On lattice codes for Gaussian interference channels

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Abstract—The usefulness of lattice codes is investigated for two-user Gaussian interference channels (IC). A coding scheme based on the compute-and-forward technique is shown to achieve the capacity region of the Gaussian IC under strong interference. The proposed scheme uses single-user decoders whereas the conventional scheme uses multi-user decoders for simultaneous decoding. The same scheme is applicable to the Gaussian Z-interference channel. A lattice-based coding scheme is also devised for the state-dependent Gaussian IC with the state sequence non-causally known to transmitters. New achievable rate regions are presented using a similar lattice-based scheme.

In this work we introduce a novel coding scheme for the two-user Gaussian IC using single-user decoders. More precisely, we show that for the Gaussian IC with strong interference, this scheme always achieves the corner points of the capacity region, and for some parameters, achieves the whole capacity region without time-sharing. We also show that it achieves the capacity region for the so-called Gaussian Z-interference channel, i.e. $g_2 = 0$ in (1). At last we study state-dependent Gaussian interference channels with the state sequence noncausally known at transmitters. New achievable rate regions are presented using a similar lattice-based scheme.

I. INTRODUCTION

Consider a two-user Gaussian IC

\[ y_1 = x_1 + g_1 x_2 + z_1 \]  
\[ y_2 = x_2 + g_2 x_1 + z_2 \]

with $x_k, y_k \in \mathbb{R}^n$ denoting the channel input at transmitter (Tx) $k$ and the channel output at receiver (Rx) $k$, $k = 1, 2$. The noises $z_1, z_2 \in \mathbb{R}^n$ are assumed to be Gaussian with unit variance per entry. Power constraints are imposed on the channel input as $\mathbb{E}([|x_k|^2] \leq nP_k$ for $k = 1, 2$. Transmitter $k$ has a message $W_k$ from the set $\{1, \ldots, 2^{nR_k}\}$ to send to the corresponding Rx $k$ and it is required that both receivers can decode their intended message reliably. We denote the received signal-to-noise ratio as $S_k := P_k$, $k = 1, 2$ and the received interference-to-noise ratio as $I_1 := g_2^2 P_2, I_2 := g_1^2 P_1$.

The capacity region of this channel is known under the strong interference condition, i.e. when it holds that

\[ I_1 \geq S_2, I_2 \geq S_1. \]

In this case the capacity region of the two-user Gaussian with strong interference is given by [1]

\[ R_1 \leq C(S_1), \quad R_2 \leq C(S_2) \] (3a)
\[ R_1 + R_2 \leq C_{\text{min}} := \min\{C(S_1 + I_1), C(S_2 + I_2)\} \] (3b)

where $C(x) := \frac{1}{2} \log(1 + x)$. An illustration is shown in Figure 1. The capacity region in this case is the intersection of capacity regions of two Gaussian multiple access channels (MAC): one composed of two Txs and Rx 1 as the receiver (call it MAC 1); and one composed of two Txs and Rx 2 as the receiver (call it MAC 2).

It is well known that for such a Gaussian IC, the capacity region can be achieved by letting both receivers perform joint decoding (also called simultaneous decoding) to decode both messages. For a Gaussian MAC, it is also well known that in addition to joint decoding, two other decoding schemes, successive cancellation decoding (SCD) with time-sharing and rate-splitting scheme [2], can achieve the capacity region with a single-user decoder. A single-user decoder is sometimes preferred in practice for various reasons including complexity issues. Since the capacity region of Gaussian IC with strong interference is the intersection of capacity regions of two MACs, we could ask if the above two low complexity methods also achieve the capacity region of a Gaussian IC. However, as it is shown in [3], the standard rate-splitting scheme is not able to achieve the whole capacity region, regardless the number of layers and the code distribution of each layer. It is also easy to see that SCD with time-sharing fails to achieve the capacity region. To overcome this difficulty, a sliding-window superposition coding scheme is proposed in [3] which achieves the joint decoding inner bound for general interference channels. Combined with time-sharing, it achieves the capacity region of Gaussian IC with strong interference.

In this work we introduce a novel coding scheme for the two-user Gaussian IC using single-user decoders. More precisely, we show that for the Gaussian IC with strong interference, this scheme always achieves the corner points of the capacity region, and for some parameters, achieves the whole capacity region without time-sharing. We also show that it achieves the capacity region for the so-called Gaussian Z-interference channel, i.e. $g_2 = 0$ in (1). At last we study state-dependent Gaussian interference channels with the state sequence noncausally known at transmitters. New achievable rate regions are presented using a similar lattice-based scheme.

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Fig. 1. The shaded region is the capacity region of a two-user IC under strong interference as the intersection of capacity regions of two Gaussian MAC: solid line for MAC 1 and dashed line for MAC 2. This example shows the case when $S_2 + I_2 \geq S_1 + I_1$, i.e., MAC 2 has a higher sum rate capacity.
II. PRELIMINARIES: COMPUTE-AND-FORWARD MULTIPLE ACCESS

Our proposed scheme is based on the compute-and-forward approach [4]. A generalized compute-and-forward scheme is used in [5] to provide a new multiple-access technique for the Gaussian MAC. For completeness we first introduce the scheme from [5] in this section.

Now consider a two-user Gaussian MAC consisting of two transmitters and one receiver

\[ y = x_1 + x_2 + z. \]  

(4)

with power constraints \( \mathbb{E}[|x_k|^2] \leq nP_k \) and unit noise variance. We represent the messages \( W_k \) of user \( k \) by points in \( \mathbb{R}^n \) denoted by \( t_k \), which are elements of the codebook \( C_k \) of user \( k \) with rate \( R_k := \frac{1}{2} \log |C_k| \) for \( k = 1, 2 \). Each transmitter is equipped with an encoder \( \mathcal{E}_k \) which maps its message to the channel input as \( x_k = \mathcal{E}_k(t_k) \). We require the receiver to decode both messages \( t_1, t_2 \) from \( y \) with arbitrarily small error.

Lattice codes are used for this Gaussian MAC. A lattice \( \Lambda \) is a discrete subgroup of \( \mathbb{R}^n \) with the property that \( t_1, t_2 \in \Lambda \), then \( t_1 + t_2 \in \Lambda \). More details about lattice codes can be found in [6]. Define the lattice quantizer \( Q_\Lambda : \mathbb{R}^n \rightarrow \Lambda \) as \( Q_\Lambda(x) = \arg\min_{t \in \Lambda} ||t - x|| \) and define the fundamental Voronoi region of the lattice to be \( V := \{ x \in \mathbb{R}^n : Q_\Lambda(x) = 0 \} \). The modulo operation gives the quantization error: \( |x| \mod \Lambda = x - Q_\Lambda(x) \).

Two lattices \( \Lambda \) and \( \Lambda' \) are said to be nested if \( \Lambda' \subseteq \Lambda \).

Let \( \Lambda_1, \Lambda_2 \) be two nested lattices which are simultaneously good in the sense of [6]. Let \( \beta_1, \beta_2 \) be positive numbers and we collect them into one vector \( \beta := (\beta_1, \beta_2) \). We can construct two coarser lattices \( \Lambda_k^\beta \) for \( k = 1, 2 \). The lattice \( \Lambda_k^\beta \) is also chosen to be simultaneously good and with second moment

\[ \frac{1}{n \text{Vol}(V_k^\beta)} \int_{V_k^\beta} ||x||^2 \, dx = \beta_k^2 P_k \]

where \( V_k^\beta \) denotes the Voronoi region of the lattice \( \Lambda_k^\beta \).

For transmitter \( k \) in the Gaussian MAC (4), we construct the codebook as \( C_k^\beta = \Lambda_k \cap V_k^\beta \). To send the message \( W_k \) represented by the lattice codeword \( t_k \), Tx \( k \) transmits

\[ x_k = \lfloor t_k/\beta_k + d_k \rfloor \mod \Lambda_k^\beta/\beta_k \]  

(5)

where \( d_k \) is called a dither which is a random vector uniformly distributed in the scaled Voronoi region \( V_k^\beta/\beta_k \).

In the Gaussian MAC, the decoder estimates two integer sums of the two codewords given as

\[ u_1 = a_1 t_1 + a_2 t_2, \quad u_2 = b_1 t_1 + b_2 t_2 \]

with the coefficient matrix \( A = \begin{pmatrix} a_1 & a_2 \\ b_1 & b_2 \end{pmatrix} \) satisfying the requirement that \( A \) is a full rank integer matrix. Let \( \hat{u}_k \) denote the decoded integer sums at the receiver and define the error probability of decoding

\[ P_e^{(n)} := P\{ \hat{u}_1 \neq u_1 \} \cup \{ \hat{u}_2 \neq u_2 \} \]  

(6)

where \( n \) is the length of the codewords. If the decoder can decode these two sums reliably, it can solve for two codewords \( t_1, t_2 \) from the two sums hence obtain both messages. This is another multiple access technique based on lattice codes. We have the following formal definition.

**Definition 1 (Achievability with CFMA):** For a two-user Gaussian MAC, we say a rate pair \((R_1, R_2)\) is achievable with compute-and-forward multiple access (CFMA), if the rate of codebook \( C_k \) is \( R_k, k = 1, 2 \) and the error probability defined in (6) can be made arbitrarily small for large enough \( n \).

One feature of this multiple access technique is that the decoder only needs to be equipped with a single-user decoder for performing lattice quantization. Furthermore time-sharing is not needed.

**Theorem 1 ([5] CFMA for the Gaussian MAC):** Consider the Gaussian MAC in (4). If it holds that

\[ \sqrt{\frac{P_1 P_2}{1 + P_1 + P_2}} \geq 1, \]  

(7)

then the capacity region is achievable using CFMA. Furthermore, the coefficients of the decoded sums are chosen to be either \( A = \begin{pmatrix} 1/2 \\ 1/2 \end{pmatrix} \) or \( A = \begin{pmatrix} 1/2 \\ 0 \end{pmatrix} \).

**Remark 1:** As shown in [5], in order to achieve a given rate pair \((R_1, R_2)\) in the capacity region, the scaling parameters \( \beta_k \) in (5) should be set to some specific values depending on this rate pair. Also notice that, a given rate pair can be achieved by decoding two sums with at least one of the two coefficient matrices above, and sometimes both of them. We need the condition on the transmitting power in (7) to make sure the whole capacity region is achievable; see [5] for more details.

III. CFMA FOR THE TWO-USER GAUSSIAN IC

In this section we show how to apply CFMA to the Gaussian interference channel. Rx \( k \) decodes two sums of the codewords in the form:

\[ u_{k1} = a_{k1} t_1 + a_{k2} t_2, \quad u_{k2} = b_{k1} t_1 + b_{k2} t_2 \]

with the coefficient matrix \( A_k = \begin{pmatrix} a_{k1} & a_{k2} \\ b_{k1} & b_{k2} \end{pmatrix} \) satisfying the requirement that \( A_k \) is a full rank integer matrix. Let \( u_{kj}, j = 1, 2 \) denote the two decoded integer sums at Rx \( k \) and define the error probability of decoding as

\[ P_e^{(n)} := P\{ \hat{u}_{k1} \neq u_{k1} \} \cup \{ \hat{u}_{k2} \neq u_{k2} \}, k = 1, 2 \]  

(8)

where \( n \) is the length of the codewords. Formally we have the following definition.

**Definition 2 (Achievability with CFMA):** For a two-user Gaussian IC, we say a rate pair \((R_1, R_2)\) is achievable with compute-and-forward multiple access (CFMA), if the rate of codebook \( C_k \) is \( R_k \) and the error probability \( P_e^{(n)} \), \( k = 1, 2 \) in (8) can be made arbitrarily small for large enough \( n \).

Notice that we do not include time-sharing in the above definition. This means if we say a certain rate pair is achievable using CFMA, only a single-user decoder is used at each receiver without time-sharing.
We focus on the Gaussian IC with strong but not very strong interference, i.e., in addition to (2), the sum rate constraint in (3b) is active. In this case the capacity region in (3) is a pentagon and we can identify two corner points \((R_1, R_2)\) as
\[
\left( \min_{S_1} C(S_1), C(S_2) \right) \quad \text{and} \quad \left( C(S_1), \min_{S_2} C(S_2) \right).
\] (9)

**Theorem 2 (CFMA for the Gaussian IC):** Consider the two-user Gaussian IC in (1) with strong but not very strong interference. If it holds that
\[
\min \left\{ \frac{S_1 I_1}{1 + S_1 + I_1}, \frac{S_2 I_2}{1 + S_2 + I_2} \right\} \geq 1
\] (10)
the corner points (9) of the capacity region are achievable using CFMA. Furthermore if it holds that
\[
I_1 \geq S_2 (1 + S_1) \quad \text{or} \quad I_2 \geq S_1 (1 + S_2),
\] (11)
the whole capacity region is achievable with CFMA.

**Remark 2:** It is well known that if it holds that
\[
I_1 \geq S_2 (1 + S_1) \quad \text{and} \quad I_2 \geq S_1 (1 + S_2),
\] (12)
the sum rate constraint in (3b) is inactive and the channel is said to be in very strong interference regime. The optimal point in its capacity region \(R_0 = C(S_k), k = 1, 2\) can be achieved by using SCD at both receivers to first decode the other user’s message. Our results show that under a weaker condition (11), where interference from only one transmitter is very strong, the proposed scheme can already achieve the whole capacity region using a single-user decoder without time-sharing.

**Proof:** The channel inputs are the same as for the Gaussian MAC as in (5). We give the proof for the case when it holds that \(S_2 + I_2 \geq S_1 + I_1\), i.e., MAC 2 has a higher sum capacity than MAC 1. The other case can be proved similarly. In this case the capacity of this Gaussian IC is depicted in Figure 2 as the intersection of capacity regions of two Gaussian MACs. The two corner points of the IC capacity region are marked as \(A\) and \(B\), and the upper corner point of the MAC 2 capacity region is marked as \(C\). We use \(A_1\) and \(A_2\) to denote point \(A\)’s coordinates on horizontal and vertical axes, respectively, and so on.

We first consider the subcase 1 on the left side of Figure 2, where \(C_1 \leq B_1\). This means \((1 + S_2 + I_2)/(1 + S_2) < 1 + S_1\), or equivalently \(I_2 < S_1 (1 + S_2)\).

In order to achieve the corner point \(A\), Rx 1 decodes two sums with coefficient matrix \(A_1 = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}\) or \(A_1 = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}\). According to Theorem 1, depending on the position of \(A\), at least one of the two coefficient matrices \(A_1\) allows Rx 1 to decode both messages at the rate \((R_1, R_2) = (A_1, A_2)\), if the condition (10) holds. Rx 2 decodes two sums with the coefficient matrix \(A_2 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\) namely the usual successive cancellation decoding. This allows Rx 2 to recover both messages if the rates satisfy \(R_1 \leq C_1\) and \(R_2 \leq C_2\). We point out that in order to let Rx 1 operate at point \(A\), the scaling parameter \(\beta_1, \beta_2\) should satisfy \(\beta_1/\beta_2 = c\) for some value \(c\) depending on \(A\). However, the usual SCD at Rx 2 works for any values of \(\beta_1, \beta_2\). Furthermore notice that \(A_1 \leq C_1\) and \(A_2 \leq C_2\) due to our assumption that MAC 2 has a higher sum capacity, this guarantees that both receivers can decode both messages reliably for the rate pair \(A\).

To achieve the corner point \(B\), we let Rx 2 decode two sums with coefficient matrix \(A_2 = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}\) or \(A_2 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\). Due to Theorem 1, at least one of the two choices on coefficient matrix \(A\) allows Rx 2 to decode both messages at the rate \(R_1 \leq B'_1\) and \(R_2 \leq B'_2\), if the condition (10) holds and parameters \(\beta_1, \beta_2\) are chosen accordingly. Here \(B'\) is the projection of point \(B\) on the dominant face of the MAC 2 capacity region along the vertical axis. Now Rx 1 performs the SCD to decode \(t_2\) and \(t_1\) at the rate \(R_1 = B_1, R_2 = B_2\). Since \(B'_1 = B_1\) and \(B'_2 \geq B_2\), both decoders can decode both messages reliably, hence achieve the corner point \(B\).

Now we consider the subcase 2 on the right side of Figure 2 when \(I_2 \geq S_1 (1 + S_2)\). In this case we have \(C_1 \geq B_1\). The same as achieving point \(A\), we let Rx 1 to decode two sums with coefficient matrix \(A_1 = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}\) or \(A_1 = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}\). Due to Theorem 1, all points on the segment \(AB\) are achievable if \(\beta_1, \beta_2\) are chosen accordingly. Rx 2 uses SCD (equivalently \(A_2 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\)) which allows it to decode both messages if the rate pair \((R_1, R_2)\) satisfies \(R_1 \leq C_1\) and \(R_2 \leq C_2\). However, this is true for all rate pairs on the segment \(AB\) in this case. This means all points on the dominant face of the capacity region can be achieved using CFMA in this case.

For the case when MAC 1 has a higher sum rate, the results can be proved in the same way and we summarize the decoding operation in Table I and II. Table I shows how receivers should decode to obtain the corner points. Table II shows the case when either one of the receivers experience very strong interference, the decoding operation at receivers for achieving the whole capacity region.

**Remark 3:** We point out that even when the one-sided very strong interference condition in (11) is not fulfilled, we can still achieve points other than the corner points on the capacity region with CFMA. As marked in Figure 2 subcase 1, using the same argument we can show that all points on segment \(AC'\) are achievable using CFMA, where \(C'\) is the projection of \(C\) along the vertical axis on the dominant face of MAC 1.
Remark 4: Another special case where CFMA achieves the whole capacity region without time-sharing is when \( g_1 = g_2 = 1 \) (equivalently \( I_1 = S_2 \) and \( I_2 = S_1 \)), which is not covered in the above theorem. In this case both decoders choose the same coefficient matrix \( A_k = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} \) or \( A_k = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \), \( k = 1, 2 \).

### TABLE I

<table>
<thead>
<tr>
<th>Corner point ((R_1, R_2))</th>
<th>(A_1) at Rx 1</th>
<th>(A_2) at Rx 2</th>
</tr>
</thead>
<tbody>
<tr>
<td>(C_{\min} - C(S_2), C(S_2))</td>
<td>\begin{pmatrix} 1 &amp; 1 \ 1 &amp; 0 \end{pmatrix} or \begin{pmatrix} 1 &amp; 1 \ 0 &amp; 1 \end{pmatrix}</td>
<td>\begin{pmatrix} 1 &amp; 0 \ 1 &amp; 0 \end{pmatrix} or \begin{pmatrix} 1 &amp; 1 \ 0 &amp; 1 \end{pmatrix}</td>
</tr>
<tr>
<td>(C(S_1), C_{\min} - C(S_1))</td>
<td>\begin{pmatrix} 0 &amp; 1 \ 1 &amp; 0 \end{pmatrix}</td>
<td>\begin{pmatrix} 0 &amp; 1 \ 1 &amp; 0 \end{pmatrix}</td>
</tr>
</tbody>
</table>

### TABLE II

<table>
<thead>
<tr>
<th>Condition</th>
<th>(A_1) at Rx 1</th>
<th>(A_2) at Rx 2</th>
</tr>
</thead>
<tbody>
<tr>
<td>(I_2 \geq S_1(1 + S_2))</td>
<td>\begin{pmatrix} 1 &amp; 1 \ 1 &amp; 0 \end{pmatrix} or \begin{pmatrix} 1 &amp; 1 \ 0 &amp; 1 \end{pmatrix}</td>
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</tr>
</tbody>
</table>

IV. CFMA FOR THE GAUSSIAN Z-INTERFERENCE CHANNEL

As a special case of the Gaussian IC, the so-called Gaussian Z-interference channel has also been studied in, for example, [7] and [8]. In this model, the channel gain \( g_2 \) in (1) is set to be zero (hence \( I_2 = 0 \)) and other setup is the same for the Gaussian IC channel. In the case \( I_1 \geq S_2 \) (notice it does not satisfy the strong interference condition in (2)), the capacity region of this channel is known to be

\[
R_1 \leq C(S_1), \quad R_2 \leq C(S_2), \quad R_1 + R_2 \leq C(S_1 + I_1)
\]

We use a similar argument to show that this capacity region is achievable with CFMA.

Theorem 3 (CFMA for Gaussian Z-interference channels): Consider the Gaussian Z-interference channel with strong interference, i.e., \( I_1 \geq S_2 \). If it holds that

\[
\sqrt{\frac{S_1I_1}{1 + S_1 + I_1}} \geq 1,
\]

the whole capacity region is achievable using CFMA (without time-sharing).

Proof: The capacity region (13) of a Gaussian Z-interference channel with strong interference is given in Figure 3. The solid line depicts the capacity region of MAC 1. The dominant face is the line segment \( AB \). Using lattice decoding, we can show that this capacity region can be achieved using the rate-splitting scheme [2]. It can be seen that if \( R_2 \) performs usual lattice decoding as in a point-to-point channel and the decoding will be successful if \( R_2 \leq C(S_2) \), which is the case for any rate pair on the line \( AB \).

Remark 5: Different from the Gaussian IC, rate pairs on the dominant face of the Z-interference channel capacity region can be achieved using the rate-splitting scheme [2]. It can be seen that if \( R_2 \) performs usual lattice decoding as in a point-to-point channel and the decoding will be successful if \( R_2 \leq C(S_2) \), which is the case for any rate pair on the line \( AB \).

V. THE TWO-USER GAUSSIAN IC WITH STATE

Now we consider a two-user Gaussian IC of the form

\[
\begin{align*}
    y_1 &= x_1 + g_1 x_2 + c_1 s + z_1, \\
    y_2 &= x_2 + g_2 x_1 + c_2 s + z_2
\end{align*}
\]

where \( s \in \mathbb{R}^n \) is a state sequence non-causally known to two transmitters but not to receivers. Each entry of \( s \) is an i.i.d. random variable with a given distribution (not necessarily Gaussian) and variance \( E[|s_i|^2] = Q \) for \( i = 1, \ldots, n \). The other setup is the same as for the normal Gaussian IC in (1).

Theorem 4 (CFMA for Gaussian IC with state): Consider the Gaussian IC with state where \( I_1 \geq S_2 \). If it holds that

\[
\sqrt{\frac{S_1I_1}{1 + S_1 + I_1}} \geq 1,
\]

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    y_2 &= x_2 + g_2 x_1 + c_2 s + z_2
\end{align*}
\]

where \( s \in \mathbb{R}^n \) is a state sequence non-causally known to two transmitters but not to receivers. Each entry of \( s \) is an i.i.d. random variable with a given distribution (not necessarily Gaussian) and variance \( E[|s_i|^2] = Q \) for \( i = 1, \ldots, n \). The other setup is the same as for the normal Gaussian IC in (1). To make the model slightly more general, we use two real numbers \( c_1, c_2 \) to represent the fact that two channels may suffer from differentially scaled versions of the same interference \( s \). This model has been studied in, for example, [9] [10] where various coding schemes are given.

In our scheme, the channel input for this channel is given by

\[
\begin{align*}
    x_k &= [t_k / \beta_k + d_k - \gamma_k s / \beta_k] \mod \Lambda_k / \beta_k, \\
    y_k &:= z_k + \sum_{i=1}^{2} a_{k,i} t_i, \\
    \hat{y}_k &:= \hat{z}_k + \sum_{i=1}^{2} b_{k,i} \hat{t}_i
\end{align*}
\]

with \( \hat{t}_k := t_k - Q \Lambda_k (t_k + \beta_k d_k - \gamma_k s) \). The variance \( N_{k1} \) per dimension for noise \( z_{k1} \), and variance \( N_{k2} \) for noise \( z_{k2} \) at Rx \( k \) are given in (16). Using lattice decoding, we can show the following achievable rate region for the 2-user Gaussian IC with state.

\[
\begin{align*}
    \sqrt{\frac{S_1I_1}{1 + S_1 + I_1}} \geq 1,
\end{align*}
\]
for certain parameter regimes. However the capacity result for the following special case seems not to be present in the literature.

**Remark 6:** In addition to the Gaussian IC where \( \beta_k \) are used to control the rates of two users, the extra parameters \( \gamma_k \) are used to eliminate (partially or completely) the interference \( s \). For given \( \beta_k, \gamma_k \) and \( \mathbf{A}_k \), the optimal \( \alpha_{k1}, \alpha_{k2} \) and \( \lambda_k \) which maximize the achievable rates can be given explicitly.

Depending on system parameters, the lattice-based scheme for the Gaussian IC with state can outperform the best known schemes, especially when the interference \( s \) is very strong. We show an example in Figure 4. We consider a symmetric Gaussian IC with state in (15) with parameters \( P_1 = P_2 = 5, g_1 = g_2 = 1.5, Q = 6000 \) and compare our achievable rate region with the best known result from [9, Thm. 3]. We use the capacity region in (3) as an outer bound.

The capacity region for this channel is characterized in [10] for four special cases. However the capacity result in [10] is not known for the Gaussian IC with state in (15) with parameters \( P_1 = P_2 = 5, g_1 = g_2 = 1.5, Q = 6000 \) and compare our achievable rate region with the best known result from [9, Thm. 3]. We use the capacity region in (3) as an outer bound.

The capacity region for this channel is characterized in [10] for certain parameter regimes. However the capacity result for the following special case seems not to be present in the literature.

**Lemma 1:** For the Gaussian IC with state in (15) with parameters \( g_1 = g_2 = 1 \) and \( c_1 = c_2 \), if it holds that \( \sqrt{\frac{P_1 P_2}{1 + P_1 + P_2}} \geq 1 \), the capacity region is given by

\[
R_1 \leq C(P_1), \quad R_2 \leq C(P_2), \quad R_1 + R_2 \leq C(P_1 + P_2)
\]

**Proof sketch:** The converse is obvious. For the achievability part, note that if it holds that \( g_1 = g_2 = 1 \) and \( c_1 = c_2 \), the system is equivalent to two Gaussian MACs which are exactly the same. Indeed, notice that in this case the noises \( N_{k1}, N_{k2} \) in (16) at two receivers \( k = 1, 2 \) are identical if we choose \( \alpha_{k1}, \alpha_{k2}, \lambda_k \) and \( \mathbf{A}_k \) to be the same for \( k = 1, 2 \). Further notice that for any \( \alpha_{k1}, \alpha_{k2} \) and \( \lambda_k \) we can choose \( \gamma_1, \gamma_2 \) such that the terms in (16) involving \( Q \) vanish. Hence the interference \( s \) can be canceled out completely and the system is equivalent to two identical Gaussian MAC (without interference). Using the result in Theorem 1, we know that the entire capacity region of the corresponding Gaussian MAC can be achieved with the coefficient matrices \( \mathbf{A}_k = (\frac{1}{a}, \frac{b}{a}) \) or \( \mathbf{A}_k = (\frac{1}{a}, \frac{b}{a}) \), \( k = 1, 2 \).

**REFERENCES**


