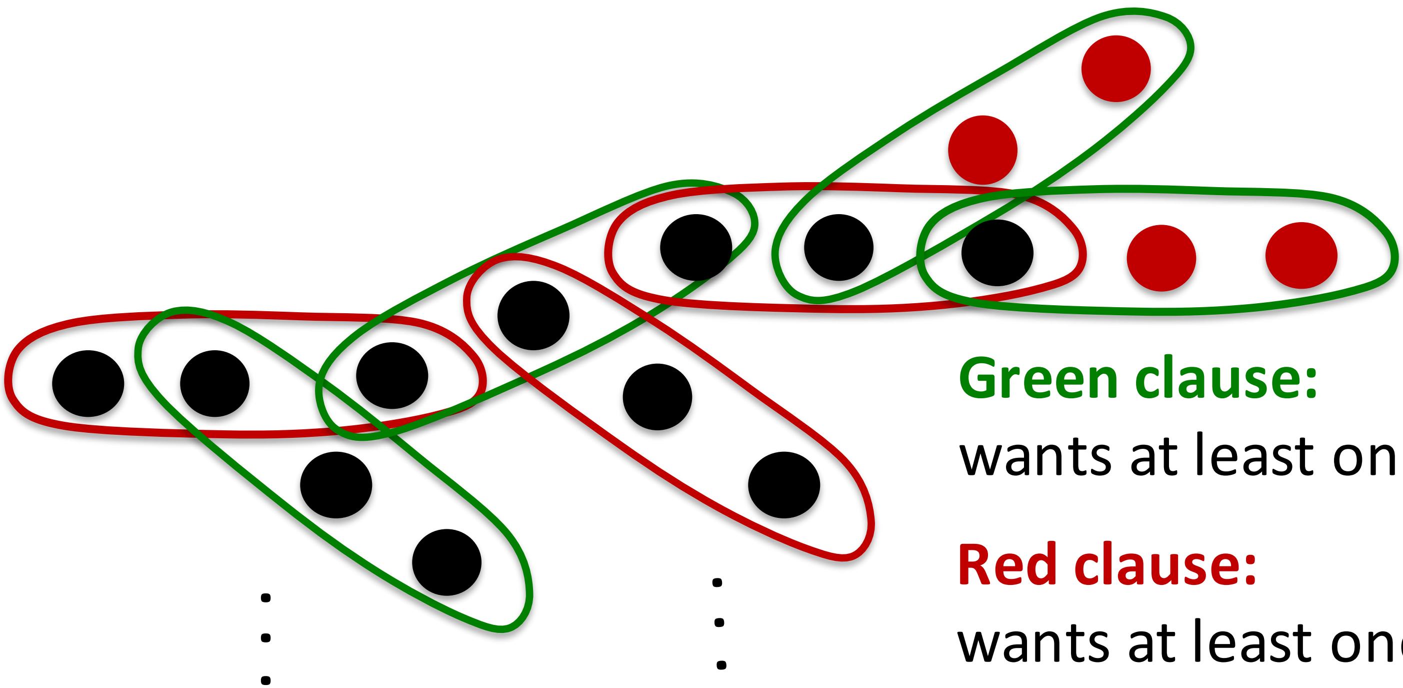

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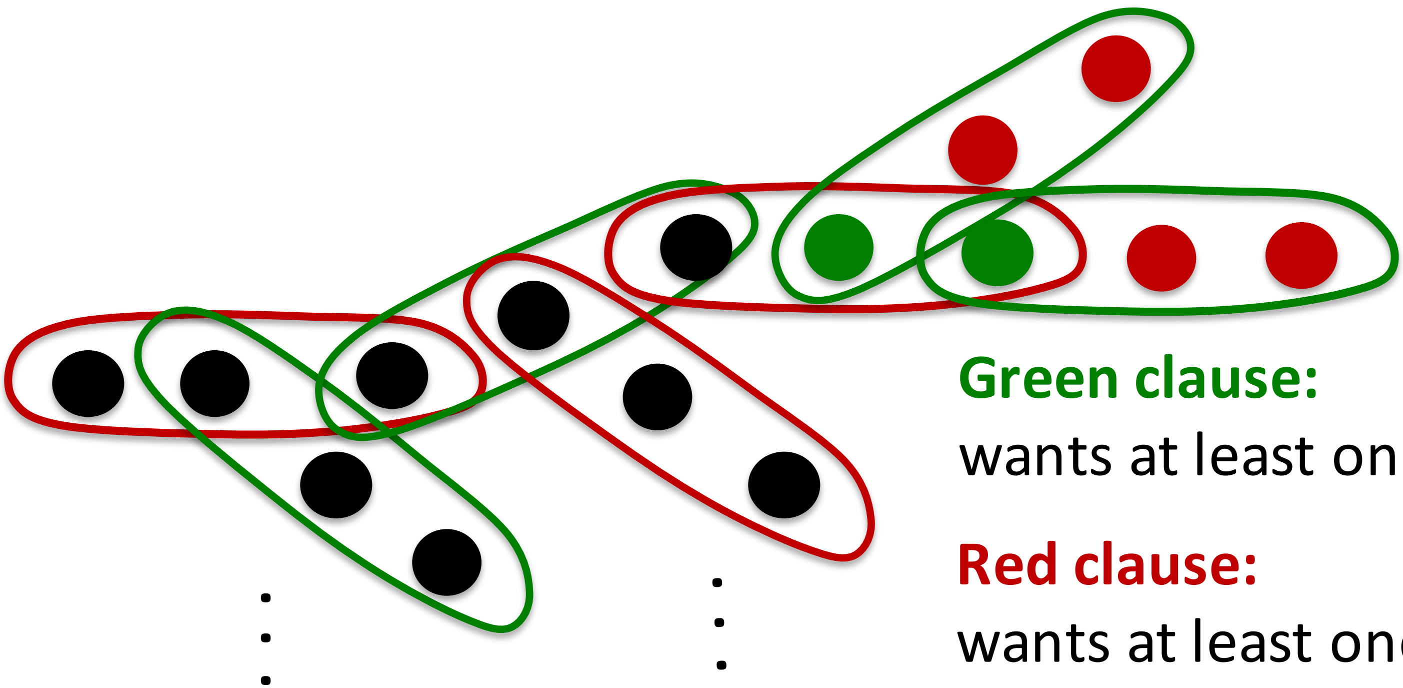

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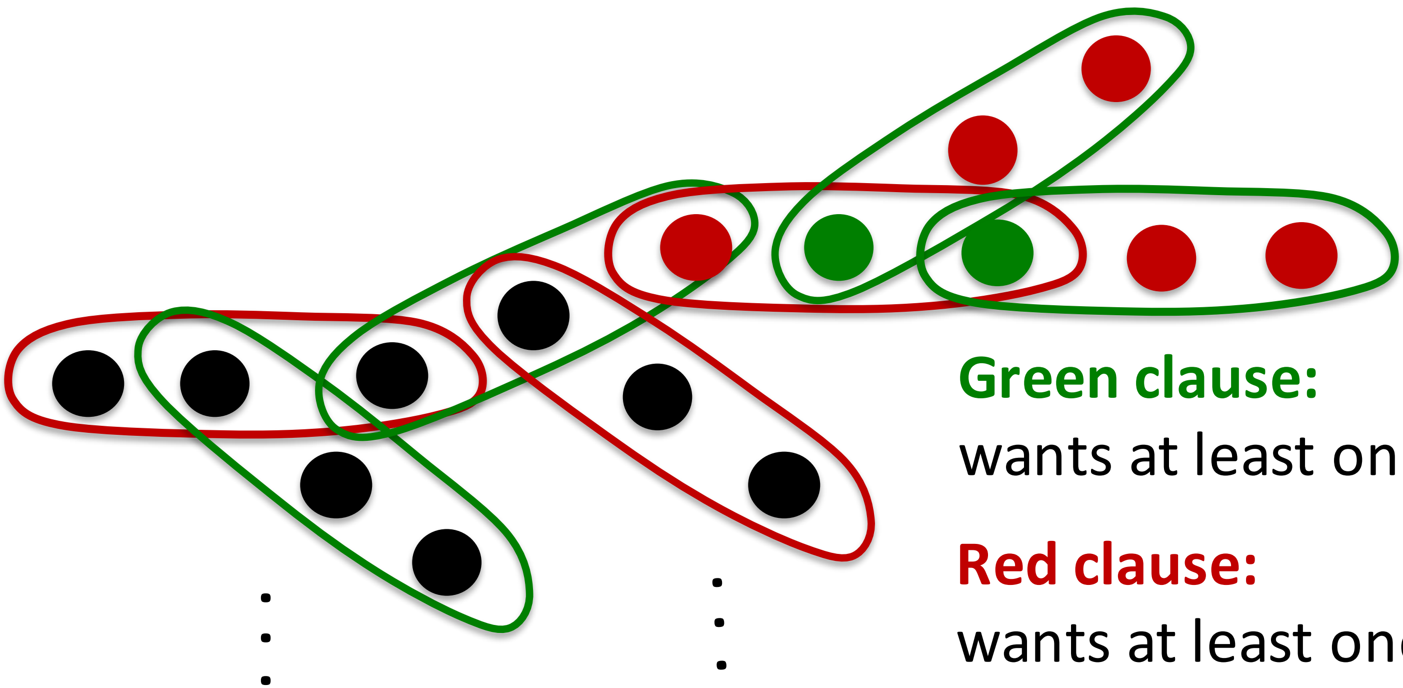

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OUR RESULTS (COUNTING)

For general CNFs with between k and 2k variables per clause

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Also extends to non-binary counting problems

e.g. **red**, **green**, **blue** assignments with NAE constraints

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i.e. the output of the algorithm is n^{-T}-close in total variation distance to the uniform distribution on satisfying assignments

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Both of these results are tight – it is NP-hard for larger d even for the types of restricted instances they consider

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(3) Use linear programming to find this special type of coupling that we now know exists

Thanks!

Main Open Question:

Is $D \leq c \cdot 2^{k/2}$ the true threshold for algorithmically counting and sampling in k-CNFs?

Any Questions?