What is an Adinkra

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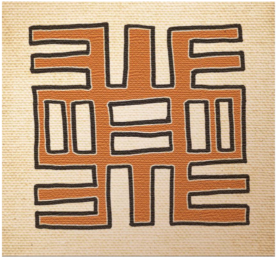

Overview

1. Physical Background
2. Classification Theorem for Chormotopology
3. Dashing
4. Ranking
5. Dessin d'enfant

Adinkras as Translator

"The use of symbols to connote ideas which defy simple verbalization is perhaps one of the oldest of human traditions. The Asante people of West Africa have long been accustomed to using simple yet elegant motifs known as Adinkra symbols, to serve just this purpose."

— Michael Faux& S. J. Gates, Jr

Physics		Combinatorics		Topology		Geometry
Generators of super Poincare algebra	⇔	Chromotopology (doubly Even Code)	⇔	Universal covering by $X(I^n)$	⇔	Belyi pair (X, β)
$+$ or $-$	⇔	Odd Dashing	⇔	Vanishing of $H^i(X(A), \mathbb{Z}_2)$	⇔	Super Riemann Surface Structure
Placement of ∂_t (Engineering dimension)	⇔	Ranking	⇔	–	⇔	Morse Divisors

The representation of algebra $\mathfrak{po}^{1|N}$

$N-$extended supersymmetry algebra in 1 dimension is generated by ∂_t and n supersymmetry generators $Q_1, Q_2, \ldots, Q_n$ with

$$\{Q_I, Q_J\} = 2i\delta_{IJ}\partial_t, [\partial_t, Q_I] = 0, \quad I, J = 1, 2, \ldots, n$$

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What's the representation on basis?

$$\{\partial_t^k \phi_I, \partial_t^k \psi_J | k \in \mathbb{N}, \quad I, J = 1.2. \ldots . m\}$$

Here the $\mathbb{R}$-valued functions $\{\phi_1, \ldots \phi_m\}$: bosons and $\{\psi_1, \ldots, \psi_m\}$: fermions. The $\#\psi = \#\phi$:*off-shell*.

Operators in $\mathfrak{po}^{1|N}$

We introduce the *engineering dimension*: operator ∂_t adds the engineering dimension by 2, and thus Q_I adds the dimension by 1(denoted by $[Q_I] = 1$). So

$$\text{either } Q_I\phi_A = \pm\psi_B \text{ or } Q_I\phi_A = \pm\partial_t\psi_B.$$

Thus $[\phi_A] + 1 = [\psi_B]$ or $[\phi_A] + 1 = \psi_B + 2$. A varies from 1 to m, then B also vary from 1 to m.

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Thus $[\phi_A] + 1 = [\psi_B]$ or $[\phi_A] + 1 = \psi_B + 2$. A varies from 1 to m, then B also vary from 1 to m. Also,

$$\text{either } Q_I\psi_B = \pm i\psi_A \text{ or } Q_I\psi_B = \pm i\partial_t\phi_A.$$

We may see ϕs and ψs as points and connect them with lines. We give line the color, dashing, ranking.

$$Q_I\phi_A(t) = c\partial_t^{\lambda}\psi_B(t) \Leftrightarrow Q_I\psi_B(t) = \frac{i}{c}\partial_t^{1-\lambda}\phi_A(t)$$

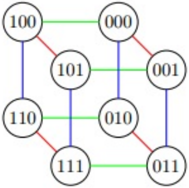
$c \in \{1, -1\}$ and $\lambda \in \{0, 1\}$

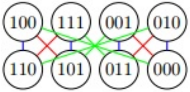
Chromotopologies

Definition

A *n−dimensional chromotopology* is a finite connected simple graph A such that

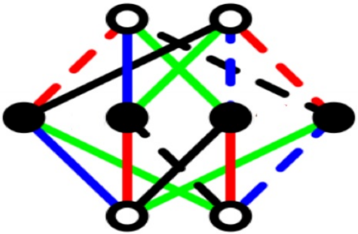
1. A is $n-$regular and bipartite(same number).
2. Elements of $E(A)$ are colored by n different colors, denoted by $[n] = \{1, 2, \ldots, n\}$
3. For any distinct i,j in $E(A)$, edges in $E(A)$ in color i and j form disjoint $4-$cycles.(2-color 4-cycle)



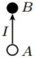
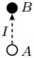
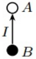
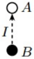


Two Structures in Chromotopologies

1. *Ranking*: A map $V(A) \to \mathbb{Z}$ that gives A the additional poset structure.
2. *Dashing*: Each edge is assigned an element in $\mathbb{Z}_2$. An *odd dashing* is a dashing that for each 2-color 4-cycle, the sum must be 1. If A is dashed by odd dashed, we call it *well-dashed*.



What's their correspondence in physics?

Adinkra	***Q*-action**	**Adinkra**	***Q*-action**
	$Q_I \begin{bmatrix} \psi_B \\ \phi_A \end{bmatrix} = \begin{bmatrix} i\dot{\phi}_A \\ \psi_B \end{bmatrix}$		$Q_I \begin{bmatrix} \psi_B \\ \phi_A \end{bmatrix} = \begin{bmatrix} -i\dot{\phi}_A \\ -\psi_B \end{bmatrix}$
	$Q_I \begin{bmatrix} \phi_A \\ \psi_B \end{bmatrix} = \begin{bmatrix} \dot{\psi}_B \\ i\phi_A \end{bmatrix}$		$Q_I \begin{bmatrix} \phi_A \\ \psi_B \end{bmatrix} = \begin{bmatrix} -\dot{\psi}_B \\ -i\phi_A \end{bmatrix}$

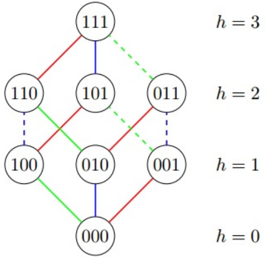
The edges are here labeled by the variable index I; for fixed I, they are drawn in the I^{th} color.

and we have the following dictionary

Adinkras	Representation of $\mathfrak{po}^{1|N}$
Vertex bipartition	Bosonic/Fermionic bipartition
Colored edges by I	Action of Q_I
Dashing	Sign in Q_I
Change of rank	power of ∂_t
Rank function	Engineering dimension

n-cube

The n-cube has a natural structure that satisfies all the requirements



We use $\mathbb{Z}_2^n$ as the points, connect by hamming distance 1(that differs exactly one element), and color the edge by the corresponding color. Use the ranking to be number of 1 in it. But for dashing, we need some induction hypothesis which will be stated later.

Adinkras

An adinkra is a ranked well-dashed chromotopology. A natural question:

How can we distinguish two Adinkra?

But the solution comes from various side, since the isomorphism of Adinkra has various unequivalent definitions! So we first consider the following:

How can we distinguish two chromotopology?

Surprisingly, the answer is coding theory!

Reminder of Codes

An *n-codeword* is a vector in $\mathbb{Z}_2^n$. Weight of the code is the number of the non-zero component, by $wt(v)$. We now have

Code

An (n, k)-binary code L is k dimensional subspace of $\mathbb{Z}_2^n$. It is *even* if for all $v \in L$, $2|wt(v)$; *doubly-even* if $4|wt(v)$.

Now we may use linear algebra to construct subspace $\mathbb{Z}_2^n/L$(later denoted by I_c^n/L).

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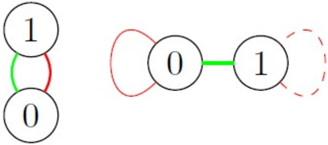
Problem

Can the equivalence class define a chromotopology?

The answer is yes, but some only when code is doubly even!

Multichromotopology

Note: we may wonder if double edge or self loop is allowed in generalization of adinkras. The former is excluded by dashing and later is excluded by ranking. But we may allow something called "multichromotopology".



Properties of codes

Lemma

1. A has a loop if and only if L contains a code word of weight 1, and a double-edge if and only if L has a word of weight 2. So A is simple if and only if all words in L has weight 3 or greater.
2.A can be ranked iff it is bipartite, and which is true iff L is even.

Reason:1. is obvious.
2.If not bipartite, then it has odd cycles, the preimage of odd cycle is an odd path from v to w, where $v - w \in L$, so L is not even.

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But the most complicated one is the dashing, which involves some Clifford algebra of the code.

The Classification Theorem

Theorem

$A = I_c^n/L$ is well-dashed if and only if L is doubly even code.

With previous theorems, we may have the following:

Chromotopology

Chromotopology is exactly $A = I_c^n/L$,where L is even code with no weight 2 word.

Also, it's easy to see that

Adinkraizable Chromotopology

Adinkraizable Chromotopology is exactly $A = I_c^n/L$,where L is is doubly even code.

Proof of one-side

Theorem

$A = I_c^n/L$ is well-dashed $\Rightarrow$ L is doubly even code.

We denote $q_I(v)$ to be the unique point connected to v that has color I. We consider the code

$$L = \{(x_1, \ldots, x_n) \in \mathbb{Z}_2^n | q_1^{x_1} \ldots q_n^{x_n}(v) = v, \forall v \in V(A)\}$$

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It's obvious that

$$q_1^{x_1+y_1} \dots q_n^{x_n+y_n}(v) = q_1^{x_1} \dots q_n^{x_n}(q_1^{y_1} \dots q_n^{y_n}(v)).$$

Also, the identity and inverse are obvious.
By a translation, we know that C is independent of choice of v.

Direct Verification

I_c^n/L is exactly A.

Reason for doubly even

First, we suppose $v \in L$.

$$Q_1^{x_1} \dots Q_n^{x_n} F_*(t) = c\partial_t^{wt(v)/2} F_*(t)$$

We must see what is c. Because

$$Q_1^{x_1} \dots Q_n^{x_n} Q_1^{x_1} \dots Q_n^{x_n} F_*(t) = c^2 \partial_t^{wt(v)} F_*(t).$$

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Using anti-commutative of Q_i we know

$$c^2 \partial_t^{wt(v)} F_*(t) = (-1)^{\binom{wt(v)}{2}} Q_1^{2x_1} \dots Q_n^{2x_n} F_*(t) = (-1)^{\binom{wt(v)}{2}} i^{wt(v)} \partial_t^{wt(v)} F_*(t).$$

This means $c^2 = 1$. But on the other hand, we recall $Q_I Q_J$ contribute one power of i, thus $c = \pm 1$ implies $wt(v) \equiv 0 \mod 4$.

The Universal Covering

Construct $X(A)$ by filling all 2-color 4 cycle with a disk. The $\mathbb{Z}_2$ complex C_0 is formal sum of vertices, C_1 is formal sum of edges, C_2 is formal sum of faces. Thus

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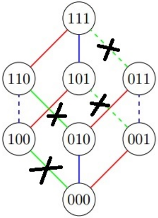
Universal covering

A is an $(n,k)-$adinkraizable chromotopology, $A = I_c^n/L$. Then $X(A) = X(I_c^n)$ as quotient complex, L acts freely on $X(A)$. We have $X(I_c^n)$ is a simply-connected covering space of $X(A)$, L is deck transformation

The reason is that $X(I_c^n)$ is $2-$skeleton of hypercube D^n, we know that H_1 and π_1 must agree.
An interpretation for dashing is, if we see $H^1(X(A), \mathbb{Z}_2)$ by sending e to $d(e)$. Then $H^2(X, \mathbb{Z}_2)$ vanish if and only if the dashing is odd.

Decomposition of Adinkra

Delete all edge with single color may create some separate adinkras



Motivated by this, we say adinkra is *i-decomposable* if removing these edge i create two separate parts $A = A_0 \coprod A_1$.

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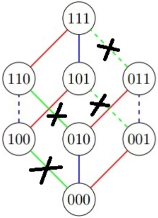

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Lemma

Color i decompose A if and only if for all $v \in L(A)$, the i-th digit of v is 0.

A direct result is, I_c^n is decomposable by all i.
Intuitive fact: A is (n, k) chromotopology, then A_0, A_1 is $(n-1, k)$ chromotopology.

Dashing on I_c^n

So far, we have not given the dashing on I_c^n! So the question is:

How can we find dashing of I_c^n?

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If there exists dashing, then

How many distinct odd dashing $o(A)$ are there on I_c^n?

Answer: The same number as even dashing, with $|o(A)| = |e(A)| = 2^{2^{n-k}+k+1}$ on an adinkraizable $(n,k)-$chromotopology.
Surprising fact: Number does not depend on the code!

Equivalence of Odd and Even Dashing

Equivalence

$|e(A)| = |o(A)|$ if A is adinkraizable chromotopology.

Proof.

$l = |E(A)|$, and see all dashing as vector space in $\mathbb{Z}_2^l$, with solid 0 and dashed 1.

1. Even dashing create a vector space.

2.$o(A)$ is not a vector space. But $odd + even = odd$. So if odd dashing exists, $|o(A)| = |e(A)|$

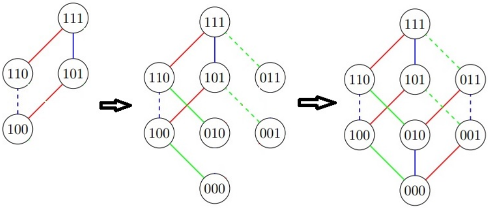
3.$|o(A)| > 0$ since adinkraizable. □

Construct odd dashing by decomposition

Theorem

If A has l edges colored i, and $A = A_0 \coprod_i A_1$, then each even(odd) dashing and 2^l dashing of i-colored edge uniquely determine an even(odd) dashing

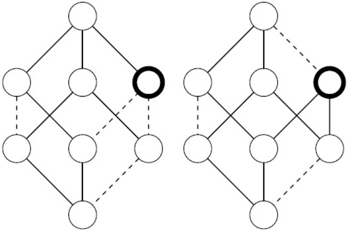
Here's an intuitive approach



So by induction, $|e(I_c^n)| = 2^{2^n - 1}$.

Labeled Switching Class

There's an operation called vertex switching



The labeled switching class(LSC) are the orbits(or equivalent class under vertex switching).

Computation of dashing in LSC

Proposition

In an adinkraizable (n, k) chromotopology A, there're exactly $2^{2^{n-k}-1}$ dashing in each LSC

Proof :

1. Vertex switch has order 2 and commutative, so give $\mathbb{Z}_2$ vector space.
2. If an operator fix a dashing, then each edge must have its vertex both switched or unswitched
3. Since connected, so all switched or all unswitched.

So 2^{n-k} vertices has $2^{2^{n-k}-1}$ ways of different switching. A corollary is that I_c^n has only one LSC.

Homological Computation of all dashing

We may count orbit of even dashing.

Proposition

Let A be an adinkraizable (n, k)-chromotopology, then there're exactly 2^k LSCs on A

We consider the complex $0 \to C_2 \xrightarrow{d_2} C_1 \xrightarrow{d_1} C_0 \to 0$ The even dashing is exactly $Im(d_2)^\perp$, since as a formal sum of edge with $\mathbb{Z}_2$, its inner product with all 2-color 4-cycle is 0. Hence

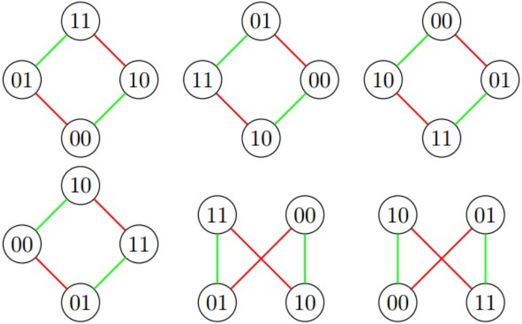
$$\dim((Im(d_2)^\perp)) = \dim(C_1) - \dim(Im(d_2)) = \dim(H_1) + \dim(C_0) - \dim(H_0)$$

Since $\dim(C_0) = 2^{n-k}$, $\dim(H_0) = 1$, thus the dimension for switching class is exactly $\dim(H_1)$. But $\pi_1(X(A)) = L$. By H_1 is abelianization of π_1, we know $H_1 = \mathbb{Z}_2^k$, and $\dim(H_1) = k$.

$$|e(A)| = |o(A)| = 2^{2^{n-k}+k-1}$$

The Rank Family

The set of all rankings of A is the *ranking family* $R(A)$. For I^2, we have



The natural question is

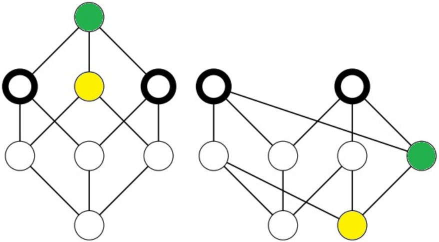
What is enumerative property of $R(A)$?

Hanging Garden Theorem

The main structure theorem

For a bipartite graph A, if $S \subset V(A)$ and $h_S : S \to \mathbb{Z}$ satisfies

1. h_S has odd value on bosons and even on fermions.
2. For distinct $s_1, s_2 \in S$, $D(S_1, s_2) \geq |h_S(s_1) - h_S(s_2)|$ Then there's a unique ranking h of A such that h agrees with h_S on S and sink of h are exactly S. (S can also be source by symmetry)



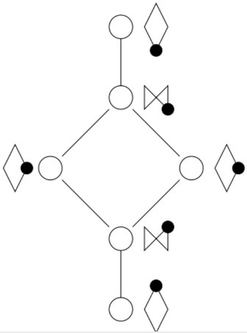
We take A^v to be the graph having only v as sink.

Rank Family Poset

We define two operators, the *vertex lowering* D_s and *vertex raising* U_s. The lowering operator acts only on the sink and change the ranking $h'(s) = h(s) - 2$. Similar with raising. Then

Theorem

Any two ranking can be obtained from a sequence of vertex-lowering and vertex-raising process.



Counting Ranking on hypercube

Let $A = A_0 \coprod_i A_1$, we first define $inc(b_1 b_2 \ldots b_{n-1}, j \to i) = b_1, b_2 \ldots b_{i-1} j b_i \ldots b_{n-1}$, just an inserting. Let $z_0 = (\vec{0}, i \to 0)$ and $z_1 = (\vec{0}, i \to 1)$, then $|h(z_1) - h(z_0)| = 1$. We denote $A = A_0 \nearrow_i A_1$ when $h(z_1) = h(z_0) + 1$ and $A = A_0 \searrow_i A_1$ otherwise.

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Lemma

For (n, k)-ranking A and $(n-1, k)$ ranking A_0 and A_1, we have $A = A_0 \nearrow_n A_1$ if and only if the colors and vertex labeling of three ranking are consistent and following condition: for each $c \in \mathbb{Z}_2^{n-1}$ and pair of $s_0 = inc(c, 0 \to n)$ and $s_1 = inc(c, 1 \to n)$ that at least one of s_0 or s_1 is a sink, we have $|h(s_0) - h(s_1)| = 1$

Counting Algorithm

The counting of ranking

1. Start with ranking of $R(I_c^1)$.
2. Given the ranking of $R(I_c^{n-1})$, iterate all pair of ranking (A, B) in $R(I_c^{n-1}) \times R(I_c^{n-1})$
   1. Consider ranking B' identical to B and $h_{B'}(\vec{0}) = h_B(\vec{0}) + 1$
   2. For each sink $s \in S(A) \cup S(B')$, verify $|h_A(s) - h_{B'}(s)| = 1$
   3. If true, put $A \nearrow_n B'$ in $R(I_c^n)$.

Embedding of surface

Let X be a compact connected oriented surface without boundary(denoted by surface) and $\mathcal{G}$ a bipartite graph.
A *two-cell embedding*(or bipartite map) $\mathcal{B}$ is embedding of $\mathcal{G}$ to X such that $X \backslash \mathcal{G} = \cup D_i$. $D_i \cong D$.

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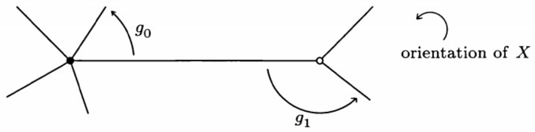
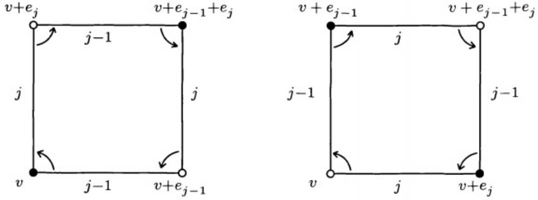
A *two-cell embedding*(or bipartite map) $\mathcal{B}$ is embedding of $\mathcal{G}$ to X such that $X \backslash \mathcal{G} = \cup D_i$. $D_i \cong D$. We mark the bipartite as black or white point, and construct a group:



By natural orientation on X induce a cyclic permutation of edge attached to each point. Denote the local rotation of black by g_0 and of white by g_1. The group $\langle g_0, g_1 \rangle$ is called *monodromy group* G. Note that fixed point of $g_\infty^l = (g_0 g_1)^{-l}$ is $2l$-gon.

Monodromy group of Hypercube

We take the graph I_c^n and the cyclic ordering of black point to be $(123\ldots n)$



The faces are 4−gons, and there're 2^n vertices, $n2^{n-1}$ edges and $n2^{n-2}$ faces, so genus is $1+(n-4)2^{n-3}$

Upperplane as an universal covering

We would also like to find the “universal covering” for these embeddings. We consider the upper half plane $\mathcal{U}$ in hyperbolic geometry and the modular group $\Gamma = PSL_2(\mathbb{Z})$ consists of the Möbius transform

$$T : z \mapsto \frac{az+b}{cz+d}, (a, b, c, d \in \mathbb{Z}, ad - bc = 1).$$

Now we acts transitively on $P^1(\mathbb{Q}) = \mathbb{Q} \cup \{\infty\}$, so we consider the extended hyperbolic plane

$$\bar{\mathcal{U}} = \mathcal{U} \cup \mathbb{Q} \cup \{\infty\}$$

Upperplane as an universal covering

Consider three disjoint set

$$[0] = \left\{ \frac{a}{b} \in \mathbb{Q} \cup \{\infty\} \middle| \ a \text{ is even and } b \text{ is odd} \right\}$$
$$[1] = \left\{ \frac{a}{b} \in \mathbb{Q} \cup \{\infty\} \middle| \ a \text{ and } b \text{ are both odd} \right\}$$
$$[\infty] = \left\{ \frac{a}{b} \in \mathbb{Q} \cup \{\infty\} \middle| \ a \text{ is odd and } b \text{ is even} \right\}$$

And we take $\infty = 1/0$. Stablizer of $[0]$ is $\Gamma_0(2) = \{T \in \Gamma | c \equiv 0 \mod 2\}$ and stablizer of three sets is $\Gamma(2) = \{T \in \Gamma | b \equiv c \equiv 0 \mod 2\}$.

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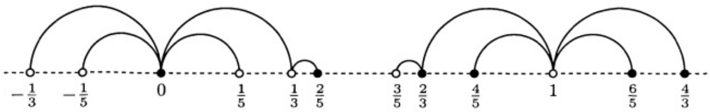

Which edge are hyperbolic geodesic combining a/b and c/d, where $ad - bc = \pm 1$. a and c has different parity, thus bipartite.

The Belyi pair

The automorphism of $\hat{\mathcal{B}}$ is $\Gamma(2)$, generated freely by $t_0 = z \mapsto \frac{z}{-2z+1}$ and $t_1 : z \mapsto \frac{z-2}{2z-3}$. Thus we use the map

$$\Gamma(2) \to G, T_0 \to g_0, , T_1 \to g_1$$

The stabilizer of edge is subgroup B of index $N = |E|$ in $\Gamma(2)$. And G acts transitively if and only if B is normal in $\Gamma(2)$. One can regard $\hat{\mathcal{B}}/B$ as $\mathcal{B}$.

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The stabilizer of edge is subgroup B of index $N = |E|$ in $\Gamma(2)$. And G acts transitively if and only if B is normal in $\Gamma(2)$. One can regard $\hat{\mathcal{B}}/B$ as $\mathcal{B}$. So we consider the compact Riemann Surface $X = \bar{\mathcal{U}}/B$ and a mapping

$$\hat{\mathcal{B}} \to \hat{\mathcal{B}}/B \cong \mathcal{B} \to \hat{\mathcal{B}}/\Gamma(2) \cong \mathcal{B}_1$$

The map $\beta : \mathcal{B} \to \mathcal{B}_1$ is from the graph to a line in $\Sigma \cong \bar{\mathcal{U}}/\Gamma(2)$, which is a sphere. (X, β) is called Belyi pair, ramified at most on $\{0, 1, \infty\}$.

The Belyi pair for Adinkras

Intuitively, the Belyi pair for Adinkra is just Riemann surface that fill in the 2-color 4-cycle with colors $\{i, i+1\}$, where $n+1=1$. So the genus of Riemann surface is just like what we have calculated before, that is, $1+2^{n-k-3}(n-4)$ for (n,k)-adinkraizable chromotopology. Particularly, for $n=4, k=0$, genus is 1, which means this is an elliptic curve.

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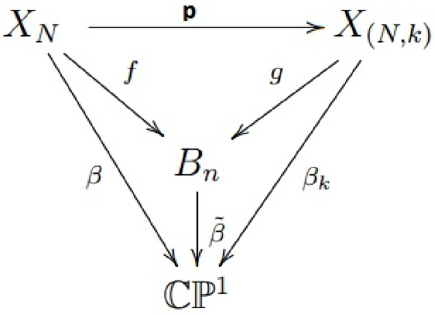
Jones,1997

Belyi pair (X_n, β) for $n-$cube factors through (B_n, β), B_n is $\Sigma \cong \mathbb{CP}^1$ with one vertex at 0, one at ∞ and one edge of each color connecting these vertices with angle $\frac{2\pi i}{n}$, the Belyi map is

$$\tilde{\beta}(x) = \frac{x^n}{x^n+1}$$

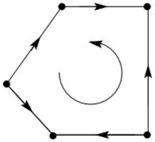
Inverse covering

The Belyi pair for (n, k) Adinkra $(X_{(n,k)}, \beta_k)$ and (X_n, β) has the following factor through property:



Kasteleyn Orientation and Spin Structure

A *Kasteleyn Orientiation* for graph G embedded in X is an orientation of edge so that you go around the boundary of X counterclockwise you go against odd number of edges.



This straightly fit odd dashing.

Cimasoni,Reshetikhin,2007

The Kasteleyn Orientation corresponds to spin structure on X.

Super Riemann Surface

A super Riemann surface X is locally $\mathbb{C}^{1|1}$ (Locally (x, θ) with $x\theta = \theta x, \theta^2 = 0$) and whose tangent bundle TX has a totally nonintegrable $0|1$ subbundle $\mathcal{D}$. (This means $\frac{1}{2}\{\mathcal{D}, \mathcal{D}\}$ is independent of $\mathcal{D}$)
A typical example is D takes the form $D_\theta = \frac{\partial}{\partial\theta} + \theta\frac{\partial}{\partial z}$. Now $D_{\theta^2} = \frac{\partial}{\partial z}$, and $D_\theta, D\theta^2$ span TX.

Adinkras as super Riemann Surface

The superconformal change of coordinate is

$$\tilde{Z} = u(z) + \theta\eta(z)\sqrt{u'(z)}, \tilde{\theta} = \eta(z) + \theta\sqrt{u'(z) + \eta(z)\eta'(z)}$$

Now we need on U_α and U_β, $z_\alpha = u_{\alpha\beta}(z_\beta), \theta_\alpha = [u'_{\alpha\beta}]^{1/2}\theta_\beta$. A choice of sign correspond to spin structure. Thus odd dashing implies super Riemann.

Future development

1. Which Adinkraic representation is irreducible?
2. When two adinkras are isomorphic?
3. How to interpret Clifford algebra in Adinkras?
4. How to generalize to $\mathfrak{po}^{p|q}$?

References

Zhang, Yan X.
"Adinkras for Mathematicians."
DMTCS Proceedings 01 (2013): 457-468.

Doran, Charles F., et al.
"Codes and supersymmetry in one dimension."
Advances in Theoretical and Mathematical Physics 15.6 (2011): 1909-1970.

Doran, Charles, et al. arXiv preprint arXiv:1311.3736 (2013).
"Geometrization of N-Extended 1-Dimensional Supersymmetry Algebras."
arXiv preprint arXiv:1311.3736 (2013).

Jones, Gareth A.
"Maps on surfaces and Galois groups."
Mathematica Slovaca 47.1 (1997): 1-33.

Thank you for Coming!

Any Questions or Remarks?