A. BOUNDS ON MULTIPLE-THRESHOLD FUNCTION

This report presents some preliminary results regarding multiple-threshold functions. A lower bound on the number of thresholds required to realize all functions of n variables will be derived.

Multiple-threshold functions will be defined as follows.

**DEFINITION 1:** A Boolean function $f(x_1, \ldots, x_n)$ is \( k \)-threshold threshold realizable if there exists a set of real numbers $w_1, \ldots, w_n, T_1, \ldots, T_k$ such that

\[
\prod_{j=1}^{k} \left( \sum_{i=1}^{n} w_i x_i - T_j \right) > 0 \iff f(x_1, \ldots, x_n) = q
\]

(1)

\[
\prod_{j=1}^{k} \left( \sum_{i=1}^{n} w_i x_i - T_j \right) < 0 \iff f(x_1, \ldots, x_n) = \overline{q},
\]

where $q = 0$ or 1. Thus a given set of $w_i$ and $T_j$ define one function with $q = 1$ and the complement of that function with $q = 0$. It is also clear that if a function is realizable with $k$ thresholds it is realizable with $m$ thresholds for $m$ greater than $k$.

It is of substantial theoretical and practical interest to determine the minimum number of thresholds required to realize any function of $n$ variables. We shall give a lower bound for this minimum number. To the author's knowledge no one has exhibited an $n$-variable function that requires more than $n$ thresholds. We will show that for sufficiently large $n$ such functions must exist.

We see that

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\[ \prod_{j=1}^{k} \left( \sum_{i=1}^{n} w_i x_i - T_j \right) = 0 \] (2)

will be satisfied iff Eq. 3 holds.

\[ \sum_{i=1}^{n} w_i x_i - T_j = 0 \quad \text{for some } j, \ 1 \leq j \leq k. \] (3)

Consider an \((n+k)\)-dimensional space (called the realization space) with axes labeled \(w_1, \ldots, w_n, T_1, \ldots, T_k\). Each point in this space corresponds to multiple-threshold realizations of a function and its complement. Both these realizations require \(k\) or fewer thresholds. Using the vectors \(\mathbf{W} = (w_1, \ldots, w_n)\) and \(\mathbf{X} = (x_1, \ldots, x_n)\), we can write Eq. 3 as

\[ \mathbf{W} \cdot \mathbf{X} - T_j = 0. \] (4)

For any particular \(\mathbf{X}\), Eq. 4 is the equation of a hyperplane passing through the origin of the realization space.

For a given \(\mathbf{X}\), the \(k\) hyperplanes defined by Eq. 5 below divide the realization space into a finite number of regions, the exact number depending on the relative orientations of the hyperplanes.

\[ \mathbf{W} \cdot \mathbf{X} - T_j = 0 \quad 1 \leq j \leq k \] (5)

The coordinates of any point on any hyperplane are such that

\[ \prod_{j=1}^{k} \left( \mathbf{W} \cdot \mathbf{X} - T_j \right) = 0. \] (6)

The coordinates of a point that is not on any hyperplane (internal to a region) are such that either

\[ \prod_{j=1}^{k} \left( \mathbf{W} \cdot \mathbf{X} - T_j \right) > 0 \] (7)

or

\[ \prod_{j=1}^{k} \left( \mathbf{W} \cdot \mathbf{X} - T_j \right) < 0. \]

Furthermore, the coordinates of all points internal to a given region will yield the same sign for the product in Eq. 7.
Now let $\overline{X}$ be a vector in $n$-dimensional switching space. Each of the $2^n$ possible $\overline{X}$'s generates $k$ hyperplanes. Thus all $2^n \overline{X}$ vectors generate $k^2 n$ hyperplanes, which divide the realization space into a finite number of regions. The coordinates of a point internal to a given region specify a Boolean function and its complement, both of which require $k$ or fewer thresholds for their realizations. The coordinates associated with all points in a given region correspond to realizations of the same two functions. It is possible, however, that different regions of the realization space may correspond to the same two functions.

Let $S(k, n)$ be the maximum number of regions into which the realization space can be divided by $k^2 n$ hyperplanes, all passing through the origin. Then $2S(k, n)$ is an upper bound to $T(k, n)$, the number of $n$-variable Boolean functions that are realizable with $k$ or fewer thresholds. Using a result of Cameron, $^3$ we have

$$S(k, n) = 2 \sum_{\ell=0}^{n+k-1} \binom{k^2 n - 1}{\ell}$$

This gives

**THEOREM 1:**

$$T(k, n) < 4 \sum_{\ell=0}^{n+k-1} \binom{k^2 n - 1}{\ell}.$$  \hspace{1cm} (8)

Employing a bound of Winder $^4$ and then using Stirling's approximation, we have

$$T(k, n) < \frac{4(k^2 n)^{n+k-1}}{(n+k-1)!} < \frac{2}{\sqrt{\pi}} \left( \frac{ekn}{n + k - 1} \right)^{n+k-1}.$$ \hspace{1cm} (9)

Let $K(n)$ be the smallest number of thresholds required to realize all $2^{2^n}$ functions of $n$ variables. $K(n)$ must be such that

$$\frac{2}{\sqrt{\pi}} \left( \frac{eK(n)}{n + K(n) - 1} \right)^{n+K(n)-1} > T(K(n), n) \geq 2^{2^n}$$ \hspace{1cm} (10)

Using the fact $^2, 5$ that $K(n) \geq n$ and $K(n) \leq 2^n$ and a series of manipulations on the left-most term of Eq. 11, we can establish

**THEOREM 2:**

$$K(n) > \frac{2^{n-2}}{n} \text{ for } n \geq 2.$$ \hspace{1cm} (12)

Thus for values of $n \geq 8$, $K(n) > n$, and hence there must exist functions of 8 variables
that require more than 8 thresholds.

Also, with reference to Spann\textsuperscript{6} we have shown the following.

**THEOREM 3:** For $n \geq 10$ the class of Modular Threshold functions does not contain all functions.

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**References**


