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BERTRAND-EDGEWORTH EQUILIBRIA IN FINITE EXCHANGE ECONOMIES

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Bertrand-Edgeworth Equilibria in Finite Exchange Economies

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Abstract

We introduce a framework for analyzing Bertrand-Edgeworth equilibria in finite Arrow-Debreu exchange economies. A key feature of the framework is the way trade takes place. There are two main stages. In the first stage agents simultaneously choose prices and quantities of commodities they want to sell. In the second stage they enter the markets as buyers sequentially and choose only quantities of commodities they want to buy. We show that, under certain Lipschitz conditions on demands, the equilibria obtained are generally different from the Walrasian ones.

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1 Introduction

In the opening paragraph of the Theory of Value, Debreu quotes as the two central problems being presented in his monograph "(1) the explanation of the prices of commodities resulting from the interaction of the agents of a private ownership economy through markets, (2) the explanation of the role of prices in an optimal state of an economy."[1] By these he means (1) the question of existence of perfectly competitive equilibria and (2) the question of equivalence between Pareto optima and perfectly competitive equilibria. In fact, throughout the entire analysis, all agents are assumed to take prices as given. A natural question to ask is, What happens to the analysis when agents do not all take prices as given? To answer this question, a framework is required that allows at least some agents in the economy to influence prices. In this paper, we are interested in frameworks where agents that want to sell a given commodity also set the price of that commodity, while agents that want to buy a given commodity can do so only at the prices specified by the sellers, and only to the extent that the commodity is available. For simplicity, we restrict ourselves to frameworks that are formulated in finite Arrow-Debreu exchange economies.

The first models that allowed agents to choose both prices and quantities as strategic variables in Arrow-Debreu exchange economies are the ones by Schmeidler (1980), Dubey (1982), and Simon(1984).[2] These models are known as **strategic market games**, and are in fact very similar to the **market games** of Shubik (1973), Shapley (1976), and Shapley and Shubik (1977), which are also formulated in an exchange economy framework but which allow agents to only choose quantities as strategic variables. Both types of models take the form of games that are formulated in an exchange economy framework with well-defined payoffs for the agents in any situa-

[1] Debreu (1959), p. ix.

[2] Earlier models by Negishi (1961, 1972), Fitzroy (1974), Nikaidô (1975) and others also allow for price-setting agents. However, they are not formulated in an exchange economy framework and, moreover, do not allow for more than one agent to control or set the price of any given commodity. Surveys of the literature of imperfect competition in general equilibrium are Hart (1985) and Bonanno (1990).

tion. However, while in the market games agents only choose quantities, and prices are determined by some hypothesized price function that depends on the quantity choices of the agents, (which therefore allows agents to have an influence on prices), in the strategic market games the price function is avoided and prices are set directly by the agents. An outcome function that gives the outcomes as a function of the strategies chosen by the agents, i.e., prices and quantities, is introduced.

It may seem that it is the property of allowing agents to also choose prices as strategic variables, that leads to astonishingly competitive outcomes. In fact, the strategic market games yield equilibria that, under very weak assumptions, (apart from the outcome function), are exactly the perfectly competitive ones.[3] This is very different from the generally inefficient outcomes of the quantity-setting market games.[4] The latter obtain perfectly competitive equilibria as the Nash equilibria of their games only in the limit of replications, as the number of agents goes to infinity, or if there are a continuum of agents.[5] So the question is whether the fact that the Nash equilibria of the strategic market games are exactly the perfectly competitive ones is due only to the fact that agents are setting prices and therefore facing much higher elasticities of payoffs, or whether it is also due to the outcome functions that are specified as part of the games. We believe that the results obtained in a partial equilibrium framework, mostly in the industrial organization literature,[6] support the hypothesis that the efficiency of the equilibria obtained in the strategic market games really comes from the outcome functions specified than from the fact that prices are strategic variables. For example, Allen and Hellwig (1986a, 1986b) obtain as the Nash equilibria of a game with n firms choosing quantities and prices simultaneously and supplying the quantities produced to consumers who take the prices as given, mixed strategy equilibria with prices between a perfectly competitive price and a monopoly price. That is, not only do they obtain non-perfectly competitive equilibria but also Pareto inefficient ones.

[3]For example Dubey (1982) and Simon (1984) obtain this with only two active agents on both sides of each market.

[4]Dubey (1980) shows that the Cournot-Nash equilibria of one of the variants of the quantity-setting market game are generically finite and inefficient.

[5]See for example Dubey, Mas-Colell and Shubik (1980) or Mas-Colell (1982).

[6]See for example Kreps and Scheinkman (1983), Davidson and Deneckere (1986), Allen and Hellwig (1986a, 1986b), Maskin(1986), and Tirole (1988), Chapter 5.

In this paper we introduce a simple model of an Arrow-Debreu exchange economy in which agents choose both prices and quantities of commodities they want to supply and just quantities of commodities they want to buy. A key feature of the model is the way trade takes place. There are two main stages. In the first stage, simultaneously choose prices and quantities of commodities they want to put for sale, then, in the second stage, they enter the markets as buyers, one by one according to a given order, observe all that is available, at what prices, and buy whatever is available and affordable at the prices specified in the first stage, and maximizes their utility. What we obtain is an extensive form game that serves as a general equilibrium model of imperfect competition that is very close in spirit to industrial organization models of Bertrand-Edgeworth competition. In particular, we show that, under certain Lipschitz continuity conditions on demands, the Bertrand-Edgeworth equilibria of our game are Walrasian if and only if initial endowments are already Walrasian allocations, in which case the resulting equilibria are no-trade equilibria.

The paper is organized as follows. The next section contains the basic framework that is used throughout the paper. Section 3 provides examples of two, three, and four agent economies. Section 4, which is the main section of the paper, contains the general results, and Section 5 indicates some possible extensions to the framework of Section 2. The appendix shows some further propositions for a restricted class of exchange economies.

2 The Model

Consider an exchange economy $\mathcal{E} = ((u^i)_{i\in I}, (\omega_0^i, \omega^i)_{i\in I})$ with m agents, l regular commodities and one nonregular commodity, money, defined by:

> for each agent $i \in I = \{1, \ldots, m\}$, a von Neumann and Morgenstern utility function $u^i : \mathbb{R}_+^{l+1} \to \mathbb{R}$,[7] that is assumed to be strictly increasing, strictly quasi-concave, and twice continuously differentiable, and an endowment vector $(\omega_0^i, \omega^i) \in \mathbb{R}_+^{l+1}$ consisting of a money endowment $\omega_0^i \in \mathbb{R}_{++}$ and a regular commodity endowment $\omega^i \in \mathbb{R}_+^l$.

[7]See for example Kreps (1988), Chapter 5.

Notice that agents are assumed to derive utility from both the regular commodities and money. Money will also serve as numeraire and medium of exchange. Denote by $L = \{1, \ldots, l\}$ the set of regular commodities.

The model is ultimately static, but it is easier to think of it as representing a process of trade that takes place in two main stages. In the first stage, agents make decisions as sellers, and choose prices and quantities of commodities they want to sell. In the second stage, they act as buyers and are allowed in the markets one by one according to a given order. They observe all that is available, at what prices, and buy whatever is available and affordable at the prices specified by the agents in the first stage, and maximizes their utility function.

So suppose there is a market for each regular commodity, that regular commodities are sold for money, and suppose that in the first stage each agent chooses how much of her endowment of the regular commodities she is willing to sell, and at what prices. Then agent i's strategy, $i \in I$, in the first stage can be written as:

$$(p^i, s^i) \in \mathbb{R}^{2l}_+,$$

where $p^i \in \mathbb{R}^l_+$ is a nonnegative price vector, and $s^i \in \mathbb{R}^l_+$, $s^i \leq \omega^i$, is a nonnegative vector of quantities of the l regular commodities that does not exceed the initial endowment ω^i. Such a strategy (p^i, s^i) is to be interpreted as agent i's willingness to sell at prices p^i anything up to the quantities specified by the vector s^i. We will see that an agent may not be able to sell all of s^i if there is insufficient demand. For convenience, we denote the strategy space of agent i in the first stage by $P_i \times S_i \subset \mathbb{R}^{2l}_+$, and the space of strategy profiles in the first stage by $P \times S = P_1 \times S_1 \times \cdots \times P_m \times S_m \subset \mathbb{R}^{2lm}_+$ with generic element $(p, s) = ((p^1, s^1), \ldots, (p^m, s^m))$. We will see how further restrictions on the space of strategy profiles for the first stage arise naturally from agents' preferences and endowments.

In the second main stage agents act as buyers and enter the markets one by one according to a given order. More specifically, the **order** in which the agents enter the markets as buyers is defined by a bijective function $\tau : I \to I$ that assigns to each agent $i \in I$, a number $\tau(i) \in I$, that denotes when agent i enters the markets as a buyer. So if $\tau(i)$ is, say, 2, then i is the second agent to enter the markets as a buyer. For simplicity, it is assumed that the agents enter in a known pre-specified order, hence τ is fixed. Without loss of generality we assume that agents are arranged so that

τ is actually the identity function. When entering as buyers, we assume agents to have full information about all the utility functions $(u^i)_{i\in I}$, the endowments $({\omega_0}^i, \omega^i)_{i\in I}$, the strategies in the first stage $(p^i, s^i)_{i\in I}$, as well as the order τ in which they are to enter the markets. In this stage, agent 1 is the first agent to enter as a buyer. Her strategy in this stage is a function:

$$b^1 : P \times S \to \mathbb{R}_+^{lm},$$

satisfying, for all $(p,s) \in P \times S$, the conditions:

$$\sum_{j\neq 1} p^j b^{1j}(p,s) \leq \omega_0^1 \text{ and } 0 \leq b^{1j}(p,s) \leq s^j, \forall j \in I,$$

where the notation $b^{1j}(p,s)$ is to be interpreted as denoting what agent 1 buys from agent j given prices and quantities for sale $(p,s) \in P \times S$.[8] So $b^1(p,s) = (b^{1j}(p,s))_{j\in I}$ is a nonnegative vector of quantities that agent 1 can afford to buy from what is available for sale on the markets. Taking into account the fact that agent 2 enters the markets after agent 1, and agent 3 after agent 2, and so on, and assuming that each agent entering the markets knows what has already been bought, we can write agent i's strategy in the second stage, for $i = 2, 3, \ldots, m$, as a function:

$$b^i : P \times S \times (\times_{h<i} B) \to \mathbb{R}_+^{lm},$$

satisfying, for all $(p, s; (b^h)_{h<i}) \in P \times S \times (\times_{h<i} B)$, the conditions:

$$\sum_{j\neq i} p^j b^{ij}(p,s;(b^h)_{h<i}) \leq \omega_0^i \tag{1}$$

and

$$0 \leq b^{ij}(p,s;(b^h)_{h<i}) \leq s^j - \sum_{h<i} b^{hj}, \forall j \in I, \tag{2}$$

where $B = \times_{i\in I}[0, \omega^i] \subset \mathbb{R}_+^{lm}$. The expressions $\sum_{h<i} b^{hj}$ denote what agents entering the markets before i, i.e., agents $h < i$, have already bought from what agent j has put for sale. So conditions (1) and (2) can be interpreted

[8] Notice that $b^{1j}(p,s)$ is a vector in $\mathbb{R}_+^l$, and that it is multiplied by p^j in the first inequality, since it is bought at prices specified by j in the first stage. Notice also that we are not allowing agents to borrow, i.e., to buy with money that they may receive from what they are putting for sale.

respectively as a standard budget constraint and a rationing constraint. Furthermore, we use the notation $\mathcal{B}_i$ for the space of functions mapping from $P \times S \times (\times_{h<i} B)$ into $\mathbb{R}_+^{lm}$ and satisfying conditions (1) and (2). Let $\mathcal{B} = \mathcal{B}_1 \times \cdots \times \mathcal{B}_m$.

Combining the strategies of the first stage with the strategies of the second stage yields agent i's strategy for the overall game:

$$a^i = (p^i, s^i; b^i) \in P_i \times S_i \times \mathcal{B}_i.$$

We denote the strategy space of agent i by $A_i \subset P_i \times S_i \times \mathcal{B}_i$, and denote the space of strategy profiles $A_1 \times \cdots \times A_m \subset P \times S \times \mathcal{B}$ by A with generic element $a = (a^1, \ldots, a^m)$. Further restrictions on the space A will be seen to follow in Sections 3 and 4.

Finally, we are in a position to define the payoffs of the game. Each agent has an endowment vector $(\omega_0^i, \omega^i) \in \mathbb{R}_+^{l+1}$ and a utility function $u^i : \mathbb{R}_+^{l+1} \to \mathbb{R}$. Therefore, agent i's payoff function is given by:

$$\Pi^i : A \to \mathbb{R}, \Pi^i(a) = u^i(x_0^i(a), x^i(a)),$$

where $(x_0^i, x^i) : A \to \mathbb{R}_+ \times \mathbb{R}_+^l$ is the final allocation function that maps any strategy profile to a possible final allocation of money and regular commodities, that is defined by:

$$x_0^i(a) = \omega_0^i + \sum_{j \neq i} p^i b^{ji}(p, s; (b^h)_{h<j}) - \sum_{j \neq i} p^j b^{ij}(p, s; (b^h)_{h<i})$$

and

$$x^i(a) = \omega^i - \sum_{j \neq i} b^{ji}(p, s; (b^h)_{h<j}) + \sum_{j \neq i} b^{ij}(p, s; (b^h)_{h<i})$$

The payoff function for the overall game is simply $\Pi : A \to \mathbb{R}^m$, where $\Pi(a) = (\Pi^i(a))_{i \in I} \subset \mathbb{R}^m$. This makes the game ? $= (A, \Pi)$ consisting of the space of strategy profiles A and the payoff function Π well-defined.

Given the game ?, the question arises of what equilibrium concept to use. Although many concepts are conceivable, we find the subgame perfect Nash equilibrium concept of Selten (1965) to be the most natural candidate. Throughout the paper we allow agents to choose mixed strategies in the first main stage but not in the second main stage. Let $\wp(P \times S)$ denote the set of probability measures on the set $P \times S$ with generic element $(p, s)_\wp$. We say the strategy profile $((p,s)_\wp; b) \in \wp(P \times S) \times \mathcal{B}$ is a **subgame perfect Nash equilibrium** of the game ? if $((p,s)_\wp; b)$ is a Nash equilibrium of ? and if, for $i \in I$, $(b_i, b_{i+1}, \ldots, b_m) \in \mathcal{B}_i \times \mathcal{B}_{i+1} \times \cdots \times \mathcal{B}_m$ induces a Nash equilibrium of the subgame starting when agent i enters the markets as a buyer.[9] Some questions we ask are: Do pure strategy subgame perfect equilibria exist? Do mixed strategy equilibria exist? Are there relationships between the subgame perfect equilibria obtained and general equilibrium concepts such as the perfectly competitive equilibria or the set of Pareto efficient allocations? We turn to these questions in the next sections by looking first at some examples with two, three, and four agents, and then at the more general case described above. Some of the examples illustrate more general propositions to be stated in Section 4.

3 Three Examples

In this section we look at three examples to see how the model of Section 2 works and what its equilibria may look like. The first example is a two agent economy, the second is a replication of that economy, and the third example is a three agent economy. Section 4 also contains a fourth example that shows how coordination failures can arise in the model of Section 2 from the quantity-setting part of agents' strategies when they are restricted to charging perfectly competitive prices.

3.1 A First Example

Consider the case $l = 1$, $m = 2$, where the utility functions are of the Cobb-Douglas form, $u^i : \mathbb{R}^2_+ \to \mathbb{R}$, $u^i(x^i_0, x^i) = x^i_0 x^i$, $i = 1, 2$, and the initial endowments are given by $(\omega^1_0, \omega^1) = (1, 3)$ and $(\omega^2_0, \omega^2) = (3, 1)$. Suppose the order in which agents enter is $\tau(i) = 2 - i$ for $i = 1, 2$. This example does not capture the role of price competition between two or more agents, but it does give an idea of how the model works in a very simple case. It

[9] We explicitly distinguished two main stages. However, the second main stage should be seen as consisting of m stages, where in each stage an agent, say i, enters according to the order τ, and makes purchases $b^i \in \mathcal{B}_i$.

also shows an extreme case of how the subgame perfect Nash equilibria of the game described in Section 2 relate to the perfectly competitive equilibria and the set of Pareto efficient allocations.

First, we notice that the marginal rates of substitution between the regular commodity and money are 1/3 and 3 for agents 1 and 2 respectively. This means that, at initial endowments, agent 1 will not buy anything that is more expensive than 1/3, but will be willing to sell at prices higher than 1/3, and agent 2 will not buy anything that is more expensive than 3, but will buy at prices below 3. Therefore, we do not loose much by restricting the strategy spaces to:

$$P_1 \times S_1 = [1/3, 3] \times [0, 1] \text{ and } P_2 \times S_2 = \{3\} \times \{0\}.$$

In other words, agent 1 can be viewed as the seller of the commodity, and agent 2 as the buyer. We can write 2's maximization problem as:

$$max_{b^{21}} u^2(\omega_0^2 - p^1 b^{21}, \omega^2 + b^{21}) = (\omega_0^2 - p^1 b^{21})(\omega^2 + b^{21})$$

subject to

$$p^1 b^{21} \leq \omega_0^2 \text{ and } 0 \leq b^{21} \leq s^1,$$

where b^{21} is what 2 buys from 1, i.e., her demand for what 1 has put for sale. Solving this problem leads to 2's demand function for the regular commodity:

$$b^{21} = \frac{\omega_0^2 - p^1 \omega^2}{2p^1}.$$

Given this demand function, and after noticing that agent 1's demand function is the zero function, and agent 2's strategy choice in the first stage always is $(p^2, s^2) = (3, 0)$, one computes 1's optimal price and quantity decision by solving the maximization problem:

$$max_{(p^1, s^1)} u^1(\omega_0^1 + p^1 s^1, \omega^1 - s^1) = (\omega_0^1 + p^1 s^1)(\omega^1 - s^1)$$

subject to

$$p^1 \geq 0 \text{ and } 0 \leq s^1 \leq min\{\omega^1, b^{21}\}$$

which has the unique solution:[10]

$$(p^1, s^1) = ((\frac{\omega_0^2(\omega_0^2 + \omega_0^1)}{\omega^2(\omega^2 + 2\omega^1)})^{1/2}, b^{21}) \approx (1.46, 0.52).$$

[10]The values given are only 2-digit approximations.

Hence we have that the subgame perfect Nash equilibrium leads to:

$$(p^N; x^N) \approx (1.46; (1.76, 2.48), (2.24, 1.52)),$$

which is clearly different from the unique Walrasian equilibrium:

$$(p^W; x^W) = (1; (2, 2), (2, 2)).$$

Given that the agents have the same utility function and symmetric endowments,[11] and that it does not matter in which order the agents enter as buyers, it is worth noting:

Remark 1 *The subgame perfect Nash equilibrium is not symmetric.*

Furthermore, after noting that the set of Pareto efficient allocations is given by:

$$\{x = ((x_0^1, x^1), (x_0^2, x^2)) : x_0^i = x^i, i = 1, 2, \text{ and } \sum_{i=1,2} x_0^i = \sum_{i=1,2} x^i = 4\},$$

we have:

Remark 2 *The subgame perfect Nash equilibrium is not Pareto efficient.*

The implications of the second remark are obvious, in particular, as we noted above, it implies that the subgame perfect Nash equilibrium need not be perfectly competitive. This is in line with much of the industrial organization literature (e.g., Kreps and Scheinkman (1983), Allen and Hellwig (1986a, 1986b), and Davidson and Deneckere (1986)), which often obtains inefficient outcomes in single markets with a finite number of price-setting firms. At the same time, it is in sharp contrast with the results of the strategic market game literature (e.g., Schmeidler (1980), Dubey (1982), and Simon (1984)), which always obtains Walrasian equilibria as the result of price and quantity-setting agents in a pure exchange economy. Although this conclusion may seem premature since some of the strategic market game models require at least three agents to obtain perfectly competitive outcomes (e.g., Schmeidler (1980)), we interpret Remark 2 as evidence for the hypothesis that it is the specific form of the outcome function in the strategic market game models

[11]Symmetric here is to be understood in the sense of economies where all commodities are treated equally.

that yields the perfectly competitive equilibria, and not the fact that agents are choosing prices as strategic variables. First, it would not change our results qualitatively if we added a third or fourth agent that had preferences and endowments like agent 2, i.e., that was a buyer, since agent 1 would still be the only seller of the regular commodity while the other agents would still only be buyers. Second, in the next example we will see how even adding agents with preferences and endowments like agent 1, i.e., adding an extra seller, does not lead to perfectly competitive outcomes. In fact, since we will be replicating the economy of this example, we will satisfy the assumption that there are two active agents on each side of each market, which is what most strategic market games require to obtain perfectly competitive outcomes (e.g., Dubey (1982) and Simon (1984)).

The implications of the first remark on the other hand may not be so obvious. There are at least two: One is that it matters which of the two commodities is the regular commodity, (i.e., the commodity that is traded), and which commodity is used as medium of exchange. This may seem related but is different from the well-known fact in the literature of imperfect competition that the choice of numeraire has an influence on the outcome of the game. In our case it is not the choice of the numeraire that matters but the commodity that is chosen as the medium of exchange. The reason that in many models of imperfect competition the choice of numeraire makes a difference is that usually there are firms that maximize profits which are measured in units of the numeraire. The problem should not arise if the agents are all maximizing utility functions independent of the numeraire chosen.[12]

Another implication of the first remark is obtained by further noticing how agent 1 is better off than agent 2, which suggests that she has more power than agent 2 in determining the final outcome. This is due to the fact that she determines the terms of trade, i.e., the price as well as the quantity she wants to put for sale, while 2 only decides how much he will buy from 1, given what 1 has put for sale and at what prices. This is not at all surprising if one views agent 1 as being a monopolist and agent 2 as a price-taker.

Another feature of this example that is worth noting is that if one is to repeat the game, using the Nash equilibrium allocations of the previous phase

[12]Grodal (1992) analyzes this problem for a production economy with quantity-setting firms. She also hints at the fact that the problem arises even with price-setting firms. As a solution to the problem, she proposes to introduce utility functions in the models as the objective functions for the firms. See also Böhm (1994).

as the new endowments and leaving the rules otherwise unchanged,[13] then one obtains, as the new subgame perfect Nash equilibrium, a point that is strictly closer to the set of Pareto efficient allocations. Furthermore, if one is to repeat this procedure, in the limit, the subgame perfect Nash equilibrium will be Pareto efficient. We illustrate this point.

Solving the same maximization problems as above with the new endowments $(\omega_0^1, \omega^1) \approx (1.76, 2.48)$ and $(\omega_0^2, \omega^2) \approx (2.24, 1.52)$, one obtains as the Nash equilibrium of the repeated game:

$$(p^{N_2}; x^{N_2}) \approx (1.14; (2.01, 2.26), (1.99, 1.74)).$$

Repeating this procedure yields as the equilibria of the further repeated games:

$$(p^{N_3}; x^{N_3}) \approx (1.05; (2.09, 2.18), (1.91, 1.82)),$$
$$(p^{N_4}; x^{N_4}) \approx (1.02; (2.12, 2.15), (1.88, 1.85)),$$
$$(p^{N_5}; x^{N_5}) \approx (1.01; (2.13, 2.14), (1.87, 1.86)),$$

and, finally:

$$(p^{N_6}; x^{N_6}) \approx (1.00; (2.135, 2.135), (1.865, 1.865)),$$

which leads to the remark:

Remark 3 *Repeating the game leads to lower prices being charged and, in the limit, to a subgame perfect Nash equilibrium that is Pareto efficient.*

This result is not very surprising if one notices that after every round the marginal rates of substitution of the two agents always get closer. However, one could think that after one round there would no longer be an incentive to trade. But this is not the case here.

[13]This would imply that agents act myopically in the sense that they do not think that the game will be repeated, i.e. they act as if they were always playing a single period game.

3.2 A Second Example

The example of the previous section did not cover the case where two or more agents compete in setting the prices of one homogeneous commodity. This is the purpose of the following example, which is obtained by replicating the economy of the previous example. Consider the case $l = 1, m = 4$, with the endowments $(\omega_0^i, \omega^i) = (1, 3)$ for $i = 1, 2$, and $(\omega_0^j, \omega^j) = (3, 1)$ for $j = 3, 4$, and the same utility functions $u^i : \mathbb{R}_+^2 \to \mathbb{R}, u^i(x_0^i, x^i) = x_0^i x^i$ for $i = 1, 2, 3, 4$. Suppose agents enter in the order $\tau(i) = 5 - i, i = 1, 2, 3, 4$.

Using slightly more involved arguments, it is possible to show that all the remarks of the previous section extend also to this example. However, in this example, we notice:

Remark 4 *There exists no pure strategy subgame perfect Nash equilibrium for the replicated economy.*

Proof. As above, the marginal rates of substitution between the regular commodity and money are 1/3 for agents 1 and 2, and 3 for agents 3 and 4. Again, this means that, at initial endowments, agents 1 and 2 will not buy anything that is more expensive than 1/3, but will be willing to sell at prices higher than 1/3, while agents 3 and 4 will not buy anything that is more expensive than 3, but will buy at prices below 3. This allows us to reduce the first stage part of the game from a four player game to a two player game by noticing that we do not loose much by reducing the strategy spaces of the first stage to:

$$P^i \times S^i = [1/3, 3] \times [0, 1], P^j \times S^j = \{3\} \times \{0\},$$

for agents $i = 1, 2$, $j = 3, 4$. Furthermore, for the subgame perfect demands in the second stage, we have $b^i = 0$ for $i = 1, 2$, and, for agents 3 and 4, we have:

$$b^{31} + b^{41} = \begin{cases} min(s^1, \frac{\omega_0^3 - p^1\omega^3}{2p^1} + \frac{\omega_0^4 - p^1\omega^4}{2p^1}) & \text{if } p^1 < p^2 \\ min(s^1, \eta(\frac{\omega_0^3 - p^1\omega^3}{2p^1} + \frac{\omega_0^4 - p^1\omega^4}{2p^1})) & \text{if } p^1 = p^2 \\ min(s^1, \frac{\omega_0^3 - p^1\omega^3}{2p^1} + \frac{\omega_0^4 - p^1\omega^4}{2p^1} - b^{32} - b^{42}) & \text{if } p^1 > p^2 \end{cases}$$

and

$$b^{32} + b^{42} = \begin{cases} min(s^2, \frac{\omega_0^3 - p^2\omega^3}{2p^2} + \frac{\omega_0^4 - p^2\omega^4}{2p^2}) & \text{if } p^1 > p^2 \\ min(s^2, (1-\eta)(\frac{\omega_0^3 - p^2\omega^3}{2p^2} + \frac{\omega_0^4 - p^2\omega^4}{2p^2})) & \text{if } p^1 = p^2 \\ min(s^2, \frac{\omega_0^3 - p^2\omega^3}{2p^2} + \frac{\omega_0^4 - p^2\omega^4}{2p^2} - b^{31} - b^{41}) & \text{if } p^1 < p^2 \end{cases}$$

where $\eta \in [0,1]$.[14] These are the total market demands that agents 1 and 2 face respectively. Next consider the following functions:

$$\bar{s}^i : [1/3, 3] \to [0, 3], \bar{s}^i(p^i) = \frac{p^i\omega^i - \omega_0^i}{2p^i},$$

which are obtained from:

$$\bar{s}^i(p^i) = \{s^i \in [0,3] : \frac{\partial u^i(\omega_0^i + p^i s^i, \omega^i - s^i)}{\partial s^i} = 0\},$$

for $p^i \in [1/3, 3]$, $i = 1, 2$. These functions should be interpreted as giving the optimal quantities agents 1 and 2 want to sell as a function of the price they charge.

We show by contradiction that there is no strategy profile $(p, s) \in P \times S$ that is a pure strategy Nash equilibrium of the first stage. Suppose (p, s) is such a Nash equilibrium, we distinguish three cases: (i) $p^1 < p^2$, (ii) $p^1 = p^2$, and (iii) $p^1 > p^2$. By symmetry, cases (i) and (iii) are the same, which leaves cases (i) and (ii). First, we show that no agent charges prices $p^i \in [1/3, 1)$, $i = 1, 2$, at a Nash equilibrium of the first stage.[15] To see this, suppose agent 2 charges some price $p^2 \in [1/3, 3]$. Then, if $p^2 \geq 1$, for any $p^1 \in [1/3, 1)$, agent 1 faces demand:

$$b^{31} + b^{41} = min(s^1, \frac{3 - p^1}{p^1}),$$

while the quantity he would optimally want to sell is:

$$\bar{s}^1(p^1) = \frac{3p^1 - 1}{2p^1} < \frac{3 - p^1}{p^1},$$

[14] Strictly speaking η should be a function of the strategies chosen in the first stage. However, for this proof, there is no loss in assuming it to be a constant. This will be clear from the arguments that follow.

[15] The number 1.46 is the 2-digit approximation to the price p^N obtained in the previous example.

for $p^1 \in [1/3, 1)$. This means that if p^1 is less than one, then agent 1 can raise his price without having to sell any less than he would like to sell and be better off, since his utility function is strictly monotonic in money. Therefore, it cannot be optimal for him to charge $p^1 \in [1/3, 1)$. If, on the other hand, $p^2 < 1$, then a similar argument shows that also in this case, agent 1 wants to raise his price. This shows that agents 1 and 2 will never charge prices from $[1/3, 1)$ at a Nash equilibrium. Now we consider the cases (i) and (ii) above and show that the strategy profile $(p, s) \in [1, 3]^2 \times [0, 3]^2$ cannot be a pure strategy Nash equilibrium.

Case (i) $p^1 < p^2$: Agent 1 will want to deviate by increasing his price until he slightly undercuts p^2. Similarly, unless $(p^1, p^2) = (1, 1.46)$, in which case agent 2 does not want to decrease his price since he is exactly facing the demand agent 1 was facing in the previous example and where the optimal price was 1.46, in all other cases, agent 2 also always wants to slightly undercut p^1.

Case (ii) $p^1 = p^2$: For the same reasons as above the agent with the smallest coefficient η or $1 - \eta$ will surely be better off undercutting the other agent's price, unless they are both charging the competitive price, $p^i = 1$, $i = 1, 2$, in which case both agents will want to deviate by charging the price 1.46 since they both individually face the same demand agent 1 faced in the previous example. But this shows that there is no pure strategy profile (p, s) that is a Nash equilibrium of the first stage of the game, and so the game has no pure strategy subgame perfect Nash equilibria. □

This result is quite striking in that it implies that there are no single prices that the two sellers will always agree to charge. However, there will be a range of prices, (in this case, at most the interval [1, 3]), that the sellers will charge according to some distribution function. Without going into the details of the distribution function, we can state:

Remark 5 *There exists a subgame perfect Nash equilibrium in mixed strategies.*

Proof. As in the proof of the previous remark, we first compute the subgame perfect demands, then consider the first stage of the game. By making the same reductions of the strategy spaces as above, we can view the first stage part of the game as a two player game, where the payoffs are given by the final allocations obtained with the subgame perfect demands.

Existence of a subgame perfect Nash equilibrium then follows by showing that a Nash equilibrium for the first stage game exists. A Nash equilibrium for the first stage game can be shown to exist from Dasgupta and Maskin (1986), Theorem 5b, p. 16. The conditions of the theorem need to be verified for this game. Their verification is standard. □

This result guarantees the existence of a mixed strategy subgame perfect Nash equilibrium for the replicated economy.[16] The fact that the strategies are mixed strategies in the strict sense, can be interpreted as saying that, in particular, there will be some form of price dispersion. In fact, the sellers will end up charging prices between the Walrasian price and a monopolistic price, possibly higher than the one charged in the previous example, according to distributions that are absolutely continuous with respect to Lebesgue measure, since there will be no point masses in the equilibrium distributions. Furthermore, it should be possible to show that replicating the economy another time leads to similar mixed strategy equilibria, where the prices charged will be again in some interval between the Walrasian and a monopolistic price. It should also be the case that further replicating the economy any finite number of times, while not shrinking the range of prices charged in equilibrium down to the degenerate interval containing just the Walrasian price, should change the probability with which prices are being charged. In particular, one would expect the probability of lower prices being charged to increase with the number of replications until, in the limit, the competitive price is charged with probability one.[17]

Again, the results obtained with this example are very much in line with the industrial organization literature, which, in comparable settings, often obtains nonexistence of pure strategy Nash equilibria but existence of mixed strategy Nash equilibria. At the same time, they once again differ quite fundamentally from the strategic market game literature, which typically obtains pure strategy Nash equilibria.

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[16]Recall that mixed strategies really refers to the strategies in the first stage, i.e., prices and quantities for sale.

[17]See Allen and Hellwig (1986b, 1989) for a very detailed account of the case of a single market with n firms. See also Varian (1980) and Börgers (1992).

3.3 A Third Example

This third example shows to what extent the order in which agents enter the markets as buyers can matter. It also shows that it need not always be optimal for a buyer to purchase a given commodity at the lowest possible price even if it is feasible to do so. Consider the case $l = 2, m = 3$, with endowments $(\omega_0^1, \omega^1) = (2, 1, 4), (\omega_0^2, \omega^2) = (2, 4, 1), (\omega_0^3, \omega^3) = (2, 8, 4)$, and the utility functions $u^i : \mathbb{R}_+^3 \to \mathbb{R}, u^i(x_0^i, x^i) = x_0^i x_1^i x_2^i + 2x_i^i$, for $i = 1, 2$ and $u^i(x_0^i, x^i) = x_0^i x_1^i x_2^i$, for $i = 3$. Suppose prices and quantities for sale have already been determined in the first stage and are:

$$(p, s) = ((p^i, s^i)_{i=1,2,3}) = (((4, 1), (0, 2)), ((1, 4), (2, 0)), ((5/6, 4), (2, 0))).$$

Suppose furthermore that agents enter in the order $\tau(i) = i$, for $i = 1, 2, 3$. Then we have:

Remark 6 *Although agent 3 charges a strictly lower price for commodity 1 ($p_1^3 = 5/6$) than agent 2 ($p_1^2 = 1$), it is subgame perfect for agent 1 to buy a positive amount of the commodity from agent 2 at the higher price even when she could buy the same amount at a lower price.*

Proof. It suffices to show that agent 1 can increase her utility by purchasing a positive amount of commodity 1 from agent 2. It follows directly from the quantities that are for sale that agent 1 could purchase the same amount from agent 3. First, we note that agent 3 doesn't make any purchases. Then we compute agent 2's demand for agent 1's commodities. It is obtained from the maximization problem:

$$max_{b_2^{21}} u^2(\omega_0^2 + p_1^2 b_1^{12} - p_2^1 b_2^{21}, \omega_1^2 - b_1^{12}, \omega_2^2 + b_2^{21})$$

$$= (2 + b_1^{12} - b_2^{21})(4 - b_1^{12})(1 + b_2^{21}) + 2(1 + b_2^{21})$$

subject to

$$0 \leq b_2^{21} \leq 2,$$

which has the solution:

$$b_2^{21} = \frac{6 + 3b_1^{12} - (b_1^{12})^2}{8 - 2b_1^{12}}.$$

Next we consider agent 1's maximization problem:

$$max_{b_1^{12}, b_1^{13}} u^1(\omega_0^1 + p_2^1 b_2^{21} - p_1^2 b_1^{12} - p_1^3 b_1^{13}, \omega_1^1 + b_1^{12} + b_1^{13}, \omega_2^1 - b_2^{21})$$

$$= (2 + b_2^{21} - b_1^{12} - 5/6 b_1^{13})(1 + b_1^{12} + b_1^{12})(4 - b_2^{21}) + 2(1 + b_1^{12} + b_1^{13})$$

subject to

$$0 \leq b_1^{12} \leq 2, 0 \leq b_1^{13} \leq 2, b_1^{12} + 5/6 b_1^{13} \leq 2,$$

where

$$b_2^{21} = \frac{6 + 3b_1^{12} - (b_1^{12})^2}{8 - 2b_1^{12}}.$$

Without actually computing the solution, we show that it cannot be optimal for agent 1 not to make a positive purchase from agent 2 at the higher price even when she could make the purchase from agent 3 at the lower price.[18] We simply compute 1's maximum utility when $b_1^{12} = 0$, and then when $b_1^{12} = 2 > 0$. Agent 1's maximization problem with the extra condition $b_1^{12} = 0$ is maximized by $b_1^{13} \approx 1.52$, which leads to a utility level of $u^1 \approx 17.2$. But if $b_1^{12} = 2 > 0$ and $b_1^{13} = 0$, then $u^1 = 18$. This shows that buying some positive quantity of commodity 1 from agent 2 at the higher price of 1, although the desired quantity of the same commodity is available at the lower price of 5/6, makes agent 1 better off. □

Notice how the remark no longer holds when agents enter in the order $\tau(i) = 4 - i$, for $i = 1, 2, 3$. In this case, agent 3 who goes first still does not make any purchase. But agent 1 who goes last will no longer buy the higher priced commodity since agent 2 has already made his decision when 1 goes in. Although we have analyzed equilibrium behavior only for the second main stage of the game and have taken the strategies from the first stage as given, we can always restrict the strategy spaces of the first stage to contain just these strategies, i.e., $P \times S = \{(p, s)\}$, where (p, s) is the vector of strategies specified above. Therefore we can state:

Remark 7 *Changing the order in which agents enter as buyers may yield different subgame perfect Nash equilibrium final allocations.*

[18] Note that a solution clearly exists since the function to be maximized is continuous on the compact set defining the possible values of b_1^{12} and b_1^{13}.

This is not very surprising although one could think of allocation mechanisms where the order of entry of buyers does not matter. It would be interesting to see whether it can only be to one's advantage to enter the markets early.

4 Some General Propositions

In this section we state and prove some general results concerning the game described in Section 2. We consider the game ? $= (A, \Pi)$ consisting of the space of strategy profiles $A = P \times S \times \mathcal{B}$ and the payoff function $\Pi : A \to \mathbb{R}^m$, all defined in Section 2.

4.1 Existence

To simplify things, we consider the version of the game ?, where the strategies chosen by the agents in the first stage are chosen from finite sets. More specifically, we consider strategy spaces where P and S are both finite subsets of $\mathbb{R}_+^{lm}$.[19] It is important to point out, however, that the following existence result is sufficient for our purposes, since it provides games with equilibria, to which the propositions of the next subsection apply.

Proposition 1 *Let $?_F$ denote the game ? where the sets P and S are finite. Then $?_F$ possesses a subgame perfect Nash equilibrium.*

Proof. The proof is in two steps. First we show that for any strategy profile chosen in the first stage, $(p, s) \in P \times S$, there exist demand functions $b \in \mathcal{B}$ that are subgame perfect equilibria of the subgame starting in the second stage. Second we show that by replacing subgames starting in the second stage with any subgame perfect equilibrium payoffs yields a one-stage game which possesses a Nash equilibrium. This will yield the desired subgame perfect equilibrium of the entire game.

[19]Without this assumption the proof of the existence theorem becomes very difficult. In fact, it is not clear that subgame perfect equilibria always exist in this case. A way out of the problem is by broadening the equilibrium concept used to that of correlated equilibrium. See for example Harris, Reny, and Robson (1995).

Step 1: This is a direct application of Harris (1985), Theorem 1, p. 618. We only need to verify that the following conditions hold: (i) the functions $b^i \in \mathcal{B}_i$ map into compact Hausdorff spaces, (ii) the conditions defining elements of $\mathcal{B}_i$ are lower-hemi continuous on $P \times S \times (\times_{h<i} B)$, (iii) the set of histories, i.e., possible realizations of the functions b^i, $i \in I$, is closed, and, (iv) the payoffs are continuous on the set of histories. Clearly (i) is satisfied because demands are always bounded below by 0 and above by agents' total endowments, i.e., we can take all functions $(b^i)_{i\in I}$ to map into the set $B = \times_{j\in I}[0, \omega^j] \subset \mathbb{R}_+^{lm}$, which is compact and Hausdorff. To see that (ii) is satisfied, consider some closed subset $G \subset B$. We need to show that for any $i \in I$, the set:

$$F_i = \{(b^h)_{h<i} \in \times_{h<i} B : b^i \text{ satisfies } (1), (2) \text{ implies } b^i \in G\}$$

is closed. Pick a sequence $((b^h)_{h<i})^n$ in F_i converging to some point $(\bar{b}^h)_{h<i}$. To see that the limit is in F_i, consider some convergent sequence $(b^i)^n$ of demands satisfying conditions (1) and (2) for each n, where these conditions now vary with n. These demands all lie also in G. Because the conditions are simple weak inequalities, the limit $\bar{b}^i$ of the sequence $(b^i)^n$ must also satisfy the conditions (1) and (2) under the limit $(\bar{b}^h)_{h<i}$. But then $\bar{b}^i$ must also be in G because G is closed. This proves (ii). To prove (iii), we need some extra notation. Let $H \subset B^m$ denote the set of all possible realizations of demands. Again, this set is closed because the conditions on demands are weak inequalities, so that if h^n is a sequence of histories in H, then they all satisfy the relevant conditions (1) and (2), and hence so does the limit. This proves (iii). Finally, the continuity of the payoffs on H follows from the continuity of the utility functions, and the fact that the final allocations are linear combinations of the histories. We have shown that conditions (i)-(iv) are satisfied. This allows us to apply Harris's theorem which guarantees the existence of a subgame perfect equilibrium of the subgame starting at stage two for any choice of strategies $(p, s) \in P \times S$.

Step 2: Fix one subgame perfect equilibrium for each strategy profile chosen in the first stage. Let agents' payoffs be the payoffs associated to this subgame perfect equilibrium. Then we only need to show that there is a Nash equilibrium of the resulting game. But this follows immediately from Nash's (1950) theorem on the existence of Nash equilibria for finite games, because the strategies in the first stage are taken from $P \times S$, which we have assumed to be finite.

The Nash equilibrium of Step 2 combined with the subgame perfect equilibria of Step 1 yield a subgame perfect equilibrium of the game $?_F$. This completes the proof. $\square$

4.2 Characterization

In this section we consider a fixed exchange economy $\mathcal{E}$ with fixed endowments and preferences. For this exchange economy we can obtain the set of perfectly competitive equilibria by letting money be a regular commodity. Fix the price of money to be one and denote by $W(\mathcal{E})$ the set of perfectly competitive equilibria of the exchange economy $\mathcal{E}$ with generic element $(p^W, x^W) \in \mathbb{R}^l_+ \times \mathbb{R}^{(l+1)m}_+$, consisting of a price vector and a vector of final allocations.[20] We want to see whether forcing agents to choose perfectly competitive prices in the first stage will necessarily lead to final allocations that are consistent with the perfectly competitive allocations, i.e., whether forcing agents to choose p^W in the first stage will also lead to final allocations x^W. Notice that in order to ensure that the perfectly competitive allocations are feasible, we have to modify the budget constraints[21] to allow them to buy with money they may receive from sales of parts of their endowments.

Proposition 2 *Let $(p^W, x^W) \in W(\mathcal{E})$ be a perfectly competitive equilibrium of the exchange economy $\mathcal{E}$, and let $?_{p^W}$ denote the game $?$ where $P = \{p^W\}^m$ and where agents are allowed to borrow up to $p^i s^i \in \mathbb{R}_+$, for $i \in I$. Then there always exists a subgame perfect Nash equilibrium of $?_{p^W}$ that yields the final allocation x^W. If, furthermore, $?_{(p^W, s^W)}$ denotes the game $?_{p^W}$ further restricted by $S = \{s^W\}$, where*

$$(s^i_k)^W = \ max \ \{0, \omega^i_k - (x^i_k)^W\}, \forall k \in L, i \in I,$$

then any subgame perfect Nash equilibrium of $?_{(p^W, s^W)}$ always yields the final allocation x^W.

Proof. The proof is in two steps. First we show that the game $?_{p^W}$ has a subgame perfect Nash equilibrium with final allocations x^W. Then we

[20] Notice that in p^W we are omitting the entry of the price of money which we have assumed to be fixed at one. This is why p^W is an l-vector.

[21] See equation (1) in Section 2.

show that in all subgame perfect Nash equilibria of the game $?_{(p^W,s^W)}$ final allocations are x^W.

Step 1: Notice first that we are not restricting agents to choose from a finite set S, so the proposition of the previous subsection cannot be applied to obtain existence. Consider the choice of quantities in the first stage s^W defined by:

$$(s^i_k)^W = max\{0, \omega^i_k - (x^i_k)^W\}, i \in I, k \in L.$$

These are feasible choices since they lie in $[0, \omega^i_k]$, $k \in L, i \in I$. We want to show that if agents choose strategies $(p^i, s^i) = (p^W, ((s^i_k)^W)_{k\in L}) \in P_i \times S_i$, for $i \in I$, in the first stage, then there exists a continuation, say $b^W \in \mathcal{B}$, that together with $(p^W, s^W) \in P \times S$ constitutes a subgame perfect equilibrium of $?_{p^W}$ and also yields final allocations x^W. We construct the vector of demand functions b^W starting from the last agent to enter the markets as a buyer, agent m. His maximization problem at this stage is:[22]

$$max_{(b^{mj})_{j\in I}} u^m(x^m_0((b^{mj})_{j\in I}), x^m((b^{mj})_{j\in I})$$

subject to

$$\sum_{j\in I} p^j b^{mj} \leq \omega^m_0 + p^m s^m$$

$$0 \leq b^{mj} \leq s^j - \sum_{h<m} b^{hj}, \forall j \in I,$$

where

$$x^m_0((b^{mj})_{j\in I}) = \omega^m_0 + \sum_{j\neq m} p^m b^{jm} - \sum_{j\neq m} p^j b^{mj},$$

$$x^m((b^{mj})_{j\in I}) = \omega^m - \sum_{j\neq m} b^{jm} + \sum_{j\neq m} b^{mj}.$$

By compactness and continuity, this maximization problem has a solution for each element $(p, s; (b^h)_{h<m}) \in P \times S \times (\times_{h<i} B)$. By definition of perfectly competitive equilibrium, if agents entering before agent m, i.e., agents $h < m$, make purchases that allow agent m to make purchases where he can obtain the final allocation $(x^m)^W$, then it will be subgame perfect for him to make such a purchase. Fix such a choice in each of these cases. If, on the other hand, it is not possible for agent m to make such a purchase, then there will

[22]Notice that, since we are allowing agents to borrow up to $p^i s^i = p^W (s^i)^W \in \mathbb{R}_+, \forall i \in I$, agents' budget constraints are just like in Walrasian economies.

be quantities that solve his maximization problem, but these will yield lower utility levels than the ones leading to the final allocation $(x^m)^W$. This follows from the definition of perfectly competitive equilibrium together with the strict quasi-concavity of the utility function and the constraints on possible demands. Fix some utility maximizing choice also for these cases. This gives a function $b^m : P \times S \times (\times_{h<m} B) \rightarrow \mathbb{R}_+^{lm}$ that is a subgame perfect strategy for agent m. Now consider the agent entering before m, i.e., agent $m-1$. Her maximization problem is similar to agent m's with the difference that $m-1$ has not observed m's demand yet, but knows what m will do given what has happened before m enters as a buyer. Her maximization problem is:

$$max_{(b^{m-1,j})_{j\in I}} u^{m-1}(x_0^{m-1}((b^{m-1,j})_{j\in I}), x^{m-1}((b^{m-1,j})_{j\in I}))$$

subject to

$$\sum_{j\in I} p^j b^{m-1,j} \leq \omega_0^{m-1} + p^{m-1}s^{m-1}$$

$$0 \leq b^{m-1,j} \leq s^j - \sum_{h<m-1} b^{hj}, \forall j \in I,$$

where

$$\begin{aligned} x_0^{m-1}(b^{m-1}) = \omega_0^{m-1} + \sum_{j<m-1} p^{m-1}b^{j,m-1} &+ p^{m-1}b^{m,m-1}(p,s;(b^h)_{h<m}) \\ &- \sum_{j\neq m-1} p^j b^{m-1,j}, \end{aligned}$$

$$\begin{aligned} x^{m-1}(b^{m-1}) = \omega^{m-1} - \sum_{j<m-1} b^{j,m-1} &- b^{m,m-1}(p,s;(b^h)_{h<m}) \\ &+ \sum_{j\neq m-1} b^{m-1,j}. \end{aligned}$$

Again, she will make purchases that lead to the final allocation $(x^{m-1})^W$ when it is possible, and make some other purchase that will yield a lower utility level if it is not possible. An optimal choice always exists by Harris' theorem.[23] Fixing some utility maximizing choices in each of the cases gives a demand function $b^{m-1} : P \times S \times (\times_{h<m-1} B) \rightarrow \mathbb{R}_+^{lm}$, that is a subgame perfect strategy for agent $m-1$. Using the same line of reasoning for agents

[23] Harris (1985), Theorem 1, p. 618.

$m-2, m-3$, and so on until agent 1, gives us demand functions for all agents such that whenever it is possible for an agent to make a choice leading to the perfectly competitive allocations, then they will do so. These are all feasible by definition of perfectly competitive equilibrium and because we are assuming agents to choose the quantities $((s^i)^W)_{i \in I}$ in the first stage. But, although putting more for sale may, in certain cases, yield the same final allocation as putting the quantities $((s^i)^W)_{i \in I}$ for sale, the latter choice is always optimal. This is because at the competitive prices even if there is more for sale there will be no demand for more, so that putting more for sale cannot increase any agent's utility. Hence, we have constructed a subgame perfect Nash equilibrium that yields the final allocations x^W. This leads to the next step.

Step 2: Here we need to show that any subgame perfect Nash equilibrium leads to the allocations x^W when agents are choosing competitive prices and quantities (p^W, s^W). But this is straightforward, given the fact that there is a subgame perfect equilibrium that yields final allocations x^W. By definition these constitute the most preferred outcome for all agents, given the feasibility restrictions. Therefore, starting from the first agent to enter the markets, agent 1, then going to agents $2, 3, \ldots, m$, one can show that because utility functions are strictly quasi-concave, any other strategy profile that does not lead to allocation x^W as a final allocation will lead to a lower utility level for at least some agents, and for no agent to a higher utility level. Therefore, if they can, agents entering as buyers will always choose a strategy that will make the allocation x^W arise. But because agents in the first stage are choosing strategies from $P \times S = \{p^W\} \times \{s^W\}$, this says that any subgame perfect Nash equilibrium leads to the final allocations x^W. This completes the proof. □

This proposition provides an alternative way of viewing the perfectly competitive equilibrium, namely, as the outcome of an economy, in which agents are forced not only to choose competitive prices but also to put at least competitive quantities for sale. However, it also shows that if competitive prices are charged, then there will always be a subgame perfect equilibrium that yields a perfectly competitive final allocation. What the proposition does not show is that agents are not going to choose perfectly competitive prices in the first stage, unless they are forced to do so. This question is addressed in the next proposition. What follows now is a brief example that shows that forcing agents to choose competitive prices is not enough to guarantee that the final outcome will be the perfectly competitive one. It is a typical example of a coordination failure where, if one of the two agents does not put anything for sale, then neither will the other agent.[24]

Example Consider the case $l = 2, m = 2$, with endowments $(\omega_0^1, \omega^1) = (2, 5-\epsilon, 0+\epsilon)$ and $(\omega_0^2, \omega^2) = (2, 0+\epsilon, 5-\epsilon)$, for any $\epsilon \in (0,1)$, and the utility functions $u^i : \mathbb{R}^3_+ \to \mathbb{R}, u^i(x_0^i, x^i) = x_0^i x_1^i x_2^i + 2x_i^i$, for $i = 1, 2$. Suppose agents enter in the order $\tau(i) = i$, for $i = 1, 2$.

The unique Walrasian equilibrium of this exchange economy is given by:

$$(p^W; x^W) = ((1,1); ((2,3,2),(2,2,3)).$$

However, it is not the case that restricting prices to be perfectly competitive, i.e., $p^i = p^W$ for $i = 1, 2$, necessarily leads to subgame perfect final allocations x^W. In fact:

Remark 8 *Let $P = \{p^W\}^2 = \{(1,1)\}^2$ contain only the perfectly competitive prices, then there are subgame perfect Nash equilibria that do not lead to the corresponding perfectly competitive allocation $x^W = ((2,3,2),(2,2,3))$.*

Proof. To see this we show that the initial endowments can be supported as final allocations of a subgame perfect Nash equilibrium. In fact, the strategies

$$(p^i, s^i; b^i) = ((1,1),(0,0); b^i) \text{ for } i = 1, 2,$$

where $b^i \in \mathcal{B}_i$ is i's subgame perfect demand function,[25] form a subgame perfect Nash equilibrium. To see that the quantities put for sale $s^i, i = 1, 2$, are optimal, notice that both agents' marginal rates of substitution between

[24]In fact, the example can be used to show that, despite the fact that agents are forced to charge perfectly competitive prices for commodities they put for sale, there are arbitrarily large finite economies (with agents of two types) in which there are subgame perfect equilibria that are not perfectly competitive. To see this, simply take replications of the economy of the following example and use Remark 8. The strategy profile where agents put zero quantities for sale should always be a subgame perfect Nash equilibrium for the replicated economy. Maybe it is possible to use the example to show that such equilibria persist even with infinitely many agents.

[25]They can be calculated by maximizing the agents' utility functions subject to the budget and rationing constraints. See equations (1) and (2) in Section 2 above.

the regular commodity they are endowed with, i.e., commodity i for agent i, and money at initial endowments are:

$$MRS^i_{k,0} = \frac{2\epsilon + 2}{(5-\epsilon)\epsilon} > 1, \forall \epsilon \in (0,1), i, k = 1, 2, k = i.$$

Similarly, the marginal rates of substitution between the other regular commodity, i.e., commodity 2 for agent 1 and commodity 1 for agent 2, and money are:

$$MRS^i_{k,0} = \frac{2(5-\epsilon)}{(5-\epsilon)\epsilon} = \frac{2}{\epsilon} > 1, \forall \epsilon \in (0,1), i, k = 1, 2, k \neq i.$$

Therefore, at competitive prices $p^i = (1,1), i = 1, 2$, it is a subgame perfect Nash equilibrium when agents put quantities $s^i = (0,0), i = 1, 2$, for sale, so that, in other words, if one agent does not put anything for sale, then it is optimal for the other agent to also not put anything for sale. □

Before stating our next proposition, we need to introduce some notation and an assumption regarding subgame perfect demands. Let $B = \times_{i \in I}[0, \omega^i] \in \mathbb{R}^{lm}_+$ denote the space from which agents choose demands and, for $i \in I$, let $B_i : P \times S \times (\times_{h<i} B) \to B$ denote the correspondence that maps a point $(p, s; (b^h)_{h<i}) \in P \times S \times (\times_{h<i} B)$ to the set of demands $(b^{ij})_{j \in I} \in B$ that satisfy the conditions:[26]

$$\sum_{j \in I} p^j b^{ij} \leq \omega^i_0 \text{ and } 0 \leq b^{ij} \leq s^j - \sum_{h<i} b^{hj}, j \in I,$$

which are precisely the budget and rationing constraints defined in Section 2 by equations (1) and (2). We now formally define the notion of a subgame perfect demand correspondence. As we saw in the proof of Proposition 1, it follows from a theorem by Harris (1985)[27] that for any strategy profile $(p, s) \in P \times S$ for the first stage, there always exist demand functions $b \in \mathcal{B}$ that are subgame perfect equilibria of the associated game starting in the second stage. Therefore there is a well-defined correspondence $\beta : P \times S \to \mathcal{B}$

[26] In these conditions we write agents' budget constraints as in the original form of the model, where agents are not allowed to borrow. Allowing agents to borrow, say, up to the amounts $(p^i s^i)_{i \in I}$ does not change the result of the next lemma.

[27] Theorem 1, p. 618.

mapping from the space of strategy profiles in the first stage into the space of demand functions in the second stage that assigns to each strategy profile in the first stage the set of all possible subgame perfect demand functions with generic element $(b_1(p,s), b_2(p,s;f_1), \ldots, b_m(p,s;(f_h)_{h<m})) \in \mathcal{B}$, where $f_i \in \mathcal{B}_i$ and $(p,s) \in P \times S$. If we set:

$$\hat{b}_1(p,s) = b_1(p,s)$$

and

$$\hat{b}_i(p,s) = b_i(p,s;(\hat{b}_h(p,s))_{h<i}), i = 2, \ldots, m,$$

where $(b_1(p,s), b_2(p,s;f_1), \ldots, b_m(p,s;(f_h)_{h<m})) \in \beta(p,s)$, then this defines a correspondence $\hat{b} : P \times S \to B^m$ mapping from the space of strategy profiles in the first stage to actual quantities demanded in the second stage. We call $\hat{b}$ the **subgame perfect demand correspondence** of the given exchange economy $\mathcal{E}$. The following lemma states a basic property of this demand correspondence.

Lemma 1 *The subgame perfect demand correspondence $\hat{b} : P \times S \to B^m$ of any given exchange economy $\mathcal{E}$ is upper-hemi continuous and compact-valued.*

Proof. This is a direct application of Börgers (1991), Corollary 1, p. 99. In order to apply the result we need to show that (i) payoffs are continuous on the space $P \times S \times B^m$, and (ii) the correspondences $B_i : P \times S \times (\times_{h<i} B) \to B$ defining elements of $\mathcal{B}_i$ are continuous on $P \times S \times (\times_{h<i} B)$, $i \in I$, using the topology induced by the Hausdorff metric on the space of nonempty closed subsets of B.[28] Condition (i) was already verified in the proof of Proposition 1 and condition (ii) is clear by inspection of the conditions defining the correspondences B_i. Notice that the space of nonempty closed subsets of B with the Hausdorff metric is a compact metric space. □

The next proposition requires the subgame perfect demand correspondence to satisfy a somewhat stronger condition, namely:

Premise 1 *The subgame perfect demand correspondence $\hat{b} : P \times S \to B^m$ is such that, at any perfectly competitive point $((p^W, (s^i)^W)_{i \in I}) \in P \times S$, the*

[28] This topology allows us to view the correspondences B_i as functions mapping points in $P \times S \times (\times_{h<i} B)$ to nonempty closed subsets of B, so that the notion of continuity makes sense here. See Hildenbrand (1974), p. 16, for a definition of the Hausdorff metric.

correspondences defined by:

$$\varphi^i : P \times S \to \mathbb{R}^l_+, (p, s) \mapsto \sum\nolimits_{j \neq i} \hat{b}^{ji}(p, s)$$

and

$$\psi^i : P \times S \to \mathbb{R}^l_+, (p, s) \mapsto \sum\nolimits_{j \neq i} (\hat{b}^{ji}(p, s) - \hat{b}^{ij}(p, s)),$$

for $i \in I$, are Lipschitz continuous.[29]

Originally this assumption was intended to be stated as a lemma. However, we can only show that the correspondences $(\varphi^i)_{i \in I}$ and $(\psi^i)_{i \in I}$ are continuous at competitive points. To see this, we use Proposition 2 to show that, given perfectly competitive strategies in the first stage, there is a unique possible final allocation $x^W \in \mathbb{R}^{(l+1)m}_+$ that is subgame perfect. This implies that the quantities $\varphi^i(p, s)$ and $\psi^i(p, s) \in \mathbb{R}^l_+$ are uniquely determined, for $i \in I$, at the perfectly competitive point $(p, s) = ((p^W, (s^i)^W)_{i \in I})$,[30] which means that the correspondences $(\varphi^i)_{i \in I}$ and $(\psi^i)_{i \in I}$ are single-valued. Using Lemma 1, we have that they are sums of upper-hemi continuous and compact-valued correspondences, so that applying Hildenbrand (1974), Proposition 5, p. 25, shows that they are also upper-hemi continuous and compact-valued. But a correspondence that is upper-hemi continuous, compact-valued, and single-valued at a point is also continuous at that point.

Now we turn to the question of whether, without imposing any restrictions on the strategy spaces, as we did in Proposition 2, we should expect the

[29] Following Clarke (1983), p. 113, we say a correspondence $F : X \to Y$ is **Lipschitz continuous** at $x \in X$ if there exists an open set $U \subset X$ with $x \in U$ such that, for all $x' \in U$, and for all $y' \in F(x')$, there exists $y \in F(x)$ such that:

$$d_Y(y, y') \leq K d_X(x, x'),$$

and for all $x' \in U$, and for all $y \in F(x)$, there exists $y' \in F(x')$ such that:

$$d_Y(y, y') \leq K d_X(x, x'),$$

where $K < \infty$ is some constant and d_X and d_Y are metrics on the spaces X and Y respectively. In all our cases the spaces X and Y are always subsets of Euclidean space so that the metric employed is always the standard Euclidean one.

[30] Notice that we are not saying that the demands $(\hat{b}^{ij})_{j \in I}$ are uniquely determined but that the total quantities sold φ^i and total net quantities traded ψ^i are uniquely determined, for $i \in I$.

subgame perfect Nash equilibria of the game ? to be Walrasian. By Walrasian we mean that agents are actually choosing Walrasian strategies at all stages, we do not just mean that the final allocations are Walrasian. In particular, we ask whether we should expect agents to choose Walrasian prices in the first stage of the game as part of a subgame perfect Nash equilibrium.

Proposition 3 *Let $(p^W, x^W) \in W(\mathcal{E})$ be a perfectly competitive equilibrium of the exchange economy $\mathcal{E}$, let $s^W \in \mathbb{R}_+^{lm}$ be the corresponding perfectly competitive quantities for sale, and suppose the subgame perfect demand correspondence satisfies Premise 1. If agents are allowed to borrow up to $p^i s^i \in \mathbb{R}_+, i \in I$, and if $P \times S$ contains the set $[p^W, p^W + \nu]^m \times \{s^W\}$ for some $\nu \in \mathbb{R}_{++}^l$, then (p^W, s^W) is the first stage part of a subgame perfect Nash equilibrium of the game ? if and only if initial endowments are x^W.*

Proof. First we show the if part. Suppose initial endowments are the perfectly competitive final allocation x^W. Take $(p^i, s^i) = ((p^i)^W, 0), \forall i \in I$, then by Proposition 2 there will be subgame perfect equilibrium demand functions $(b^i)_{i \in I} \in \mathcal{B}$ such that the strategies $(p^i, s^i; b^i)_{i \in I} \in P \times S \times \mathcal{B}$ constitute a subgame perfect Nash equilibrium of the entire game ?. This shows the if part of the proposition. For the only if part, we need to show that, if $\forall i \in I, (p^i, s^i) = (p^W, (s^i)^W)$ are subgame perfect equilibrium strategies, then initial endowments must be x^W. We show this by contradiction. Suppose agents are choosing perfectly competitive strategies in the first stage, and initial endowments are not Pareto efficient. Then from Proposition 2, we know that there exists $k \in L$ and $i \in I$ such that $s_k^i > 0$, i.e., some trade must take place. We show that agent i can increase his utility by charging a price for commodity k that is higher than the perfectly competitive price p_k^i. We make use of the following lemma:

Lemma 2 *At any perfectly competitive equilibrium $((p^W, (x_0^i, x^i)^W)_{i \in I}) \in W(\mathcal{E})$, the functions defined by:*

$$\zeta^i : \mathbb{R}^l \to \mathbb{R}, \epsilon \mapsto u^i((x_0^i, x^i)^W) - u^i((x_0^i)^W + p^W \epsilon, (x^i)^W - \epsilon),$$

for $i \in I$, vanish and are continuously differentiable at $\epsilon = 0_l$, and are positive everywhere else. Furthermore, all first-order partial derivatives vanish at $\epsilon = 0_l$.

The proof of this lemma follows from the utility maximizing property of the competitive allocations at competitive prices, the twice differentiability and strict quasi-concavity of the utility functions, and because composing differentiable functions yields a differentiable function.

We use the last lemma to construct a $\delta > 0$ such that, if agent i raises the price of commodity k by δ, then he will increase his utility. First, we introduce some notation. Let (p, s) denote the point (p^W, s^W) and let $(p, s)_\delta$ denote the same point except for the entry p^i_k which is now $p^i_k + \delta$. By Premise 1, φ^i and ψ^i are continuous at the perfectly competitive strategies, so for any $\epsilon > 0$ there exists a $\delta > 0$ such that φ^i and ψ^i take values within an ϵ-neighborhood of the values $\varphi^i(p, s)$ and $\psi^i(p, s)$ for any strategy profile in a δ-neighborhood of a perfectly competitive point. Since we are only concerned with points within this neighborhood, we treat φ^i and ψ^i as if they were functions, i.e., $\varphi^i((p, s)_\delta)$ and $\psi^i((p, s)_\delta)$ denote arbitrary elements of $\varphi^i((p, s)_\delta)$ and $\psi^i((p, s)_\delta)$, and it does not matter which one they denote since they are all sufficiently close. More specifically, if we take $\epsilon \in \mathbb{R}^l_{++}$ to be, say, $\epsilon_k = 1/4\varphi^i_k(p, s) > 0$, then there exists $\bar{\delta} > 0$ such that for all $k \in L$:

$$|\varphi^i_k(p, s) - \varphi^i_k((p, s)_{\bar{\delta}})| < \epsilon_k$$

or equivalently,

$$3/4\varphi^i_k(p, s) < \varphi^i_k((p, s)_{\bar{\delta}}) < 5/4\varphi^i_k(p, s).$$

Similarly, let $(x^i_0, x^i) = (x^i_0, x^i)^W \in \mathbb{R}^{l+1}_+$ denote agent i's perfectly competitive allocation, and let $(x^i_0, x^i)_\delta$ denote agent i's allocation when he charges $p^i_k + \delta$ for commodity k, and all other strategies are as in the perfectly competitive case. Define the correspondence: $\varepsilon : \mathbb{R}_+ \to \mathbb{R}^l_+, \varepsilon(\delta) = \psi^i((p, s)_\delta) - \psi^i(p, s)$, giving changes in net quantities traded. Then the change in agent i's utility when increasing p^i_k by any $\delta \in (0, \bar{\delta})$ is given by:

$$\begin{aligned}
& u^i((x^i_0, x^i)_\delta) - u^i(x^i_0, x^i) \\
&= u^i(x^i_0 + p^W\varepsilon(\delta) + \delta\varphi^i_k((p, s)_\delta), x^i - \varepsilon(\delta)) - u^i(x^i_0, x^i) \\
&= u^i(x^i_0 + p^W\varepsilon(\delta) + \delta\varphi^i_k((p, s)_\delta), x^i - \varepsilon(\delta))
\end{aligned}$$

$$
\begin{aligned}
& -u^i(x_0^i + p^W\varepsilon(\delta), x^i - \varepsilon(\delta)) \\
& +u^i(x_0^i + p^W\varepsilon(\delta), x^i - \varepsilon(\delta)) \\
& -u^i(x_0^i, x^i) \\
= \quad & \partial_0 u^i(x_0^i + p^W\varepsilon(\delta), x^i - \varepsilon(\delta))\delta\varphi_k^i((p,s)_\delta) + o(\delta\varphi_k^i(p_k^i + \delta)) \\
& -\partial_0 u^i(x_0^i, x^i)p^W\varepsilon(\delta) + \sum_{k\in L}\partial_k u^i(x_0^i, x^i)\varepsilon_k(\delta) + o(\varepsilon(\delta)) \\
= \quad & \partial_0 u^i(x_0^i + p^W\varepsilon(\delta), x^i - \varepsilon(\delta))\delta\varphi_k^i((p,s)_\delta) \\
& +o(\delta\varphi_k^i((p,s)_\delta)) + o(\varepsilon(\delta)) \\
= \quad & \varrho + o(\delta) > 0,
\end{aligned}
$$

where $\varrho > 0$ is some positive constant. The first and second equalities are obvious, the third follows from a first-order Taylor expansion, and the fourth equality follows directly from Lemma 2. To see the last equality, notice that $\partial_0 u^i(x_0^i + p^W\varepsilon(\delta), x^i - \varepsilon(\delta)) > 0$ follows from the strict monotonicity of the utility function, and that $\varphi_k^i((p,s)_\delta)$ is bounded below and above by positive constants given our choice of δ. Finally, to see that $o(\delta) = o(\varepsilon(\delta))$, notice that ε is Lipschitz since ψ^i is Lipschitz by Premise 1, and therefore:

$$
\frac{|o(\varepsilon(\delta))|}{|\delta|} \leq \frac{|o(K\delta)|}{|\delta|} \overset{\delta\to 0}{\to} 0,
$$

for some Lipschitz constant $K > 0$.

This shows that if agent i charges the higher price $p_k^i + \delta$, then he is better off. Since any smaller $\delta > 0$ will do the same job, and since the strategies available do allow agent i to charge prices $[p^W, p^W + \nu]$ for some $\nu \in \mathbb{R}^l_{++}$, we have showed that the perfectly competitive strategy $(p^i, s^i) = (p^W, (s^i)^W)$, $i \in I$, is not part of a subgame perfect Nash equilibrium strategy profile of the game ?. This contradicts the assumption made above and hence completes the proof. $\square$

This proposition says that, under Premise 1, Walrasian strategy profiles that are subgame perfect Nash equilibria are very rare. In fact, they arise only when endowments are already Walrasian allocations, so that no trade takes place at the subgame perfect equilibrium. The intuition for requiring Premise 1 is best seen in the simple case of a monopolist selling a single commodity and facing a demand curve that is not Lipschitz continuous in the price he charges at the point of intersection of the demand curve he faces with his marginal cost curve. In such a case it is profit-maximizing for the monopolist to charge the perfectly competitive price. This example also extends to an exchange economy.

Notice also that, while the last proposition is formulated for games in which the strategy space is continuous, the existence result of the previous subsection assumes the space $P \times S$ to be finite. However, we can use the existence result of the previous subsection to obtain games that have equilibria and to which the last proposition applies, in the sense that deviations from the strategies necessary to yield the Walrasian allocations can be made possible by letting the strategy spaces be large enough while still finite.

5 Extensions

In this section we only mention some natural extensions of the framework developed in Section 2. We do not make any formal statements about what the consequences for the equilibria of the resulting games would be.

1. Throughout the paper, we have taken the order in which agents enter the markets as buyers to be exogenously given. Moreover, we have not provided any explanation about how such an order can be determined or how it may arise. It may be of some interest, therefore, to allow agents to enter the markets randomly, i.e., to let the order τ be a random variable whose realization is announced to the agents as they enter the markets. While this should not complicate the analysis much, it may also be useful in determining which equilibria are more robust to small changes in the order in which agents enter the markets as buyers.[31]

[31] We would like to remark that the statement of Proposition 3 remains valid also in the case where τ is a random variable. To see this, notice first that quantities traded at any perfectly competitive equilibrium do not in any way depend on the order in which

2. Another way of extending the framework is to allow agents to enter the markets so that more than one agent at a time can enter as a buyer. The framework developed in Section 2 only allows one agent at any point in time to make purchases on all markets. In reality, one observes that many people simultaneously make purchases on one or more markets at any given point in time. A way that allows for a more general procedure of trade is the following. Let a pair $(i,k) \in I \times L$ denote a **market**, namely the market where agent $i \in I$ sells commodity $k \in L$.[32] Suppose we allow agents to enter the markets and trade simultaneously, while requiring that they enter any given market, say, market (i,k), one at a time. The generalized order we get in this case is a map:

$$\tau : I \to (I \times L)^t,$$

that satisfies:

$$(\tau_\nu(i) = \tau_\nu(j), \nu \in T, i, j \in I) \text{ implies } (i = j),$$

where $T = \{1, \ldots, t\}$ and $t \in \mathbb{N}$ is some sufficiently large integer. The above condition says that no two agents can, at any given point in time $\nu \in T$, enter the same market $\tau_\nu(i) \in I \times L$ simultaneously. This more general order no longer has agents always making decisions one after the other in the second stage of the game. In particular, at a given point in time, agents need not know what is currently happening on other markets. Such a generalized order will be different from the one defined in the basic framework of Section 2, only if it matters to an agent what is being bought at any given point in time on another market, for example, if that agent is heading towards the other market in a coming period and would like to know whether there will still be sufficient quantities left for sale. A natural candidate for the equilibrium concept to be employed in this case is the concept of sequential equilibrium of Kreps and Wilson (1982). It may be necessary to consider finite spaces B in the second stage of the game ? to obtain existence of equilibria.

3. A further extension is to allow other ways in which agents set prices and quantities in the first stage. For example, one could let agents first decide

agents enter the markets as buyers. Second, since the set of agents I is finite, and since an order is like a permutation on the set I, there are at most finitely many possible orders. Therefore, the expected demand correspondence is a (finite) linear combination of demand correspondences for different orders, which are all assumed to be Lipschitz continuous, and so is also Lipschitz continuous. The claim now follows from the proof of Proposition 3.

[32] Depending on the strategies chosen in the first stage, some markets may not be open.

the quantities to put for sale and then let them set the prices at which to sell the quantities. Conversely, one could let them first set the prices and then the quantities. Another possibility would be to let them choose both prices and quantities in some more general order, maybe even allowing them to revise their decisions as agents enter as buyers. Once again, it would be interesting to see how changing the way prices and quantities are set affects the resulting equilibria, and, in particular, whether a statement like Proposition 3 still holds.[33]

4. One could also allow agents to charge prices in a nonlinear fashion, for example, by letting them charge different prices depending on the quantities being bought by the buyers or depending on the total quantities being sold by a given seller. They could also be allowed to charge entrance fees. It would be interesting to see whether making such changes will yield Pareto efficient allocations as the final allocations of a nontrivial subgame perfect Nash equilibrium or under what circumstances Pareto efficient allocations may arise.

A Appendix

A.1 The Game Γ_A

The subgame perfect demand correspondence plays a central role in analyzing the equilibria of the model developed in Section 2. A property that is

[33]It should be pointed out that a main feature of the framework developed in Section 2, and of Bertrand-Edgeworth competition in general, is the way agents can make commitments when choosing quantities and prices in the first main stage, while only being allowed to make purchases from what is left over at the already specified prices in the second main stage. Letting agents revise their decisions both as sellers and as buyers need not violate this basic feature. In fact, we make the following conjecture: If agents, as sellers, act as in the first stage of the game Γ, i.e., choosing prices and quantities of commodities they want to put for sale, and if, as buyers, they act as in the second stage of Γ, i.e., purchasing from quantities available at the time they enter the markets, and affordable at prices previously specified by sellers, then, regardless of the order in which agents make decisions as buyers and sellers, under certain regularity conditions on demands, the subgame perfect or sequential equilibria of the resulting game will be Walrasian if and only if initial endowments are Walrasian allocations.

particularly important in the proofs of Proposition 3 and Corollaries A1 and A2 below is that certain sums of the subgame perfect demand correspondence be Lipschitz continuous. In what follows we show that this property holds for a restricted version of the game ? defined in Section 2. We also show a further proposition (Corollary A2) on the efficiency of the Bertrand-Equilibria for this restricted class of games.

Proposition A 1 *Let $?_A$ denote the game ? where the space $\mathcal{B}$ satisfies the additional condition:*

$$b^{ii} = s^i - \sum_{j<i} b^{ji}, \forall i \in I, \tag{3}$$

and let $\mathcal{E}_A$ denote the exchange economy $\mathcal{E}$ where utility functions satisfy the additional condition:

$$\begin{vmatrix} \partial^2 u^i(x) & \partial u^i(x) \\ (\partial u^i(x))^T & 0 \end{vmatrix} \neq 0, \forall x \in \mathbb{R}_+^{l+1}, \forall i \in I, \tag{4}$$

then the subgame perfect demand correspondence $\hat{b} : P \times S \to B^m$ of any game $?_A$, for any given exchange economy $\mathcal{E}_A$, is upper-hemi continuous, compact-valued, and convex-valued, and its restriction to $P_D \times S$, where $P_D = \{p \in P : \forall k \in L, \forall i, j \in I, i \neq j, p_k^i \neq p_k^j\}$, is Lipschitz continuous and single-valued. Furthermore, at any strategy profile of the first stage $(\bar{p}, \bar{s}) \in P \times S$ that leads to a unique final allocation $\bar{x} \in \mathbb{R}_+^{(l+1)m}$, the correspondences defined by:

$$\varphi^i : P \times S \to \mathbb{R}_+^l, (p, s) \mapsto \sum\nolimits_{j \neq i} \hat{b}^{ji}(p, s),$$

$$\psi_0^i : P \times S \to \mathbb{R}_+, (p, s) \mapsto \sum\nolimits_{j \neq i} (p^i \hat{b}^{ji}(p, s) - p^j \hat{b}^{ij}(p, s)),$$

and

$$\psi^i : P \times S \to \mathbb{R}_+^l, (p, s) \mapsto \sum\nolimits_{j \neq i} (\hat{b}^{ji}(p, s) - \hat{b}^{ij}(p, s)),$$

for $i \in I$, are Lipschitz continuous.

Condition (4) is purely technical and is used to prove Lipschitz continuity of the given correspondences.[34] Condition (3) is much more restrictive and says that agents entering the markets as buyers have to buy whatever is left over of what they have put for sale. This guarantees that agents entering

[34]See Mas-Colell (1985), Chapter 2, Sections 2.5 and 2.6, and also Debreu (1972).

after them will not buy anything from them. It is a rather strong assumption to make. However, there are many economies where such an assumption is naturally satisfied, for example, if endowments are such that the economy consists of agents that are either only buyers or only sellers. It does not have to be as restrictive as that but if condition (3) is satisfied, there will always be one agent that is effectively just a buyer, namely agent 1, and there will always be an agent that is just a seller, namely agent m. The rest of this section is devoted to showing the above proposition. Before starting with the proof, we state two corollaries.

Corollary A 1 *Let $(p^W, x^W) \in W(\mathcal{E}_A)$ be a perfectly competitive equilibrium of the exchange economy $\mathcal{E}_A$ and let $s^W \in \mathbb{R}_+^{lm}$ be the corresponding perfectly competitive quantities for sale. If agents are allowed to borrow up to $p^i s^i \in \mathbb{R}_+, i \in I$, and if $P \times S$ contains the set $[p^W, p^W + \nu]^m \times \{s^W\}$ for some $\nu \in \mathbb{R}_{++}^l$, then (p^W, s^W) is the first stage part of a subgame perfect Nash equilibrium of the game $?_A$ if and only if initial endowments are x^W.*

Proof. This follows immediately from Propositions 2, 3, and A1. □

Similarly, we ask when to expect final allocations of the game $?_A$ to be Pareto efficient. The set of Pareto efficient allocations of the exchange economy $\mathcal{E}_A$ can be obtained by letting money be a regular commodity and fixing its price to be one. We denote this set by $PE(\mathcal{E}_A)$ with generic element $x^{PE} \in \mathbb{R}_+^{(l+1)m}$.

Corollary A 2 *Let $x^{PE} \in PE(\mathcal{E}_A) \cap \mathbb{R}_{++}^{(l+1)m}$ be a Pareto efficient allocation of the exchange economy $\mathcal{E}_A$ and let $s^{PE} \in \mathbb{R}_+^{lm}$ be the corresponding quantities for sale. If agents put at most quantities $(s^i)^{PE}$ for sale, $i \in I$, and if $P \times S$ contains the set $[0, p^{PE} + \nu]^m \times \{s^{PE}\}$ for some $\nu \in \mathbb{R}_{++}^l$, where $p^{PE} \in \mathbb{R}_+^l$ is given by:*

$$p_k^{PE} = MRS_{k,0}^i((x_0^i, x^i)^{PE}), k \in L, \text{ any } i \in I,$$

then x^{PE} is the final allocation of a subgame perfect Nash equilibrium of the game $?_A$ if and only if initial endowments are x^{PE}.

Proof. See the next section of this appendix. □

This proposition says that a subgame perfect Nash equilibrium of the game $?_A$ will yield final allocations that are Pareto efficient, when agents put at most certain quantities s^{PE} for sale, if and only if initial endowments are already Pareto efficient, in which case no trade takes place at the subgame perfect equilibrium.

*Proof of Proposition A*1. We need to consider the maximization problems agents face when acting as buyers. The proof is in three steps. First, we show some general facts about agent i's maximization problem, for arbitrary $i \in I$. Second, we show the statements concerning single-valuedness, compact-valuedness, and convex-valuedness. Upper-hemi continuity is already shown in Lemma 1. Third, we show the statements concerning the Lipschitz continuity of the correspondences $\hat{b}$, $(\varphi^i)_{i\in I}$, $(\psi_0^i)_{i\in I}$, and $(\psi^i)_{i\in I}$. We use the same notation as in the rest of the paper.

Step 1: For $i \in I$, let $u^i : \mathbb{R}_+^{l+1} \to \mathbb{R}$ be a smooth,[35] strictly increasing, and strictly quasi-concave utility function, and let $(\omega_0^i, \omega^i) \in \mathbb{R}_+^{l+1}$ be an endowment vector consisting of a money endowment $\omega_0^i \in \mathbb{R}_+$ and a commodity endowment $\omega^i \in \mathbb{R}_+^l$.

Agent i faces the optimization problem:

$$max_{(b^{ij})_{j\in I}} u^i(\omega_0^i + \sum_{j\neq i} p^i b^{ji} - \sum_{j\neq i} p^j b^{ij}, \omega^i - \sum_{j\neq i} b^{ji} + \sum_{j\neq i} b^{ij})$$

subject to

$$\sum_{j\neq i} p^j b^{ij} \leq \omega_0^i \text{ and } 0 \leq b^{ij} \leq s^j - \sum_{h<i} b^{hj} \text{ for all } j \in I.$$

To simplify the notation in the proofs, let $\Upsilon_i = P \times S \times B^{i-1}$ with generic element $v = (p, s, (b^h)_{h<i})$. Then, if we write x instead of $(b^{ij})_{j\in I}$ so that $x_k^j = b_k^{ij}$, $k \in L, j \in I$, we can rewrite the maximization problem as:

$$max_x f(v, x) \text{ subject to } g(v, x) \leq c(v), x \geq 0, \text{ for } v \in \Upsilon_i, \tag{5}$$

where $f : \Upsilon_i \times \mathbb{R}_+^{lm} \to \mathbb{R}$ is defined by:

$$f(v, x) = u^i(\omega_0^i + \sum_{j\in I} p^i b^{ji} - \sum_{j\in I} p^j x^j, \omega^i - \sum_{j\in I} b^{ji} + \sum_{j\in I} x^j),$$

[35] Assuming smoothness here is by no means restrictive in the sense that Proposition A1 still holds if the utility functions are assumed to be twice continuously differentiable.

$g : \Upsilon_i \times \mathbb{R}_+^{lm} \to \mathbb{R}_+^{lm+1}$ is defined by:

$$g(v, x) = (\sum_{j \neq i} p^j x^j, x),$$

and $c : \Upsilon_i \to \mathbb{R}_+^{lm+1}$ by:

$$c(v) = (\omega_0^i, s^1 - \sum_{j<i} b^{j1}, \ldots, s^m - \sum_{j<i} b^{jm}).$$

We can state the following results.

Lemma A 1 *For $v \in \Upsilon_i$, the set $\chi_v = \{x \in \mathbb{R}_+^{lm} : g(v, x) \leq c(v)$ and $x \geq 0\}$ is a nonempty, compact, and convex subset of Euclidean space. Furthermore, $g : \Upsilon_i \times \mathbb{R}_+^{lm} \to \mathbb{R}_+^{lm+1}$ is smooth.*

Proof. This is trivial. □

Lemma A 2 *The function $f : \Upsilon_i \times \mathbb{R}_+^{lm} \to \mathbb{R}$ is smooth and the function $f_v : \mathbb{R}_+^{lm} \to \mathbb{R}, x \mapsto f(v, x)$ is quasi-concave for all $v \in \Upsilon_i$.*

Proof. Clearly, f is smooth because u^i is smooth and its arguments are multilinear functions of the components of (v, x). Fix $v \in \Upsilon_i$ and let $x, y \in \mathbb{R}_+^{lm}, x \neq y$ be such that $f(v, x) = f(v, y)$. We need to show that $f(v, x\alpha y) \geq f(v, x)\alpha f(v, y) = f(v, x)$, where $x\alpha y = \alpha x + (1 - \alpha)y$ and $\alpha \in [0, 1]$. Let $(t_0, t) \in \mathbb{R}_+^{l+1}$ be some constant vector, then, by definition of f, we have:

$$\begin{aligned} f(v, x\alpha y) &= u^i(t_0 - \sum_{j \neq i} p^j (x^j \alpha y^j), t + \sum_{j \neq i} (x^j \alpha y^j)) \\ &= u^i((t_0 - \sum_{j \neq i} p^j x^j)\alpha(t_0 - \sum_{j \neq i} p^j y^j), (t + \sum_{j \neq i} x^j)\alpha(t + \sum_{j \neq i} y^j)) \\ &\geq u^i(t_0 - \sum_{j \neq i} p^j x^j, t + \sum_{j \neq i} x^j) \\ &= f(v, x). \end{aligned}$$

The inequality follows because u^i is strictly quasi-concave. □

Step 2: Now we show that the correspondence $\hat{b}$ is compact-valued and convex-valued, and that its restriction to $P_D \times S$ is single-valued. Both compact-valuedness and convex-valuedness follow immediately from Lemmas A1 and A2. Single-valuedness on $P_D \times S$ follows from the following lemma.

Lemma A 3 *The function* $f_v : \mathbb{R}^{lm}_+ \to \mathbb{R}, x \mapsto f(v,x)$ *has a unique maximum on the set* $\chi_v = \{x \in \mathbb{R}^{lm}_+ : g(v,x) \leq c(v)$ *and* $x \geq 0\}$, *for all* $v \in P_D \times S \times B^{i-1}$.

Proof. Suppose $x, y \in \chi_v$, $x \neq y$, are two maximizers of f_v, where $v \in P_D \times S \times B^{i-1}$. By Lemmas A1 and A2, both x and y must lead to the same final allocation, i.e., $\sum_{j \neq i} x^j = \sum_{j \neq i} y^j$. Therefore, the vector $z \in \mathbb{R}^{lm}_+$ that satisfies $\sum_{j \neq i} z^j = \sum_{j \neq i} x^j$ and is defined by combining the entries in the vectors x and y and taking the cheapest possibility of obtaining the quantities $\sum_{j \neq i} x^j$, is contained in the set χ_v and yields a strictly higher utility than x or y since prices are in P_D and utility functions are strictly increasing. This contradicts x and y being distinct maximizers. □

Step 3: Lemmas A1 and A2 allow us to view the maximization problem above as a classical nonlinear programming problem with not necessarily unique solution. The Kuhn-Tucker conditions for this maximization problem are:[36]

$$(\frac{\partial f(v,x)}{\partial x} - \lambda \frac{\partial g(v,x)}{\partial x}) \leq 0, (\frac{\partial f(v,x)}{\partial x} - \lambda \frac{\partial g(v,x)}{\partial x})x = 0, x \geq 0$$

and

$$(c(v) - g(v,x)) \geq 0, \lambda(c(v) - g(v,x)) = 0, \lambda \geq 0,$$

where $\lambda \in \mathbb{R}^{lm+1}$ is the vector of Lagrange multipliers. We use these conditions to derive some Lipschitz continuity properties of the subgame perfect demand correspondence of agent i. In what follows, we assume that agent i's first constraint, i.e., the budget constraint is not binding. If this constraint is binding the same arguments with some minor alterations lead to the same conclusions.[37]

[36] See for example Intriligator (1971), Chapter 4, Section 4.2. We need not worry about constraint qualifications since the functions $g(v,\cdot)$ are linear. See for example Mangasarian (1969), Chapter 7, Section 3.4.

[37] This is where condition (4) is used.

Consider partitions of x and λ of the form:

$$x = (x_0, x_1, x_2) \in \{0\}^{lm-r-s} \times \mathbb{R}^r_+ \times \{s_2\}$$

and

$$\lambda = (\lambda_0, \lambda_1, \lambda_2) \in \{0\}^{lm+1-s-t} \times \mathbb{R}^t_+ \times \mathbb{R}^s_+,$$

where $s_2 \in \mathbb{R}^s_+$ is a subvector of $s - \sum_{h<i} b^h \in \mathbb{R}^{lm}_+$, and r, s, and t are nonnegative integers such that $r + s \leq lm$ and $s + t \leq lm + 1$. In this notation all elements of the vector x_2 are equal to the corresponding element in the vector s_2, and the Lagrange multiplier for this constraint is the corresponding element of λ_2. Similarly, elements of x_1 are quantities, say, x^j_k such that there is a tangency of the indifference surface with the constraint set in direction of the kth commodity bought from agent j. Given that besides the constraints $x^j_k \leq s^j_k$ and the nonnegativity constraints, there is only the constraint $\sum_{j \neq i} p^j x^j \leq \omega^i_0$, λ_1 denotes the multiplier associated to the latter constraint. Since we are considering the case where the first constraint is not binding, the relevant parts of the Kuhn-Tucker conditions are:[38]

$$\frac{\partial f(v,x)}{\partial x_1} - \lambda \frac{\partial g(v,x)}{\partial x_1} = 0, \frac{\partial f(v,x)}{\partial x_2} - \lambda \frac{\partial g(v,x)}{\partial x_2} = 0, c_2(v) - g_2(v,x) = 0,$$

where c_2 and g_2 denote the parts of c and g associated with λ_2, here $c_2(v) = s_2$ and $g_2(v,x) = x_2$. We introduce the notions of a regular partition of x and of a regular set of Kuhn-Tucker equations.

Definition A 1 *Let $x \in \mathbb{R}^{lm}_+$ be a solution to the optimization problem (5) with corresponding vector of Lagrange multipliers $\lambda \in \mathbb{R}^{lm+1}_+$. We say the partition $x = (x_0, x_1, x_2) \in \mathbb{R}^{lm}_+$ is* ***regular*** *if the subvectors $x_0 \in \mathbb{R}^{lm-r-s}_+$ and $x_2 \in \mathbb{R}^s_+$ satisfy:*

$$x_0 = 0_{lm-r-s} \text{ and } x_2 = s_2,$$

and the subvector $x_1 \in \mathbb{R}^r_+$ satisfies:

$$\frac{\partial f(v,x)}{\partial x_1} - \lambda \frac{\partial g(v,x)}{\partial x_1} = 0,$$

[38] Notice that there are $r + 2s$ equations and $r + 2s$ unknowns, i.e., $(x_1, x_2, \lambda_2) \in \mathbb{R}^{r+2s}_+$.

and contains, for every $k \in L$, at most one element $x_k^j \in \mathbb{R}_+$, for some $j \in I$. Furthermore, we say the set of Kuhn-Tucker equations of (5), $F(v,x,\lambda) = 0_{r+2s}$, where $F : \Upsilon_i \times \mathbb{R}^{2lm+1} \to \mathbb{R}^{r+2s}$ is given by:

$$F(v,x,\lambda) = (\frac{\partial f(v,x)}{\partial x_1} - \lambda\frac{\partial g(v,x)}{\partial x_1}, \frac{\partial f(v,x)}{\partial x_2} - \lambda\frac{\partial g(v,x)}{\partial x_2}, c_2(v) - g_2(v,x)),$$

*is **regular** if the corresponding partition of x is regular.*

This definition rules out sets of Kuhn-Tucker equations where there is a tangency between the indifference surface and the constraint set in direction of, say, the kth commodity bought from agent j, and in direction of the same commodity bought from agent h, where $j, h \in I$, $j \neq h$. We will see that regular sets of Kuhn-Tucker equations have nice properties.

Lemma A 4 *Let $F : \Upsilon_i \times \mathbb{R}_+^{2lm+1} \to \mathbb{R}_+^{r+2s}$ define a regular set of Kuhn-Tucker equations of (5), $F(v,x,\lambda) = 0_{r+2s}$, then x can be solved, locally, as a smooth function of v.*

Proof. This is a direct application of the Implicit Function Theorem[39] to the function F. In order to be able to apply the theorem, we need to check that the Jacobian matrix $D_{(x_1,x_2,\lambda_2)}F$ has full rank at the point $(v,x,\lambda) \in \Upsilon_i \times \mathbb{R}^{2lm+1}$ satisfying $F(v,x,\lambda) = 0$, i.e., we need to show that the $(r+2s) \times (r+2s)$ matrix:

$$D_{(x_1,x_2,\lambda_2)}F(v,x,\lambda) = \begin{pmatrix} \frac{\partial^2 f(v,x)}{\partial x_1^2} & \frac{\partial^2 f(v,x)}{\partial x_1 \partial x_2} & 0 \\ \frac{\partial^2 f(v,x)}{\partial x_2 \partial x_1} & \frac{\partial^2 f(v,x)}{\partial x_2^2} & -\frac{\partial g_2(v,x)}{\partial x_2} \\ 0 & -(\frac{\partial g_2(v,x)}{\partial x_2})^T & 0 \end{pmatrix}$$

has full rank $r+2s$. It is sufficient to show that the matrix can be decomposed in the form:

$$D_{(x_1,x_2,\lambda_2)}F(v,x,\lambda) = A^T H A,$$

where H is an $(r+2s+1) \times (r+2s+1)$ matrix of rank $r+2s+1$ and A is a $(r+2s+1) \times (r+2s)$ matrix of rank $r+2s$. Consider the following choice of matrices:

$$H = \begin{pmatrix} H_{11} & H_{12} & 0 \\ H_{21} & H_{22} & -I_s \\ 0 & -I_s & 0 \end{pmatrix},$$

[39] See for example Mas-Colell (1985), Chapter 1, Section C.3, p.20.

where H_{11} is an $(r+1) \times (r+1)$ submatrix of the Hessian $\partial^2 u^i(v,x)$ with entries corresponding to the vector x_1, $H_{12} = H_{21}^T$ is an $(r+1) \times s$ matrix with entries from the Hessian $\partial^2 u^i(v,x)$ where the rows are determined from the vector x_1 and the columns from x_2, H_{22} is an $s \times s$ matrix with entries also from the Hessian $\partial^2 u^i(v,x)$ where both rows and columns are determined from the vector x_2, i.e.,

$$H_{11} = \begin{pmatrix} u_{00} & u_{01_1} & \cdots & u_{0l_1} \\ u_{1_10} & u_{1_11_1} & \cdots & u_{1_1l_1} \\ \vdots & \vdots & \ddots & \vdots \\ u_{l_10} & u_{l_11_1} & \cdots & u_{l_1l_1} \end{pmatrix}, H_{12} = \begin{pmatrix} u_{01_2} & u_{02_2} & \cdots & u_{0l_2} \\ u_{1_11_2} & u_{1_12_2} & \cdots & u_{1_1l_2} \\ \vdots & \vdots & \ddots & \vdots \\ u_{l_11_2} & u_{l_12_2} & \cdots & u_{l_1l_2} \end{pmatrix},$$

and

$$H_{22} = \begin{pmatrix} u_{1_21_2} & u_{1_22_2} & \cdots & u_{1_2l_2} \\ u_{2_21_2} & u_{2_22_2} & \cdots & u_{2_2l_2} \\ \vdots & \vdots & \ddots & \vdots \\ u_{l_21_2} & u_{l_22_2} & \cdots & u_{l_2l_2} \end{pmatrix},$$

where for simplicity all superscripts i have been omitted from the functions u^i. Here the indices $1_1, \ldots, l_1$ and $1_2, \ldots, l_2$ denote commodities of the vectors x_1 and x_2 respectively. Notice that the commodities of x_1 are all distinct while they need not be distinct in x_2. Because the utility function is strictly quasi-concave the matrix H_{11} has rank $r+1$, so it is easy to see that the matrix H has rank $r+2s+1$. Furthermore, for the matrix A we have:

$$A = \begin{pmatrix} A_{11} & A_{12} & 0 \\ 0 & I_s & 0 \\ 0 & 0 & I_s \end{pmatrix},$$

where A_{11} and A_{12} are $(r+1) \times r$ and $(r+1) \times s$ matrices respectively, that are given by:

$$A_{11} = \begin{pmatrix} -p_{1_1}^{i_1} & \cdots & -p_{l_1}^{i_1} \\ 1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & 1 \end{pmatrix} \text{ and } A_{12} = \begin{pmatrix} -p_{1_2}^{i_2} & \cdots & -p_{l_2}^{i_2} \\ 0 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & 0 \end{pmatrix},$$

where the indices $1_1, \ldots, l_1$ and $1_2, \ldots, l_2$ are as above while the indices i_1 and i_2 indicate, by slight abuse of notation in that they may vary across the commodities, the agents from which the commodities from x_1 and x_2 respectively are bought. The matrix A_{11} clearly has rank r, so it is easy to see that A has rank $r+2s$. This implies that $A^T HA$ has rank $r+2s$. It is straightforward to check that $D_{(x_1,x_2,\lambda_2)}F(\upsilon,x,\lambda) = A^T HA$. But then we can apply the Implicit Function Theorem to show that (x,λ) can locally be solved as a smooth function of υ. □

The next lemma shows that, in a neighborhood of any point $\upsilon \in \Upsilon_i$, the demand correspondence of agent i is contained in the convex hull of solutions to regular sets of Kuhn-Tucker equations.

Lemma A 5 *Let $N \subset \mathbb{N}$ denote the index set for all regular Kuhn-Tucker equations of (5), $F_\nu(\upsilon,x,\lambda) = 0_{r+2s}$, where for $\nu \in N$, $F_\nu : \Upsilon_i \times \mathbb{R}_+^{2lm+1} \to \mathbb{R}_+^{r+2s}$ and $\upsilon \in \Upsilon_i$. Let $\varphi_\nu : \Upsilon_i \to \mathbb{R}_+^{lm}$, $\nu \in N$, denote the corresponding solutions of x. Then N is a nonempty and finite set and, on some neighborhood $U \subset \Upsilon_i$ of υ, agent i's demand correspondence is contained in the convex hull of all the φ_ν's.*[40]

Proof. Clearly, the optimization problem (5) has a solution $x \in B$ at the given $\upsilon \in \Upsilon_i$. Moreover, we can write the set of all solutions to (5) as $\{y \in B : y \in \chi_\upsilon \text{ and } \sum_{j\neq i} y^j = \sum_{j \neq i} x^j\}$. This set forms a compact polyhedron that is spanned by a finite number of points. For each of these extremal points there exists at least one regular set of Kuhn-Tucker equations and there can only be at most finitely many such sets of equations, since each solution has a finite number of possible regular partitions. To see that i's subgame perfect demand correspondence is contained in the convex hull of the φ_ν's, notice that the latter contains all points $x' \in B$ that satisfy the Kuhn-Tucker conditions at a given point υ' in some sufficiently small neighborhood $U \subset \Upsilon_i$ of υ. This is because, by Lemmas A1 and A2 and the Kuhn-Tucker Theorem,[41] the Kuhn-Tucker conditions are both necessary and sufficient, and because the φ_ν's are obtained by essentially inverting the Kuhn-Tucker

[40] We say a function $f : X \to Y$ or a correspondence $F : X \to Y$ is **contained in the convex hull of the functions** $h_\nu : X \to Y$, for ν in some index set N, if respectively, for all $x \in X$:

$$f(x) \in co\{y \in Y : \exists \nu \in N, y = h_\nu(x)\} \text{ and } F(x) \subset co\{y \in Y : \exists \nu \in N, y = h_\nu(x)\},$$

where co denotes the convex hull of the given set.

[41] See for example Intriligator (1971), Chapter 4, Section 4.3.

conditions holding with equalities at v. But any point $x' \in \hat{b}^i(v')$, where $\hat{b}^i : \Upsilon_i \to B$ is agent i's demand correspondence, must satisfy the Kuhn-Tucker conditions at $v' \in U$. □

Combining this lemma with the fact that the correspondence $\hat{b}$ is single-valued and upper-hemi continuous on $P_D \times S$ shows that it is Lipschitz continuous on $P_D \times S$. Similarly, the correspondences $(\varphi^i)_{i \in I}$, $(\psi_0^i)_{i \in I}$, and $(\psi^i)_{i \in I}$ are Lipschitz continuous at points $(\bar{p}, \bar{s}) \in P \times S$ that lead to unique final allocations, since, if they lead to unique final allocations, by definition, the correspondences must also be single-valued at the given points. This completes the proof of Proposition A1. □

A.2 Proof of Corollary A2

To see the if part, let $\omega = x^{PE} \in PE(\mathcal{E}_A) \cap \mathbb{R}_{++}^{(l+1)m}$ be a Pareto efficient initial endowment. At this allocation all agents' marginal rates of substitution are equal, and we can set:

$$p_k^i = (p_k)^{PE} = MRS_{k,0}^i((x_0^i, x^i)^{PE}), k \in L, i \in I.$$

But then, if agents play strategies $(p^i, s^i) = (p^{PE}, 0) \in P \times S$, $i \in I$, since $(p^{PE}; x^{PE}) \in P \times \mathbb{R}_+^{(l+1)m}$ is a no trade perfectly competitive equilibrium, by Proposition 2 there will be a unique subgame perfect equilibrium with final allocations x^{PE}. This shows the if part of the proposition. To see the only if part, we need to show that if some $x^{PE} \in PE(\mathcal{E}_A) \cap \mathbb{R}_{++}^{(l+1)m}$ is a final allocation of some subgame perfect Nash equilibrium, then initial endowments must be equal to x^{PE}. We show that if initial endowments are not x^{PE}, then x^{PE} cannot be the final allocation of a subgame perfect equilibrium. We proceed by contradiction. Fix $(p, s) \in P \times S$ to be the first stage part of an equilibrium strategy profile that leads to some Pareto efficient final allocation x^{PE}. We distinguish four different cases for the prices that can be charged at equilibrium. But before we examine the four cases, notice first that if x^{PE} is to be the final allocation of a subgame perfect equilibrium, then quantities chosen in the first stage have to satisfy the conditions:

$$s_k^i \geq (s_k^i)^{PE} = max\{0, \omega_k^i - (x_k^i)^{PE}\}, k \in L, i \in I.$$

This is because otherwise some agent will necessarily end up with more of some commodity than is consistent with the efficient allocation $(x^i)^{PE}$. Together with the assumption that agents are putting at most s^{PE} for sale, it implies that agents are putting exactly s^{PE} for sale. Now we show that Proposition A1 applies at the strategy profile $(p, s) \in P \times S$. This follows from the following lemma:

Lemma A 6 *If $(\bar{p}, \bar{s}) \in P \times S$ is the first stage part of a subgame perfect Nash equilibrium profile that leads to the final allocation $\bar{x}^{PE} \in PE(\mathcal{E}_A) \cap \mathbb{R}^{(l+1)m}_{++}$, where $\bar{s} = \bar{s}^{PE}$, then $\bar{x}^{PE}$ is the unique subgame perfect final allocation when $(\bar{p}, \bar{s})$ is the strategy profile of the first stage.*

By Lemmas A1 and A2, all agents when entering as buyers have a unique optimal final allocation. If total quantities for sale are fixed by $\bar{s} = \bar{s}^{PE}$, and if one of the solutions leads to $\bar{x}^{PE}$, then so do all others. This lemma allows us to apply the second part of Proposition A1 because it implies that the equilibrium strategy profile of the first stage (p, s) leads to a unique final allocation, namely x^{PE}. Next consider the following lemma which will be used later.

Lemma A 7 *At any allocation $x \in \mathbb{R}^{(l+1)m}_{+}$ that is the final allocation of a subgame perfect Nash equilibrium of the game $?_A$ with strategy profile in the first stage $(p, s) \in P \times S$, no agent, $i \in I$, is purchasing anything at a price $p^j_{k'} > MRS^i_{k',0}(x^i_0, x^i), j \in I, k' \in L$.*

This is straightforward since all agents are better off decreasing their purchase of commodity k', and clearly it is always feasible to do so.

Now we come to the four cases. Let $W(\mathcal{E}_A)$ denote the set of perfectly competitive price vectors and corresponding allocations of the exchange economy $\mathcal{E}_A$.[42] We distinguish the following cases:

Case 1: $\forall i \in I, p^i = p^{PE}$ and $(p^{PE}, x^{PE}) \in W(\mathcal{E}_A)$. This cannot be part of a subgame perfect Nash equilibrium from Proposition 3.

Case 2: $\forall i \in I, p^i = p^{PE}$ and $(p^{PE}, x^{PE}) \notin W(\mathcal{E}_A)$. Because p^{PE} is not a Walrasian price vector for the economy $\mathcal{E}_A$, we know that when p^{PE} is charged by all agents, then markets do not clear. In our model, this means that there is at least one agent who is either buying more of some commodity

[42]Notice that while there always exists an economy $\mathcal{E}'_A$ with $(p^{PE}, x^{PE}) \in W(\mathcal{E}'_A)$, it need not be the case that for the given exchange economy $\mathcal{E}_A$, $(p^{PE}, x^{PE}) \in W(\mathcal{E}_A)$. See for example Debreu (1959) or Mas-Colell (1985).

than he wants to buy or who is buying less than he wants to buy, i.e., who is rationed. But neither can be if the final allocation is Pareto efficient.

Case 3: $\forall i \in I, p^i \geq p^{PE}$ and $p^i \neq p^{PE}$ for some $i \in I$. If there exists no $i \in I$ and no $k \in L$ such that $p^i_k > (p_k)^{PE}$ and $s^i_k \geq (s^i_k)^{PE} > 0$, then we are reduced to cases 1 or 2. So suppose $\exists i \in I, \exists k \in L$ such that $p^i_k > (p_k)^{PE}$ and $s^i_k \geq (s^i_k)^{PE} > 0$. From Lemma A6, we know that any agent that at equilibrium purchases some positive quantity of commodity k from agent i is better off purchasing less. So this implies that agent i must be selling less than $(s^i_k)^{PE} > 0$, which is inconsistent with x^{PE} being the final allocation.

Case 4: $\exists i \in I, \exists k \in L$, such that $p^i_k < (p_k)^{PE}$. As before, if $s^i_k \geq (s^i_k)^{PE} = 0$, then we are reduced to cases 1, 2, or 3 above. So suppose in addition $s^i_k \geq (s^i_k)^{PE} > 0$. We want to show that agent i is better off increasing the price p^i_k by some $\delta > 0$. Let $i \in I$ and $k \in L$ be fixed. We need a last lemma:

Lemma A 8 *At any strategy profile $(p,s) \in P \times S$ that is part of a subgame perfect Nash equilibrium of the game $?_A$ that leads to the Pareto efficient final allocation $x^{PE} \in PE(\mathcal{E}_A) \cap \mathbb{R}^{(l+1)m}_{++}$, there exists a $\delta > 0$ such that, if $(p,s)_\delta$ denotes the strategy profile (p,s) where p^i_k is replaced by $p^i_k + \delta$, then:*

$$\psi^i_0((p,s)_\delta) - \psi^i_0(p,s) \geq p^{PE}(\psi^i((p,s)_\delta) - \psi^i(p,s)).$$

Before we prove this lemma, as in the proof of Proposition 3, we introduce some notation. Let $(p,s;b) \in P \times S \times \mathcal{B}$ be the given equilibrium strategy profile that yields the final allocation x^{PE}, let $(x^i_0, x^i) \in \mathbb{R}^{l+1}_+$ denote agent i's Pareto efficient allocation $(x^i_0, x^i)^{PE} \in \mathbb{R}^{l+1}_+$and let $(x^i_0, x^i)_\delta \in \mathbb{R}^{l+1}_+$ denote agent i's final allocation when he charges $p^i_k + \delta$ for commodity k, and all other strategies are as in the 0 profile. Because, by Proposition A1, final allocations are continuous at the equilibrium strategy profile and because utility functions are assumed to be twice continuously differentiable, all marginal rates of substitution are continuous at the final allocation x^{PE}. We use this fact to prove this lemma. Consider first the correspondences γ^i and $(\gamma^{ij})_{j \in I}$ defined by:

$$\gamma^i : \mathbb{R}_+ \to \mathbb{R}^l_+, \delta \mapsto \sum_{j \neq i} (\hat{b}^{ji}((p,s)_\delta) - \hat{b}^{ji}(p,s))$$

and

$$\gamma^{ij} : \mathbb{R}_+ \to \mathbb{R}^l_+, \delta \mapsto \hat{b}^{ij}((p,s)_\delta) - \hat{b}^{ij}(p,s),$$

for $j \in I$. It suffices to show that equilibrium quantities of commodities bought or sold by agent i at prices other than the corresponding prices in p^{PE} do not change when $p_k^i + \delta$ is charged instead of p_k^i. We show this is the case for some sufficiently small $\bar{\delta} > 0$.

For the commodities that agent i is selling, there are four possibilities: (i) commodity k of which the price is being increased, (ii) commodities, say, $k_1 \in L$, where $p_{k_1}^i < (p_{k_1})^{PE}$, (iii) commodities $k_2 \in L$, where $p_{k_2}^i = (p_{k_2})^{PE}$, and (iv) commodities $k_3 \in L$, where $p_{k_3}^i > (p_{k_3})^{PE}$.

Similarly, for the commodities agent i is purchasing, there are three possibilities: (i) commodities $h_1 \in L$ bought from agents $j_1 \in I$ at prices $p_{h_1}^{j_1} < p_{h_1}^{PE}$, (ii) commodities $h_2 \in L$ bought from agents $j_2 \in I$ at prices $p_{h_2}^{j_2} = p_{h_2}^{PE}$, and, (iii) commodities $h_3 \in L$ bought from agents $j_3 \in I$ at prices $p_{h_3}^{j_3} > p_{h_3}^{PE}$.

We are concerned with commodities of type (i), (ii), and (iv) for the commodities sold and of type (i) and (iii) for the commodities bought. For all these commodities, by continuity of the marginal rates of substitution, we can find a $\bar{\delta} > 0$ such that $\forall j \in I$:

$$p_k^i + \bar{\delta} < MRS_{k,0}^j((x_0^j, x^j)_{\bar{\delta}}), p_{k_1}^i < MRS_{k_1,0}^j((x_0^j, x^j)_{\bar{\delta}}), p_{k_3}^i > MRS_{k_3,0}^j((x_0^j, x^j)_{\bar{\delta}}),$$

and

$$p_{h_1}^{j_1} < MRS_{h_1,0}^j((x_0^j, x^j)_{\bar{\delta}}), p_{h_3}^{j_3} > MRS_{h_3,0}^j((x_0^j, x^j)_{\bar{\delta}}),$$

which, by Lemma A6, immediately implies:

$$\gamma_{k_3}^i(\bar{\delta}) = 0 \text{ and } \gamma_{h_3}^{ij_3}(\bar{\delta}) = 0.$$

Furthermore, since budget constraints may be binding, it also implies:

$$\gamma_k^i(\bar{\delta}) \leq 0, \gamma_{k_1}^i(\bar{\delta}) \leq 0, \text{ and } \gamma_{h_1}^{ij_1}(\bar{\delta}) \geq 0,$$

This shows Lemma A7, since we now have:

$$\begin{aligned} \psi_0^i((p,s)_\delta) - \psi_0^i(p,s) &= p^i \gamma^i(\bar{\delta}) - \sum_{j \neq i} p^j \gamma^{ij}(\bar{\delta}) \\ &\geq p^{PE}(\gamma^i(\bar{\delta}) - \sum_{j \neq i} \gamma^{ij}(\bar{\delta})) \\ &= p^{PE}(\psi^i((p,s)_{\bar{\delta}}) - \psi^i(p,s)), \end{aligned}$$

for any $\delta \in (0, \bar{\delta})$, where $\bar{\delta} > 0$.

To finish the proof of the proposition, as with Proposition 3, we take $\epsilon \in \mathbb{R}^l_{++}$ to be such that $\epsilon_k = 1/4\varphi^i_k(p,s) > 0$, then we can find a $\bar{\bar{\delta}} \in (0, \bar{\delta})$ such that:

$$3/4\varphi^i_k(p,s) < \varphi^i_k((p,s)_{\bar{\delta}}) < 5/4\varphi^i_k(p,s).$$

Let $\delta \in (0, \bar{\bar{\delta}})$ and consider the correspondence defined by:

$$\varepsilon : \mathbb{R}_+ \to \mathbb{R}^l_+, \delta \to \psi^i((p,s)_\delta) - \psi^i(p,s).$$

Then we have:

$$\begin{aligned}
& u^i((x^i_0, x^i)_\delta) - u^i(x^i_0, x^i) \\
\geq \;& u^i(x^i_0 + p^{PE}\varepsilon(\delta) + \delta\varphi^i_k((p,s)_\delta), x^i - \varepsilon(\delta)) \\
& -u^i(x^i_0, x^i) \\
= \;& u^i(x^i_0 + p^{PE}\varepsilon(\delta) + \delta\varphi^i_k((p,s)_\delta), x^i - \varepsilon(\delta)) \\
& -u^i(x^i_0 + p^{PE}\varepsilon(\delta), x^i - \varepsilon(\delta)) \\
& +u^i(x^i_0 + p^{PE}\varepsilon(\delta), x^i - \varepsilon(\delta)) \\
& -u^i(x^i_0, x^i) \\
= \;& \partial_0 u^i(x^i_0 + p^{PE}\varepsilon(\delta), x^i - \varepsilon(\delta))\delta\varphi^i_k((p,s)_\delta) + o(\delta\varphi^i_k((p,s)_\delta)) \\
& -\partial_0 u^i(x^i_0, x^i)p^{PE}\varepsilon(\delta) + \sum_{k\in L}\partial_k u^i(x^i_0, x^i)\varepsilon_k(\delta) + o(\varepsilon(\delta)) \\
= \;& \partial_0 u^i(x^i_0 + p^{PE}\varepsilon(\delta), x^i - \varepsilon(\delta))\delta\varphi^i_k((p,s)_\delta) \\
& +o(\delta\varphi^i_k((p,s)_\delta)) + o(\varepsilon(\delta))
\end{aligned}$$

$$= \varrho + o(\delta) > 0,$$

where $\varrho > 0$ is some positive constant. The first inequality follows, given our choice of δ, from Lemma A7, the definition of ε, and the strict monotonicity of the utility function. The second equality is obvious. The third equality follows from a first order Taylor series expansion. To see the fourth equality, notice that, for commodities $k_2 \in L$ and commodities $h_2 \in L$ bought from agents j_2, agent i is indifferent between selling or not selling or buying or not buying an arbitrarily small extra amount at equilibrium. This is because these commodities are traded at prices $p^i_{k_2} = (p_{k_2})^{PE}$ and $p^{j_2}_{h_2} = (p_{h_2})^{PE}$ respectively. In fact, Lemma 2 applies when substituting perfectly competitive with Pareto efficient allocations and p^W with p^{PE}, so that this equality follows from Lemma 2. The last equality, finally, follows because $\partial_0 u^i(x^i_0 + p^{PE}\varepsilon(\delta), x^i - \varepsilon(\delta)) > 0$ by strict monotonicity of u^i, and because $\varphi^i_k((p,s)_\delta)$ is bounded below and above by a positive constant, given our choice of δ. Furthermore, also as in the proof of the previous proposition, $o(\delta) = o(\varepsilon(\delta))$ follows because ε is Lipschitz continuous.

This shows that if agent i charges the higher price $p^i_k + \delta$, then he is better off. Since strategies available do allow agent i to charge prices in $[0, p^{PE} + \nu]$, for some $\nu \in \mathbb{R}^l_{++}$, and since any smaller $\delta > 0$ will do the same job, we have showed that if initial endowments are not Pareto efficient, then there is no strategy profile $(p,s) \in P \times S$ that is part of a subgame perfect equilibrium of the game $?_A$. This shows the only if part of the proposition for the case where agents are allowed to play just pure strategies in the first stage of the game. To see the case where they are allowed to play mixed strategies in the first stage, suppose $(p,s)_\wp \in \wp(P \times S)$ is a mixed strategy profile of the first stage that is part of a subgame perfect Nash equilibrium that leads to the final allocation $x^{PE} \in PE(\mathcal{E}_A) \cap \mathbb{R}^{(l+1)m}_{++}$. Then all realizations (p,s) in the support of $(p,s)_\wp$ must lead to the final allocation x^{PE}. But we have showed that for any $((p^j, s^j)_{j \neq i}) \in \times_{j \neq i}(P_j \times S_j)$ that is part of a realization (p,s) in the support of $(p,s)_\wp$, it is never optimal for agent i to choose a strategy (p^i, s^i) that leads to the Pareto efficient allocation x^{PE}, since in each case he must be selling a positive amount of commodity k and therefore, in each case, would want to increase the price he charges by some positive amount. This shows that $(p,s)_\wp$ cannot be part of a subgame perfect Nash equilibrium of the game $?_A$ and so completes the proof. $\square$

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