Maximum Size of 2-factor Isomorphic Graphs

Paul Wrayno

Western Kentucky University

June 1, 2012
joint work with Ron Gould

Definitions

Definition

A 2-factor of a graph, G, is a spanning 2-regular subgraph of G. It can also be thought of as as a collection of disjoint cycles that cover the vertices of G.

Definitions

Definition

A 2-factor of a graph, G, is a spanning 2-regular subgraph of G. It can also be thought of as as a collection of disjoint cycles that cover the vertices of G.

Definition

A graph, G, is said to be 2-factor hamiltonian if all of its 2-factors are hamiltonian cycles.

Definitions

Definition

A 2-factor of a graph, G, is a spanning 2-regular subgraph of G. It can also be thought of as as a collection of disjoint cycles that cover the vertices of G.

Definition

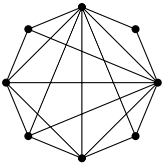
A graph, G, is said to be 2-factor hamiltonian if all of its 2-factors are hamiltonian cycles.

Definition

A graph, G, is 2-factor isomorphic if it contains a 2-factor, F, and all other 2-factors are isomorphic to F. In other words, all 2-factors in G are the same when viewed as a collection of unlabeled cycles.

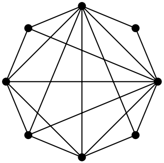
Illustrations of Definitions

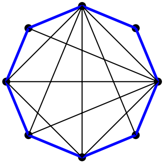
This graph is 2-factor hamiltonian and so 2-factor isomorphic:



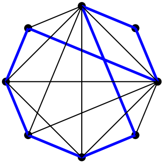
Illustrations of Definitions

This graph is 2-factor hamiltonian and so 2-factor isomorphic:





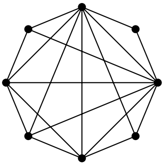
$\{C_8\}$ 2-factor

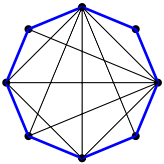


$\{C_8\}$ 2-factor

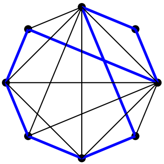
Illustrations of Definitions

This graph is 2-factor hamiltonian and so 2-factor isomorphic:



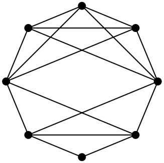


$\{C_8\}$ 2-factor



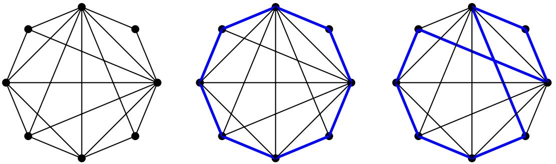
$\{C_8\}$ 2-factor

This graph is not 2-factor isomorphic



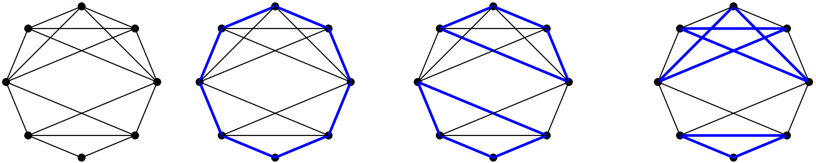
Illustrations of Definitions

This graph is 2-factor hamiltonian and so 2-factor isomorphic:



$\{C_8\}$ 2-factor $\{C_8\}$ 2-factor

This graph is not 2-factor isomorphic



$\{C_8\}$ 2-factor $\{C_4, C_4\}$ 2-factor $\{C_3, C_5\}$ 2-factor

Primary Questions

Question

What is the maximum size of a 2-factor isomorphic graph?

Primary Questions

Question

What is the maximum size of a 2-factor isomorphic graph?

Question

What do some 2-factor isomorphic graphs of this size look like?

Large 2-factor Hamiltonian Bipartite Graphs

In 2007 Faudree, Gould, and Jacobson determined the maximum size of 2-factor hamiltonian bipartite graphs.

Theorem

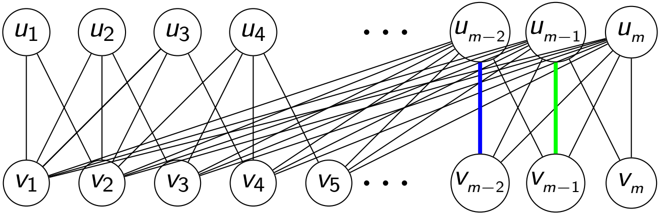
If G is a bipartite 2-factor hamiltonian graph of order n, $n \geq 8$, then

$$|E(G)| \leq \left\lceil \frac{n(n+4)}{8} \right\rceil$$

and the bound is sharp.

Large 2-factor Hamiltonian Bipartite Graphs

To demonstrate sharpness, they used this construction, which we will refer to as $B(n, V)$.



$B(n, V)$

The blue edge is only present when m is even and the green edge is only present when m is odd.

Small 2-factor Hamiltonian Bipartite Graphs

- For smaller n, the complete balanced bipartite graphs $K_{2,2}$ and $K_{3,3}$ are 2-factor Hamiltonian. For convenience, we will refer to these graphs as $B(4, V)$ and $B(6, V)$ respectively.

Small 2-factor Hamiltonian Bipartite Graphs

- For smaller n, the complete balanced bipartite graphs $K_{2,2}$ and $K_{3,3}$ are 2-factor Hamiltonian. For convenience, we will refer to these graphs as $B(4, V)$ and $B(6, V)$ respectively.
- For $n = 4$, the expression $\dfrac{n(n+4)}{8} = 4$ is still the correct maximum size.

Small 2-factor Hamiltonian Bipartite Graphs

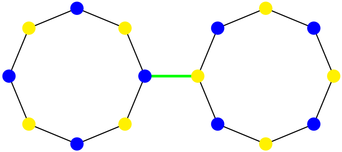
- For smaller n, the complete balanced bipartite graphs $K_{2,2}$ and $K_{3,3}$ are 2-factor Hamiltonian. For convenience, we will refer to these graphs as $B(4, V)$ and $B(6, V)$ respectively.
- For $n = 4$, the expression $\dfrac{n(n+4)}{8} = 4$ is still the correct maximum size.
- For $n = 6$, the expression $\left\lceil \dfrac{n(n+4)}{8} \right\rceil = 8$ is one less than the correct maximum size of 9.

Chords and Edges of Cycles in F

- For any 2-factor isomorphic bipartite graph, G, select a particular 2-factor F.
- For a cycle C_j, of length j in F, the induced subgraph $G[C_j]$ is 2-factor hamiltonian, and so there are, at most, $\left\lceil \frac{j(j+4)}{8} \right\rceil$ edges if $j \neq 6$ (or 9 edges if $j = 6$) between the vertices of C_j.

Edges Between Cycles in F

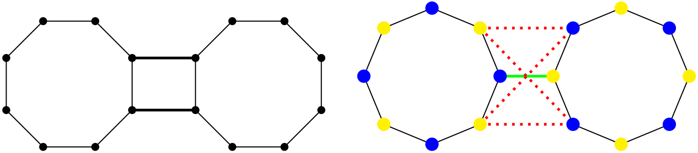
- A single edge from C_i to C_j uniquely determines the bipartition of the vertices in C_i and C_j, forbidding all monochromatic edges in the graph below.



- Call two edges between C_i and C_j paired if they, together with an edge on each of C_i and C_j form a C_4.

Edges Between Cycles in F

- If there are paired edges between two cycles, then the two cycles in F can be replaced with a single cycle, resulting in a non-isomorphic 2-factor.



- Any edge between cycles forbids the four red edges and is forbidden by any of the four red edges.
- Note that none of these four edges is monochromatic, so none of these edges were forbidden by the bipartition.

Edges Between Cycles in F

The bipartition limited the number of possible edges to $\frac{i \cdot j}{2}$ and the paired edges argument implies that at least as many of these edges are forbidden as are present. Therefore there are at most $\frac{i \cdot j}{4}$ edges between C_i and C_j.

There are then a maximum of $\sum_{C_i, C_j \in F} \frac{i \cdot j}{4}$ edges betweens cycles in F.

Combining the Bounds

Rewriting the sum over pairs as a double sum with each pair appearing twice, gives

$$\sum_{C_j \in F} \sum_{C_i \in F} \frac{i \cdot j}{8} - \sum_{C_j \in F} \frac{j^2}{8} \text{ edges.}$$

Combining the Bounds

Rewriting the sum over pairs as a double sum with each pair appearing twice, gives

$$\sum_{C_j \in F} \sum_{C_i \in F} \frac{i \cdot j}{8} - \sum_{C_j \in F} \frac{j^2}{8} \text{ edges.}$$

Combining this with the bound on the number of edges and chords of the cycles in F gives an upper bound of

$$\sum_{C_j \in F} \sum_{C_i \in F} \frac{i \cdot j}{8} - \sum_{C_j \in F} \frac{j^2}{8} + \sum_{C_j \in F} \frac{j(j+4)}{8} + \frac{c_2^*}{2} + c_6$$

where c_2^* is the number of cycles of the length $4i+2$ and c_6 is the number of C_6's in F.

Combining the Bounds

Summing over i simplifies the bound to

$$\sum_{C_j \in F} \frac{(n+4)j}{8} + \frac{c_2^*}{2} + c_6$$

and summing over j gives a final result very similar to the hamiltonian case.

Theorem

If G is a bipartite 2-factor isomorphic graph of order n, then

$$|E(G)| \leq \frac{n(n+4)}{8} + \frac{c_2^*}{2} + c_6.$$

Construction

1. Start with the specified 2-factor, F, as our graph.

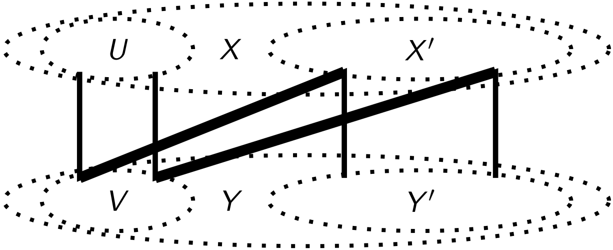
Construction

1. Start with the specified 2-factor, F, as our graph.
2. If F is a single cycle, use an appropriate $B(n, V)$ to cover the cycle.

Construction

1. Start with the specified 2-factor, F, as our graph.
2. If F is a single cycle, use an appropriate $B(n, V)$ to cover the cycle.
3. If F consists of multiple cycles, remove a C_j from F and construct a bipartite 2-factor isomorphic graph of maximum size, G', on $F - C_j$. Cover the C_j with a $B(j, V)$ and join the V partite set of $B(j, V)$ to the X' partite set of G'.

Construction



Joining V and X'

All 2-factors consist of the union of a 2-factor in $B(j, V)$ and one in G' and since both are 2-factor isomorphic, their unions must be as well.

Large 2-factor Hamiltonian Graphs

In 2007 Faudree, Gould, and Jacobson determined the maximum size of 2-factor hamiltonian graphs.

Theorem

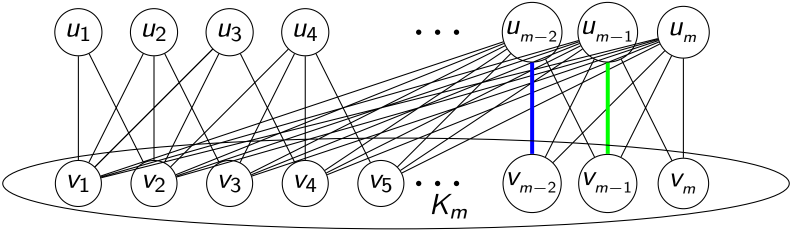
If G is a 2-factor hamiltonian graph of order $n \geq 7$, then

$$|E(G)| \leq \left\lceil \frac{n(n+1)}{4} \right\rceil$$

and the bound is sharp.

Large 2-factor Hamiltonian Graphs

To demonstrate sharpness when n is even, they completed the V partite set of $B(n, V)$, which we will refer to as $G(n, V)$.

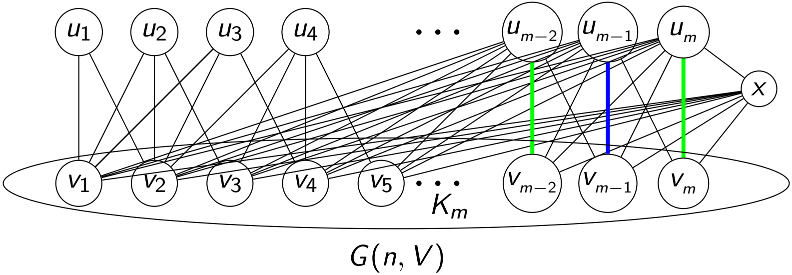


$G(n, V)$

As with $B(n, V)$, the blue edge is only present when m is even and the green edge is only present when m is odd.

Large 2-factor Hamiltonian Graphs

To demonstrate sharpness when n is odd, they constructed $G(n+1, V)$ and contracted edge $u_{m+1}v_{m+1}$ to form vertex x.



Small 2-factor Hamiltonian Graphs

- For small graphs ($n = 3, 4, 5$), the only 2-factor is the hamiltonian cycle, C_n, so complete graphs $K_3, K_4,$ and K_5 are 2-factor hamiltonian and have 3, 6, and 10 edges respectively.

Small 2-factor Hamiltonian Graphs

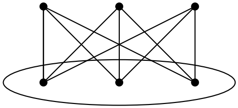
- For small graphs ($n = 3, 4, 5$), the only 2-factor is the hamiltonian cycle, C_n, so complete graphs $K_3, K_4,$ and K_5 are 2-factor hamiltonian and have 3, 6, and 10 edges respectively.
- Define $G(3, V)$ to be a copy of K_3, with one vertex serving as V.

Small 2-factor Hamiltonian Graphs

- For small graphs ($n = 3, 4, 5$), the only 2-factor is the hamiltonian cycle, C_n, so complete graphs $K_3, K_4,$ and K_5 are 2-factor hamiltonian and have 3, 6, and 10 edges respectively.
- Define $G(3, V)$ to be a copy of K_3, with one vertex serving as V.
- While $G(n, V)$'s are not of maximum size for $n = 4, 5$ they will still be useful later in building 2-factor isomorphic graphs.

Small 2-factor Hamiltonian Graphs

- For small graphs ($n = 3, 4, 5$), the only 2-factor is the hamiltonian cycle, C_n, so complete graphs $K_3, K_4,$ and K_5 are 2-factor hamiltonian and have 3, 6, and 10 edges respectively.
- Define $G(3, V)$ to be a copy of K_3, with one vertex serving as V.
- While $G(n, V)$'s are not of maximum size for $n = 4, 5$ they will still be useful later in building 2-factor isomorphic graphs.
- For $n = 6$, $G(n, V)$, a $K_{3,3}$ with one partite set completed, has 12 edges rather than the 11 given by the general formula.



Chords and Edges of Cycles in F

- For any 2-factor isomorphic graph, G, we select a particular 2-factor F.
- For a cycle C_j, of length j in F, the induced subgraph $G[C_j]$ is 2-factor hamiltonian, and so has, at most , $\left\lceil \frac{j(j+1)}{4} \right\rceil$ edges if $j \geq 7$ and 3, 6, 10, or 12 edges respectively if $j = 3, 4, 5,$ or 6.

Chords and Edges of Cycles in F

Let c_j be the number of cycles of length j in F, and let

$$c_1^* = \sum_{i=1}^{n} c_{4i+1} \text{ and } c_2^* = \sum_{i=1}^{n} c_{4i+2}.$$

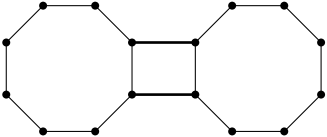
Summing over all cycles in F gives an upper bound for the number of edges that are either on a cycle in F or chords of a cycle in F:

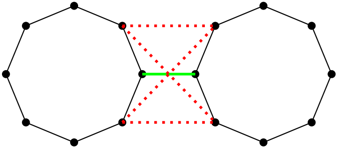
$$\sum_{C_j \in F} |E(G[C_j])| = \sum_{C_j \in F} \left(\frac{j(j+1)}{4} \right) + \frac{c_1^*}{2} + \frac{c_2^*}{2} + c_4 + 2 \cdot c_5 + c_6.$$

The c_1^* and c_2^* account for rounding up when $j = 4i+1$ or $4i+2$ and the other trailing terms account for the discrepancies between the true bound and the general bound for C_4's, C_5's, and C_6's.

Edges Between Cycles in F

- Recall that two edges between C_i and C_j paired if they, together with an edge on each of C_i and C_j form a C_4 and that paired edges are forbidden in 2-factor isomorphic graphs.
- Any edge between cycles forbids the four red edges and is forbidden by any of the four red edges.

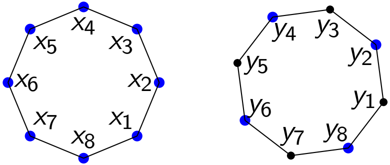




Edges Between Cycles in F

Label the vertices of C_i as $x_1, \ldots x_i$ and the vertices of C_j as $y_1, \ldots y_j$

- Forbidding paired edges gives an upper bound of $\dfrac{i \cdot j}{2}$ edges between C_i and C_j. This is the proper bound when i or j is even and can be attained by joining x_k to y_{2l} for all k and l.

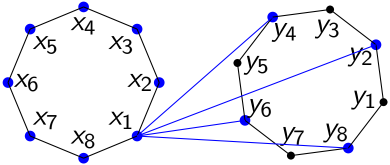


- For odd i and j, such a construction would give $i \cdot \left\lfloor \frac{j}{2} \right\rfloor$, and if $i \leq j$, this gives the sharp upper bound.

Edges Between Cycles in F

Label the vertices of C_i as $x_1, \ldots x_i$ and the vertices of C_j as $y_1, \ldots y_j$

- Forbidding paired edges gives an upper bound of $\frac{i \cdot j}{2}$ edges between C_i and C_j. This is the proper bound when i or j is even and can be attained by joining x_k to y_{2l} for all k and l.

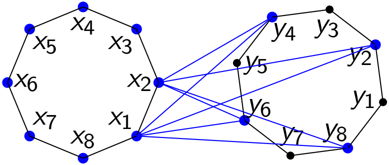


- For odd i and j, such a construction would give $i \cdot \left\lfloor \frac{j}{2} \right\rfloor$, and if $i \leq j$, this gives the sharp upper bound.

Edges Between Cycles in F

Label the vertices of C_i as $x_1, \ldots x_i$ and the vertices of C_j as $y_1, \ldots y_j$

- Forbidding paired edges gives an upper bound of $\dfrac{i \cdot j}{2}$ edges between C_i and C_j. This is the proper bound when i or j is even and can be attained by joining x_k to y_{2l} for all k and l.

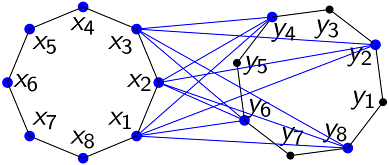


- For odd i and j, such a construction would give $i \cdot \left\lfloor \frac{j}{2} \right\rfloor$, and if $i \leq j$, this gives the sharp upper bound.

Edges Between Cycles in F

Label the vertices of C_i as $x_1, \dots x_i$ and the vertices of C_j as $y_1, \dots y_j$

- Forbidding paired edges gives an upper bound of $\dfrac{i \cdot j}{2}$ edges between C_i and C_j. This is the proper bound when i or j is even and can be attained by joining x_k to y_{2l} for all k and l.

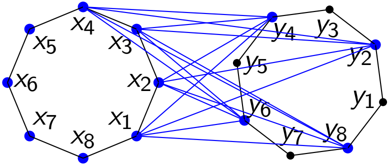


- For odd i and j, such a construction would give $i \cdot \left\lfloor \frac{j}{2} \right\rfloor$, and if $i \leq j$, this gives the sharp upper bound.

Edges Between Cycles in F

Label the vertices of C_i as $x_1, \ldots x_i$ and the vertices of C_j as $y_1, \ldots y_j$

- Forbidding paired edges gives an upper bound of $\frac{i \cdot j}{2}$ edges between C_i and C_j. This is the proper bound when i or j is even and can be attained by joining x_k to y_{2l} for all k and l.

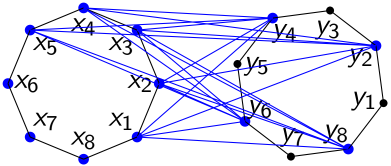


- For odd i and j, such a construction would give $i \cdot \left\lfloor \frac{j}{2} \right\rfloor$, and if $i \leq j$, this gives the sharp upper bound.

Edges Between Cycles in F

Label the vertices of C_i as $x_1, \ldots x_i$ and the vertices of C_j as $y_1, \ldots y_j$

- Forbidding paired edges gives an upper bound of $\dfrac{i \cdot j}{2}$ edges between C_i and C_j. This is the proper bound when i or j is even and can be attained by joining x_k to y_{2l} for all k and l.

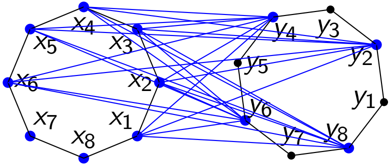


- For odd i and j, such a construction would give $i \cdot \left\lfloor \frac{j}{2} \right\rfloor$, and if $i \leq j$, this gives the sharp upper bound.

Edges Between Cycles in F

Label the vertices of C_i as $x_1, \ldots x_i$ and the vertices of C_j as $y_1, \ldots y_j$

- Forbidding paired edges gives an upper bound of $\frac{i \cdot j}{2}$ edges between C_i and C_j. This is the proper bound when i or j is even and can be attained by joining x_k to y_{2l} for all k and l.

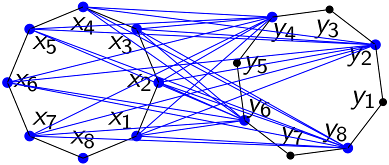


- For odd i and j, such a construction would give $i \cdot \left\lfloor \frac{j}{2} \right\rfloor$, and if $i \leq j$, this gives the sharp upper bound.

Edges Between Cycles in F

Label the vertices of C_i as $x_1, \ldots x_i$ and the vertices of C_j as $y_1, \ldots y_j$

- Forbidding paired edges gives an upper bound of $\frac{i \cdot j}{2}$ edges between C_i and C_j. This is the proper bound when i or j is even and can be attained by joining x_k to y_{2l} for all k and l.

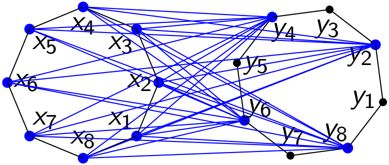


- For odd i and j, such a construction would give $i \cdot \left\lfloor \frac{j}{2} \right\rfloor$, and if $i \leq j$, this gives the sharp upper bound.

Edges Between Cycles in F

Label the vertices of C_i as $x_1, \dots x_i$ and the vertices of C_j as $y_1, \dots y_j$

- Forbidding paired edges gives an upper bound of $\dfrac{i \cdot j}{2}$ edges between C_i and C_j. This is the proper bound when i or j is even and can be attained by joining x_k to y_{2l} for all k and l.



- For odd i and j, such a construction would give $i \cdot \left\lfloor \frac{j}{2} \right\rfloor$, and if $i \leq j$, this gives the sharp upper bound.

Refining the Bounds

- While the K_4 and K_5 allowed more edges than the $G(4, V)$ and $G(5, V)$, joining from these to C_j allows at most j edges, giving fewer total edges.
- This observation implies that for all but possibly one cycle, there are at most $|E(G(j, V))|$ rather than $|K_j|$ for the C_4 and C_5 cycles.

Main Result

The maximum number of edges in a 2-factor isomorphic graph with 2-factor F is then

$$\sum_{C_j \in F} \left\lceil \frac{j(j+1)}{4} \right\rceil + \sum_{C_i, C_j \subset F} \max\left(i \cdot \left\lfloor \frac{j}{2} \right\rfloor, j \cdot \left\lfloor \frac{i}{2} \right\rfloor \right) + c_6 + c$$

where the constant c accounts for the possible presence of a K_4 or K_5 on the single cycle that is not joined from.

Main Result

The maximum number of edges in a 2-factor isomorphic graph with 2-factor F is then

$$\sum_{C_j \in F} \left\lceil \frac{j(j+1)}{4} \right\rceil + \sum_{C_i, C_j \subset F} \max\left(i \cdot \left\lfloor \frac{j}{2} \right\rfloor, j \cdot \left\lfloor \frac{i}{2} \right\rfloor \right) + c_6 + c$$

where the constant c accounts for the possible presence of a K_4 or K_5 on the single cycle that is not joined from.

This can be rewritten as:

$$\sum_{C_j \in F} \frac{j(j+1)}{4} + \frac{c_1^* + c_2^*}{2} + \sum_{C_i, C_j \in F} \frac{i \cdot j}{2} - \sum_{C_{2i+1}, C_{2j+1} \in F} \min\left(\frac{j}{2}, \frac{i}{2}\right) + c_6 + c,$$

Main Result

which can be rewritten as:

$$\sum_{C_j \in F} \frac{j(j+1)}{4} + \sum_{C_j \in F} \sum_{C_i \in F} \frac{i \cdot j}{4} - \sum_{C_j \in F} \frac{j^2}{4}$$

$$- \sum_{C_{2i+1}, C_{2j+1} \in F} \min\left(\frac{i}{2}, \frac{j}{2}\right) + c_6 + \frac{c_1^* + c_2^*}{2} + c,$$

Main Result

which can be rewritten as:

$$\sum_{C_j \in F} \frac{j(j+1)}{4} + \sum_{C_j \in F} \sum_{C_i \in F} \frac{i \cdot j}{4} - \sum_{C_j \in F} \frac{j^2}{4}$$

$$- \sum_{C_{2i+1}, C_{2j+1} \in F} \min\left(\frac{i}{2}, \frac{j}{2}\right) + c_6 + \frac{c_1^* + c_2^*}{2} + c,$$

which can be further simplified to:

$$\sum_{C_j \in F} \frac{j(n+1)}{4} - \sum_{C_{2i+1}, C_{2j+1} \in F} \min\left(\frac{i}{2}, \frac{j}{2}\right) + c_6 + \frac{c_1^* + c_2^*}{2} + c =$$

$$\frac{n(n+1)}{4} - \sum_{C_{2i+1}, C_{2j+1} \in F} \min\left(\frac{i}{2}, \frac{j}{2}\right) + c_6 + \frac{c_1^* + c_2^*}{2} + c$$

Main Result

Theorem

If G is a 2-factor isomorphic graph with 2-factor F, then

$$|E(G)| \leq \frac{n(n+1)}{4} - \sum_{C_{2i+1}, C_{2j+1} \in F} \min\left(\frac{i}{2}, \frac{j}{2}\right) + c_6 + \frac{c_1^* + c_2^*}{2} + c$$

Where c_1^* and c_2^* are the number of cycles of lengths $4i+1$ and $4i+2$ respectively for some i and

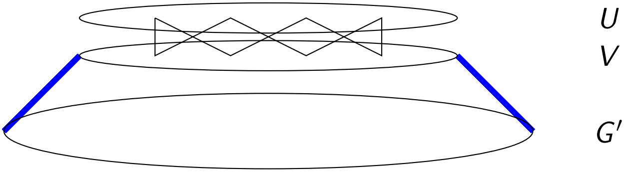
$$c = \begin{cases} 2 & c_5 > 0 \text{ and } c_3 = 0 \\ 1 & c_5 > 0 \text{ and } c_3 = 1 \\ 1 & c_4 > 0 \text{ and } c_{2i+1} = 0 \text{ for all } i \\ 0 & \text{otherwise.} \end{cases}$$

Construction

1. If F is a single cycle, then the graph must then be 2-factor hamiltonian, and we can use the 2-factor hamiltonian construction for the appropriate n.
2. If F consists of multiple cycles, we carefully order the cycles as we choose and remove the last cycle C_j. We construct a 2-factor isomorphic graph, G', with maximum number of edges and $F - C_j$ as a 2-factor, then construct $G(j, V)$ on the C_j and add all of the edges from V to G'.

Construction

Edges added between the V part of $G(j, V)$ and G'.



- This cannot produce any new 2-factors because all the adjacencies of the vertices in V in a 2-factor are used by the vertices of U and so the new edges are unusable.
- Our chosen ordering is to start with a C_5 if one or fewer C_3's and a C_5 are present, followed by the remaining odd cycles in ascending order, followed by the even cycles.

Extending to Unknown F

Using the maximum sizes of bipartite and general 2-factor isomorphic graphs that contain any specified F as a 2-factor, we can examine all possible F's to determine an overall maximum size or maximum size over all 2-factors consisting of k cycles.

Overall Maximum Size of Bipartite Graphs

For fixed n, the formula $\dfrac{n(n+4)}{8} + c_2^* + c_6$ only depends on c_2^* and c_6. Both are maximized by maximizing the number of C_6's.

Corollary

If G is a bipartite 2-factor isomorphic graph of order n then $|E(G)| \leq \dfrac{n(n+6)}{8} - c$, where

$$c = \begin{cases} 0 & n = 6k \\ 1 & n = 6k+4 \\ 2 & n = 6k+2 \end{cases}$$

The 2-factors that attain this bound consist of C_6's, a C_4 and C_6's and either two C_4's or a C_8 and C_6's respectively.

Overall Maximum Size of General Graphs

For fixed n, the formula

$$\frac{n(n+1)}{4} - \sum_{C_{2i+1}, C_{2j+1} \in F} \min\left(\frac{i}{2}, \frac{j}{2}\right) + c_6 + \frac{c_1^* + c_2^*}{2} + c$$

is maximized when:

- pairs of odd cycles are minimized,

Overall Maximum Size of General Graphs

For fixed n, the formula

$$\frac{n(n+1)}{4} - \sum_{C_{2i+1}, C_{2j+1} \in F} \min\left(\frac{i}{2}, \frac{j}{2}\right) + c_6 + \frac{c_1^* + c_2^*}{2} + c$$

is maximized when:

- pairs of odd cycles are minimized,
- the number of C_6's is maximized,

Overall Maximum Size of General Graphs

For fixed n, the formula

$$\frac{n(n+1)}{4} - \sum_{C_{2i+1}, C_{2j+1} \in F} \min\left(\frac{i}{2}, \frac{j}{2}\right) + c_6 + \frac{c_1^* + c_2^*}{2} + c$$

is maximized when:

- pairs of odd cycles are minimized,
- the number of C_6's is maximized,
- the one odd cycle (if needed) is a C_5 covered by a K_5
- if no odd cycles are needed, 2 C_4's are used rather than a C_8.

Overall Maximum Size

There are exceptions for $n = 3, 7$, but in general:

n	Max	2-factor(s)
$n \equiv 0 \mod 6$	$(n^2 + 2n)/4$	$\{C_6, \ldots, C_6\}$
$n \equiv 1 \mod 6$	$(n^2 + 2n - 3)/4$	$\{C_4, C_4, C_5, C_6, \ldots, C_6\}$
$n \equiv 2 \mod 6$	$(n^2 + 2n - 4)/4$	$\{C_4, C_4, C_6, \ldots, C_6\}$
$n \equiv 3 \mod 6$	$(n^2 + 2n + 1)/4$	$\{C_4, C_5, C_6, \ldots, C_6\}$
$n \equiv 4 \mod 6$	$(n^2 + 2n)/4$	$\{C_4, C_6, \ldots, C_6\}$
$n \equiv 5 \mod 6$	$(n^2 + 2n + 5)/4$	$\{C_5, C_6, \ldots, C_6\}$

F of k cycles for Bipartite Graphs

- We can again increase the maximum size by performing the adjustment

$$\{C_i, C_j\}, (j > 6) \rightarrow \{C_6, C_{i+j-6}\}.$$

- We start with any F consisting of k cycles.
- After exhausting this adjustment we are left with either C_4's and C_6's, C_6's alone, or C_6's and a large cycle of length $n - 6k + 6$.
- Applying the formula to these 2-factors gives the following corollary:

F of k cycles for Bipartite Graphs

Corollary

If G is a bipartite 2-factor isomorphic graph with a 2-factor consisting of k cycles, then the maximum size of G and the 2-factor that allows attainment of this maximum are given by:

Max Size	*Domain*	*2-factor*
$\frac{n^2+10n-24k}{8}$	$n < 6k$	$\{C_4, \ldots, C_4, C_6, \ldots, C_6\}$
$\left\lceil \frac{n^2+4n+12k-12}{8} \right\rceil$	$n > 6k$	$\{C_6, \ldots, C_6, C_{n-6k+6}\}$
$\frac{n^2+6n}{8}$	$n = 6k$	$\{C_6, \ldots, C_6\}$

F of k cycles for General Graphs

- When $k = 1$, only the hamiltonian 2-factor is possible so $|E| \leq \left\lceil \frac{n(n+1)}{4} \right\rceil$.
- When $k > 1$, start with any F consisting of k cycles. If at any point F contains any of the following pairs of cycles, we make an adjustment and do not decrease the number of edges.

$$
\begin{array}{ccc}
\{C_3, C_j\}(j \neq 6) & \rightarrow & \{C_4, C_{j-1}\} \\
\{C_3, C_3, C_6\} & \rightarrow & \{C_4, C_4, C_4\} \\
\{C_i, C_j\}(i + j \geq 10) & \rightarrow & \{C_6, C_{i+j-6}\} \\
\{C_6, C_j\}(j > 5, \text{odd}) & \rightarrow & \{C_5, C_{j-1}\}
\end{array}
$$

Utilizing these adjustments until they are exhausted, we can determine the general form 2-factors that maximize the number of edges.

F of k cycles for General Graphs

Corollary

If G is a 2-factor isomorphic graph with a 2-factor consisting of k cycles, then the maximum size of G and the 2-factor that allows attainment of this maximum are given by:

Max Size	*Domain of n*	*2-factor*
$3k(2n-4k+1)-\frac{n(n+1)}{2}$	$n < 4k$	$\{C_3 \ldots, C_3, C_4 \ldots, C_4\}$
$\frac{n^2}{4} + n - 3k + 1$	$4k \leq n < 6k$, *even*	$\{C_4 \ldots, C_4, C_6 \ldots, C_6\}$
$\frac{n^2}{4} + n - 3k + \frac{7}{4}$	$4k \leq n < 6k$, *odd*	$\{C_4 \ldots, C_4, C_5, C_6 \ldots, C_6\}$
$\frac{n^2+2n}{4}$	$6k = n$	$\{C_6 \ldots, C_6\}$
$\lceil \frac{n^2+n}{4} + \frac{3k-3}{2} \rceil$	$6k < n$, *even*	$\{C_6 \ldots, C_6, C_{n-6k+6}\}$
$\lceil \frac{n^2+n}{4} + \frac{3k-1}{2} \rceil$	$6k < n$, *odd*	$\{C_5, C_6 \ldots, C_6, C_{n-6k+7}\}$

Criticality

What about maximally 2-factor isomorphic graphs? For what sizes are there maximal 2-factor isomorphic graphs for either fixed or variable 2-factors?

- What is the minimum size of a maximal 2-factor hamiltonian graph?
- What is the minimum size of a maximal 2-factor isomorphic graph?

Classification

- In finding the overall bounds a number of adjustments were used to constrain the possible forms of the 2-factor, however several of these only guaranteed nondecreasing size, not increasing size. For some of these classifying which other 2-factors attain the bound would be interesting.
- In general the constructions that attain the bounds are not unique. Is there any way to classify or describe what 2-factor isomorphic graphs have maximum size? This seems like a much harder question, but simple cases will provide significant information about the general case.

Bipartite Construction Achieves the Bound

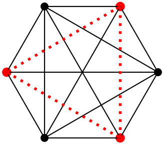
- Now we need to count the total number of edges in G. $B(j, V)$ adds at least $\dfrac{j(j+4)}{8}$. The additional edges if $j \equiv 2 \mod 4$ or $j = 6$ are accounted for when c_2^* and c_6 are adjusted.
- By induction G' attains the bound, and there are $\dfrac{j(n-j)}{4}$ edges added between V and X'.
- There are therefore

$$\frac{j(j+4)}{8} + \frac{j(n-j)}{4} + \frac{(n-j)(n-j+4)}{8} + c_2^* + c_6 = \frac{n(n+4)}{8} + c_2^* + c_6$$

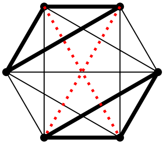
total edges in G and we do indeed attain the bound.

Determining $G(6, V)$

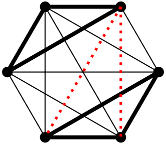
$G(6, V)$, a $K_{3,3}$ with one partite set completed, has 12 edges and does not contain two disjoint C_3's since no pair of red vertices can be in a C_3. Any graph with 13 or more edges contains both a C_6 and two disjoint C_3's.



$G(6, V)$



2 Disjoint Edges Missing



2 Adjacent Edges Missing

Sharpening the Bound when i, j odd

- To do better we must have a higher average degree, and so for some pair of consecutive vertices x_k and x_{k+1} in C_i, the degree sum must be greater than the average of $j-1$ and so at least j.

Sharpening the Bound when i, j odd

- To do better we must have a higher average degree, and so for some pair of consecutive vertices x_k and x_{k+1} in C_i, the degree sum must be greater than the average of $j-1$ and so at least j.
- $x_k y_l$ forbids its paired edges $x_{k+1}y_{l+1}$ and $x_{k+1}y_{l-1}$ and is in turn forbidden by either.

Sharpening the Bound when i, j odd

- To do better we must have a higher average degree, and so for some pair of consecutive vertices x_k and x_{k+1} in C_i, the degree sum must be greater than the average of $j-1$ and so at least j.
- $x_k y_l$ forbids its paired edges $x_{k+1}y_{l+1}$ and $x_{k+1}y_{l-1}$ and is in turn forbidden by either.
- This relationship implies that the j edges must doubly forbid each of the j non-edges.

Sharpening the Bound when i, j odd

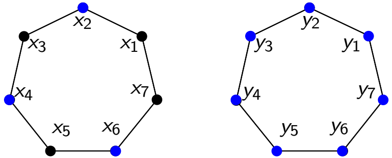
- To do better we must have a higher average degree, and so for some pair of consecutive vertices x_k and x_{k+1} in C_i, the degree sum must be greater than the average of $j-1$ and so at least j.
- $x_k y_l$ forbids its paired edges $x_{k+1}y_{l+1}$ and $x_{k+1}y_{l-1}$ and is in turn forbidden by either.
- This relationship implies that the j edges must doubly forbid each of the j non-edges.
- Doubly forbidding all non-edges implies that if $x_k y_l$ is present, $x_k y_{l+2}$ and so $x_k y_{l+2m}$ are also present for all m.

Sharpening the Bound when i, j odd

- $x_k y_{l+(j+1)} = x_k y_{l+1}$, so if an edge from x_k to C_j is present, all edges from x_k to C_j are present and no edges from x_{k-1} or x_{k+1} to C_j are present.

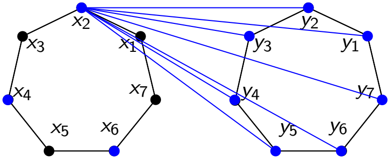
Sharpening the Bound when i, j odd

- $x_k y_{l+(j+1)} = x_k y_{l+1}$, so if an edge from x_k to C_j is present, all edges from x_k to C_j are present and no edges from x_{k-1} or x_{k+1} to C_j are present.
- Even if the remaining $i - 3$ vertices achieve the maximum average degree of $\frac{j}{2}$ only $\frac{i-1}{2}j$ total edges would be present, at most the same as the prior construction.



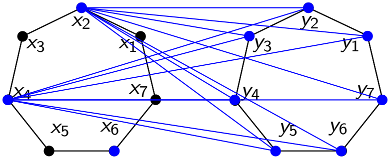
Sharpening the Bound when i, j odd

- $x_k y_{l+(j+1)} = x_k y_{l+1}$, so if an edge from x_k to C_j is present, all edges from x_k to C_j are present and no edges from x_{k-1} or x_{k+1} to C_j are present.
- Even if the remaining $i-3$ vertices achieve the maximum average degree of $\frac{j}{2}$ only $\frac{i-1}{2}j$ total edges would be present, at most the same as the prior construction.



Sharpening the Bound when i, j odd

- $x_k y_{l+(j+1)} = x_k y_{l+1}$, so if an edge from x_k to C_j is present, all edges from x_k to C_j are present and no edges from x_{k-1} or x_{k+1} to C_j are present.
- Even if the remaining $i-3$ vertices achieve the maximum average degree of $\frac{j}{2}$ only $\frac{i-1}{2}j$ total edges would be present, at most the same as the prior construction.



Sharpening the Bound when i, j odd

- $x_k y_{l+(j+1)} = x_k y_{l+1}$, so if an edge from x_k to C_j is present, all edges from x_k to C_j are present and no edges from x_{k-1} or x_{k+1} to C_j are present.
- Even if the remaining $i-3$ vertices achieve the maximum average degree of $\frac{j}{2}$ only $\frac{i-1}{2}j$ total edges would be present, at most the same as the prior construction.