

De La Vallée Poussin's Theorem  
and Weakly Compact Sets in  
Orlicz Spaces

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October 15, 1992

# 1 Introduction and Background

Recall that a subset  $\mathcal{K}$  of  $L^1(\mu)$  is called uniformly integrable if given  $\varepsilon > 0$  there is a  $\delta > 0$  so that  $\sup \{ \int_E |f| d\mu : f \in \mathcal{K} \} < \varepsilon$  whenever  $\mu(E) < \delta$ . Alternatively  $\mathcal{K}$  is bounded and uniformly integrable if and only if given  $\varepsilon > 0$  there is an  $N > 0$  so that

$$\sup \left\{ \int_{[|f|>c]} |f| d\mu : f \in \mathcal{K} \right\} < \varepsilon \text{ whenever } c \geq N.$$

The classical theorem of Dunford and Pettis [5, page 93], identifies the bounded, uniformly integrable subsets of  $L^1(\mu)$  with the relatively weakly compact sets. Another characterization of uniform integrability is given in a theorem of De La Vallée Poussin [16, pages 19-20], which states that *a subset  $\mathcal{K}$  of  $L^1(\mu)$  is bounded and uniformly integrable if and only if there is an  $N$ -function  $F$  so that  $\sup \{ \int F(f) d\mu : f \in \mathcal{K} \} < \infty$* . We refine and improve this theorem in several directions. The theorem of De La Vallée Poussin does not, for instance, specify just how well the function  $F$  can be chosen. It gives little additional information in case the set in question is relatively norm compact in  $L^1(\mu)$ . Finally it gives no information on the structure of the set in the corresponding Banach space of  $F$ -integrable functions. More specifically we establish the fact that *a subset  $\mathcal{K}$  of  $L^1$  is relatively compact if and only if there is an  $N$ -function  $F \in \Delta'$  so that  $\mathcal{K}$  is relatively compact in  $L_F^*$  (Theorem 2.2)*. Furthermore we prove that *a subset  $\mathcal{K}$  of  $L^1$  is relatively weakly compact if and only if there is an  $N$ -function  $F \in \Delta'$  so that  $\mathcal{K}$  is relatively weakly compact in  $L_F^*$  (Theorem 2.5)*. In establishing this last result, a weak compactness criterion for Orlicz spaces was used (Theorem 2.3). The technique employed to prove this criterion was mainly averaging. Thus the natural question of Orlicz spaces and their relationship to Banach-Saks types of properties arises.

Recall that a Banach space  $X$  has the Banach-Saks (weak Banach-Saks) property if every bounded (weakly null) sequence in  $X$  has a subsequence, each subsequence of which, has norm convergent arithmetic means. Subsequently we show that a large class of non-reflexive

Orlicz spaces has the weak Banach-Saks property, by establishing a result for these spaces, very similar to the Dunford-Pettis Theorem for  $L^1$ . Specifically we show that if  $F \in \Delta_2$  and its complement  $G$  satisfies  $\lim_{t \rightarrow \infty} \frac{G(ct)}{G(t)} = \infty$  for some  $c > 0$ , then any weakly null sequence in  $L_F^*$  has equi-absolutely continuous norms (Theorem 2.8). As a corollary to this theorem we have that if  $F$  is as above then a bounded set in  $L_F^*$  is relatively weakly compact if and only if it has equi-absolutely continuous norms (Corollary 2.9). Furthermore, under the same hypothesis  $L_F^*$  has the weak Banach-Saks property (Corollary 2.10). These results complement the ones of T. Ando in [2]. We proceed to give an application in convex function theory by answering negatively the following question posed in [12, page 30]: Given an N-function  $F \in \Delta'$ , is it possible to find an N-function  $H$  equivalent to  $F$  so that  $H$  satisfies the  $\Delta'$  condition, for all real  $x, y$  ?

### 1.1 Some facts about N-Functions

Here we will summarize the necessary facts about a special class of convex functions called N-functions. For a detailed account of these facts, the reader could consult the first chapter in [12].

**Definition 1.1** Let  $p : [0, \infty) \rightarrow [0, \infty)$  be a right continuous, monotone increasing function with

1.  $p(0) = 0$ ;
2.  $\lim_{t \rightarrow \infty} p(t) = \infty$ ;
3.  $p(t) > 0$  whenever  $t > 0$ ;

then the function defined by

$$F(x) = \int_0^{|x|} p(t) dt$$

is called an *N*-function.

The following proposition gives an alternative view of *N*-functions.

**Proposition 1.1** *The function  $F$  is an *N*-function if and only if  $F$  is continuous, even and convex with*

1.  $\lim_{x \rightarrow 0} \frac{F(x)}{x} = 0$ ;
2.  $\lim_{x \rightarrow \infty} \frac{F(x)}{x} = \infty$ ;
3.  $F(x) > 0$  if  $x > 0$ .

**Definition 1.2** *For an *N*-function  $F$  define*

$$G(x) = \sup\{t|x| - F(t) : t \geq 0\} .$$

*Then  $G$  is an *N*-function and it is called the complement of  $F$ .*

Observe that  $F$  is the complement of its complement  $G$ .

**Theorem 1.2 (Young's Inequality)** *If  $F$  and  $G$  are two mutually complementary *N*-functions then*

$$xy \leq F(x) + G(y) \quad \forall x, y \in \mathbb{R} .$$

**Proposition 1.3** *The composition of two *N*-functions is an *N*-function. Conversely every *N*-function can be written as a composition of two other *N*-functions.*

The following material deals with the comparative growth of *N*-functions.

**Definition 1.3** *For *N*-functions  $F_1, F_2$  we write  $F_1 \prec F_2$  if there is a  $K > 0$  so that  $F_1(x) \leq F_2(Kx)$  for large values of  $x$ . If  $F_1 \prec F_2$  and  $F_2 \prec F_1$  then we say that  $F_1$  and  $F_2$  are equivalent.*

**Proposition 1.4** *If  $F_1 \prec F_2$  then  $G_2 \prec G_1$ , where  $G_i$  is the complement of  $F_i$ . In particular if  $F_1(x) \leq F_2(x)$  for large values of  $x$  then  $G_2(x) \leq G_1(x)$  for large values of  $x$ .*

**Definition 1.4** *A convex function  $Q$  is called the principal part of an  $N$ -function  $F$ , if  $F(x) = Q(x)$  for large  $x$ .*

**Proposition 1.5** *If  $Q$  is convex with  $\lim_{x \rightarrow \infty} \frac{Q(x)}{x} = \infty$  then  $Q$  is the principal part of some  $N$ -function.*

**Definition 1.5** *An  $N$ -function  $F$  is said to satisfy the  $\Delta_2$  condition ( $F \in \Delta_2$ ) if  $\limsup_{x \rightarrow \infty} \frac{F(2x)}{F(x)} < \infty$ . That is, there is a  $K > 0$  so that  $F(2x) \leq KF(x)$  for large values of  $x$ .*

**Definition 1.6** *An  $N$ -Function  $F$  is said to satisfy the  $\Delta'$  condition ( $F \in \Delta'$ ) if there is a  $K > 0$  so that  $F(xy) \leq KF(x)F(y)$  for large values of  $x$  and  $y$ .*

**Definition 1.7** *An  $N$ -function  $F$  is said to satisfy the  $\Delta_3$  condition ( $F \in \Delta_3$ ) if there is a  $K > 0$  so that  $xF(x) \leq F(Kx)$  for large values of  $x$ .*

**Definition 1.8** *An  $N$ -function  $F$  is said to satisfy the  $\Delta^2$  condition ( $F \in \Delta^2$ ) if there is a  $K > 0$  so that  $(F(x))^2 \leq F(Kx)$  for large values of  $x$ .*

**Theorem 1.6** *Let  $F$  be an  $N$ -function and let  $G$  be its complement; then the following hold.*

- *If  $F \in \Delta'$  then  $F \in \Delta_2$ .*
- *If  $F \in \Delta_3$  then its complement  $G \in \Delta_2$ .*
- *If  $F \in \Delta^2$  then its complement  $G \in \Delta'$ .*
- *If  $F \in \Delta_2$  then there is a  $p > 1$  so that if  $H(x) = |x|^p$  then  $F \prec H$ .*

*Finally the classes  $\Delta'$ ,  $\Delta_2$ ,  $\Delta_3$  and  $\Delta^2$  are preserved under equivalence of  $N$ -functions.*

The following proposition plays an important role for the results that follow.

**Proposition 1.7** *Given any N-function  $H$  there exists an N-function  $F \in \Delta'$  so that  $F(F(x)) \leq H(x)$  for large values of  $x$ .*

## 1.2 Some facts about Orlicz Spaces

In this section we summarize the necessary definitions and results about Orlicz spaces. A detailed account can be found in chapter two of [12]. Throughout this paper  $\mu$  is assumed to be a finite measure.

**Definition 1.9** *For an N-function  $F$  and a measurable  $f$  define*

$$\mathbf{F}(f) = \int F(f)d\mu.$$

*Let  $L_F = \{f \text{ measurable} : \mathbf{F}(f) < \infty\}$ . If  $G$  denotes the complement of  $F$  let*

$$L_F^* = \{f \text{ measurable} : |\int fg d\mu| < \infty \quad \forall g \in L_G\}.$$

*The collection  $L_F^*$  is then a linear space. For  $f \in L_F^*$  define*

$$\|f\|_F = \sup\{|\int fg d\mu| : \mathbf{G}(g) \leq 1\}.$$

*Then  $(L_F^*, \|\cdot\|_F)$  is a Banach space, called an Orlicz space. Moreover, letting  $\|\cdot\|_{(F)}$  be the Minkowski functional associated with the convex set  $\{f \in L_F^* : \mathbf{F}(f) \leq 1\}$ , we have that  $\|\cdot\|_{(F)}$  is an equivalent norm on  $L_F^*$ , called the Luxemburg norm. Indeed,  $\|f\|_{(F)} \leq \|f\|_F \leq 2\|f\|_{(F)}$ , for all  $f \in L_F^*$ .*

The following theorem establishes the fact that an Orlicz space is a dual space.

**Theorem 1.8** *Let  $F$  be an N-function and let  $E_F$  be the closure of the bounded functions in  $L_F^*$ . Then the conjugate space of  $(E_F, \|\cdot\|_{(F)})$  is  $(L_G^*, \|\cdot\|_G)$ , where  $G$  is the complement of  $F$ .*

**Theorem 1.9** *Let  $F$  be an  $N$ -function and  $G$  be its complement. Then the following statements are equivalent:*

1.  $L_F^* = E_F$ .
2.  $L_F^* = L_F$ .
3. *The dual of  $(E_F, \|\cdot\|_{(F)})$  is  $(L_G^*, \|\cdot\|_{(G)})$ .*
4.  $F \in \Delta_2$ .

**Theorem 1.10 (Hölder's Inequality)** *For  $f \in L_F^*$  and  $g \in L_G^*$  we have*

$$\int |fg| d\mu \leq \|f\|_F \cdot \|g\|_{(G)}.$$

**Theorem 1.11** *If  $f \in L_F^*$  then*

$$\|f\|_F = \inf \left\{ \frac{1}{k} (1 + \mathbf{F}(kf)) : k > 0 \right\}.$$

*It follows then that  $f \in L_F^*$  if and only if there is  $c > 0$  so that  $\mathbf{F}(cf) < \infty$ .*

**Proposition 1.12** *If  $\|f\|_F \leq 1$  then  $f \in L_F$  and  $\mathbf{F}(f) \leq \|f\|_F$ .*

Comparison of  $N$ -functions, gives rise to the following result concerning their corresponding Orlicz spaces.

**Proposition 1.13** *If  $F_1 \prec F_2$  then  $L_{F_2}^* \subset L_{F_1}^*$  and the inclusion mapping is continuous.*

**Definition 1.10** *We say that a collection  $\mathcal{K} \subset L_F^*$  has equi-absolutely continuous norms if*

$$\forall \varepsilon > 0 \exists \delta > 0 \text{ so that } \sup\{\|\chi_E f\|_F : f \in \mathcal{K}\} < \varepsilon \text{ whenever } \mu(E) < \delta.$$

*For  $f \in L_F^*$  we say that  $f$  has absolutely continuous norm if  $\{f\}$  has equi-absolutely continuous norms.*

The following two results deal with the equi-absolute continuity of the norms.

**Theorem 1.14** *A function  $f \in L_F^*$  has absolutely continuous norm if and only if  $f \in E_F$ .*

**Theorem 1.15** *If  $\mathcal{K} \subset L_F^*$ ,  $\mathcal{K}$  has equi-absolutely continuous norms and  $\mathcal{K}$  is relatively compact in the topology of convergence in measure, then  $\mathcal{K}$  is relatively (norm) compact in  $L_F^*$ .*

## 2 The Main results

### 2.1 De La Vallée Poussin's Theorem revisited

**Lemma 2.1** *If  $F \in \Delta_2$  and  $\mathcal{K} \subset L_F^*$  then the following statements are equivalent:*

- I) *The set  $\mathcal{K}$  has equi-absolutely continuous norms.*
- II) *The collection  $\{F(f) : f \in \mathcal{K}\}$  is uniformly integrable in  $L^1$ .*

*Proof :* The implication “(I)  $\Rightarrow$  (II)” follows directly from the fact that

$$\int_E F(f)d\mu = \int F(\chi_E f)d\mu = \mathbf{F}(\chi_E f) \leq \|\chi_E f\|_F$$

whenever  $\|\chi_E f\|_F \leq 1$ .

Next suppose  $\{F(f) : f \in \mathcal{K}\}$  is uniformly integrable. Let  $\varepsilon > 0$  and choose  $n \in \mathbb{N}$  so that  $\frac{1}{2^{n-1}} < \varepsilon$ . Since  $F \in \Delta_2$ , there are  $K > 0$ ,  $c > 0$  so that  $F(2^n x) \leq KF(x)$  for  $x \geq c$ . Choose  $0 < \delta < \frac{1}{2F(c)}$  so that

$$\sup \left\{ \int_E F(f)d\mu : f \in \mathcal{K} \right\} < \frac{1}{2K} \text{ whenever } \mu(E) < \delta.$$

Then for  $\mu(E) < \delta$ ,  $f \in \mathcal{K}$  we have

$$\begin{aligned} \int_E F(2^n f)d\mu &\leq \int_E F(c)d\mu + \int_{E \cap \{|f| \geq c\}} F(2^n f)d\mu \\ &< \frac{1}{2} + K \int_{E \cap \{|f| \geq c\}} F(f)d\mu < 1. \end{aligned}$$

Thus  $\|2^n f \chi_E\|_F \leq \int F(2^n f \chi_E) d\mu + 1 < 2$ . So  $\|f \chi_E\|_F < \frac{1}{2^{n-1}} < \varepsilon$ . ■

From this lemma we obtain the following characterization of norm compact subsets of  $L^1$ .

**Theorem 2.2** *A subset  $\mathcal{K}$  of  $L^1(\mu)$  is relatively compact if and only if there is an  $N$ -function  $F \in \Delta'$  so that  $\mathcal{K}$  is relatively compact in  $L_F^*$ .*

*Proof :* Since the inclusion map  $L_F^* \hookrightarrow L^1$  is continuous, sufficiency follows.

Suppose  $\mathcal{K}$  is relatively compact in  $L^1$ . Then  $\mathcal{K}$  is also relatively weakly compact in  $L^1$  and so by the theorem of De La Vallée Poussin there is an  $N$ -function  $H$  so that  $\sup \{\int H(f) d\mu : f \in \mathcal{K}\} < \infty$ . Choose now an  $N$ -function  $F \in \Delta'$  with  $F(F(x)) \leq H(x)$  for large values of  $x$ . Thus  $\sup \{\int F(F(f)) d\mu \mid f \in \mathcal{K}\} < \infty$  and by De La Vallée Poussin's theorem again, we have that  $\{F(f) \mid f \in \mathcal{K}\}$  is uniformly integrable in  $L^1$ . So by Lemma (2.1)  $\mathcal{K}$  has equi-absolutely continuous norms in  $L_F^*$ . Since  $\mathcal{K}$  is relatively compact in  $L^1$ , it is also relatively compact in the topology of convergence in measure. Hence  $\mathcal{K}$  is relatively compact in  $L_F^*$ . ■

The following result deals with relative weak compactness in  $L_F^*$ . We begin by mentioning a remarkable theorem of J. Komlós [11]: *If  $(f_n)$  is bounded in  $L^1$  then there is a subsequence  $(f_{n_k})$  of  $(f_n)$  and a function  $f \in L^1$  so that each subsequence of  $(f_{n_k})$  has arithmetic means  $\mu$ -a.e. convergent to  $f$ .*

**Theorem 2.3** *Let  $\mathcal{K} \subset L_F^*$ . If  $\mathcal{K}$  has equi-absolutely continuous norms and it is norm bounded, then  $\mathcal{K}$  is a Banach-Saks set in  $L_F^*$ . In particular  $\mathcal{K}$  is relatively weakly compact in  $L_F^*$ .*

*Proof :* Since  $\mathcal{K}$  has equi-absolutely continuous norms,  $\mathcal{K} \subset E_F$ . Let  $(f_n)$  be a sequence in  $\mathcal{K}$ . Since  $(f_n)$  is bounded in  $L_F^*$ -norm, it is also bounded in  $L^1$ -norm. Hence by Komlós's

theorem, there is a subsequence  $(f_{n_k})$  of  $(f_n)$  and a function  $f \in L^1$  so that any subsequence of  $(f_{n_k})$  has  $\mu$ -a.e. convergent arithmetic means to  $f$ . Let  $G$  denote the complement of  $F$ . Note that for any measurable  $E$  and any  $g \in L_G^*$  with  $\|g\|_{(G)} \leq 1$  we have

$$\begin{aligned}
\left| \int g \chi_E f d\mu \right| &\leq \int |g \chi_E f| d\mu \\
&\leq \liminf_n \int |g \chi_E \frac{1}{n} \sum_{k=1}^n f_{n_k}| d\mu \\
&\leq \sup_n \frac{1}{n} \sum_{k=1}^n \int |g \chi_E f_{n_k}| d\mu \\
&\leq \sup_n \frac{1}{n} \sum_{k=1}^n \|g\|_{(G)} \cdot \|\chi_E f_{n_k}\|_F \\
&\leq \sup\{\|\chi_E h\|_F : h \in \mathcal{K}\}.
\end{aligned}$$

Thus  $\|\chi_E f\|_F \leq \sup\{|\int g \chi_E f d\mu| : \|g\|_{(G)} \leq 1\} \leq \sup\{\|\chi_E h\|_F : h \in \mathcal{K}\}$ . So  $f \in L_F^*$  and  $f$  has absolutely continuous norm. Let  $(h_k)$  be any subsequence of  $(f_{n_k})$  and let  $a_n = \frac{1}{n} \sum_{k=1}^n h_k$ .

We now claim that  $a_n \rightarrow f$  in  $L_F^*$ -norm. Since the inclusion map  $L_G^* \hookrightarrow L^1$  is continuous, there is a  $K > 0$  so that  $\|g\|_1 \leq K\|g\|_{(G)}$  for all  $g \in L_G^*$ . Fix  $\varepsilon > 0$  and choose  $\delta > 0$  so that  $\sup\{\|\chi_A h\|_F : h \in \mathcal{K}\} < \frac{\varepsilon}{3}$  whenever  $\mu(A) < \delta$ .

By Egorov's theorem, there is a measurable set  $E$  with  $\mu(\Omega \setminus E) < \delta$  so that  $a_n \rightarrow f$  uniformly on  $E$ . Choose  $N \in \mathbb{N}$  so that  $\|\chi_E(a_n - f)\|_\infty < \frac{\varepsilon}{3K}$  whenever  $n \geq N$ . Then for any  $g \in L_G^*$  with  $\|g\|_{(G)} \leq 1$  and  $n \geq N$  we have

$$\begin{aligned}
\left| \int g(a_n - f) d\mu \right| &\leq \int |g| \cdot |a_n - f| d\mu \\
&= \int_E |g| \cdot |a_n - f| d\mu + \int_{\Omega \setminus E} |g| \cdot |a_n - f| d\mu \\
&\leq \|g\|_1 \cdot \|\chi_E(a_n - f)\|_\infty + \|g\|_{(G)} \cdot \|\chi_{\Omega \setminus E}(a_n - f)\|_F \\
&\leq K\|g\|_{(G)} \frac{\varepsilon}{3K} + \|g\|_{(G)} (\|a_n \chi_{\Omega \setminus E}\|_F + \|f \chi_{\Omega \setminus E}\|_F) \\
&< \frac{\varepsilon}{3} + \left\| \left( \frac{1}{n} \sum_{k=1}^n h_k \right) \chi_{\Omega \setminus E} \right\|_F + \frac{\varepsilon}{3}
\end{aligned}$$

$$\begin{aligned}
&\leq \frac{2\varepsilon}{3} + \frac{1}{n} \sum_{k=1}^n \|h_k \chi_{\Omega \setminus E}\| \\
&< \frac{2\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon.
\end{aligned}$$

So the claim is established.

Thus  $\mathcal{K}$  is a Banach-Saks set in  $L_F^*$ . It also follows that  $f_{n_k} \rightarrow f$  weakly in  $L_F^*$  and so  $\mathcal{K}$  is relatively weakly compact in  $L_F^*$  thanks to the Eberlein-Smulian theorem. ■

A. Grothendieck has shown that if  $1 \leq p < \infty$  and  $X$  is a closed subspace of  $L^p(\mu)$  contained in  $L^\infty(\mu)$ , then  $X$  is finite dimensional (see [7] and [18, ch. 5]). We generalize this result as follows.

**Theorem 2.4** *Suppose that  $X \subset L^\infty(\mu)$  and suppose that  $X$  is a closed subspace of an Orlicz space  $L_F^*$ . Then  $X$  is finite dimensional.*

*Proof :* Let  $i_1 : X \hookrightarrow L^\infty(\mu)$  and  $i_2 : L^\infty(\mu) \hookrightarrow L_F^*$  be the natural inclusion maps, with  $X$  having the topology inherited from  $L_F^*$ . Let  $(f_n)$  be a sequence in  $X$  and assume that  $\|f_n - f\|_F \rightarrow 0$  for some  $f \in X$ . Also assume that  $\|f_n - g\|_\infty \rightarrow 0$  for some  $g \in L^\infty$ . The first assumption yields a subsequence  $(f_{n_k})$  of  $(f_n)$  with  $f_{n_k} \rightarrow f$   $\mu - a.e.$ . Since  $f_n \rightarrow g$  uniformly  $\mu - a.e.$  we have that  $f = g$   $\mu - a.e.$ . Thus by the closed graph theorem  $i_1$  is continuous.

Now by Theorem (2.3)  $i_2$  is weakly compact and as  $L^\infty(\mu)$  has the Dunford-Pettis property,  $i_2$  is completely continuous. Hence  $i_2 \circ i_1$  is weakly compact and completely continuous. But  $i_2 \circ i_1$  is the identity on  $X$ . Now it is not hard to see that the identity on  $X$  is compact and hence  $X$  is finite dimensional. ■

We now prove the following stronger version of De La Vallée Poussin's theorem.

**Theorem 2.5** *A set  $\mathcal{K}$  is relatively weakly compact in  $L^1$  if and only if there is  $F \in \Delta'$  so*

that  $\mathcal{K}$  is relatively weakly compact in  $L_F^*$ .

*Proof:* Since the inclusion map  $L_F^* \hookrightarrow L^1$  is continuous and thus weak-to-weak continuous, sufficiency follows. So suppose that  $\mathcal{K}$  is relatively weakly compact in  $L^1$ . By De La Vallée Poussin's theorem, there is an  $N$ -function  $H$  with  $\sup\{\int H(f)d\mu : f \in \mathcal{K}\} < \infty$ . Let  $F \in \Delta'$  with  $F(F(x)) \leq H(x)$  for large  $x$ . So  $\sup\{\int F(F(f))d\mu : f \in \mathcal{K}\} < \infty$ , and by De La Vallée Poussin's theorem once more, we have that  $\{F(f) : f \in \mathcal{K}\}$  is relatively weakly compact in  $L^1$ . Hence by Lemma (2.1),  $\mathcal{K}$  has equi-absolutely continuous norms in  $L_F^*$ . Since  $\mathcal{K}$  is obviously bounded in  $L_F^*$ , we then have that  $\mathcal{K}$  is relatively weakly compact in  $L_F^*$ , thanks to Theorem (2.3). ■

**Remark:** If  $\mathcal{K} \subset L^1$  and if there is an  $N$ -function  $F$  with its complementary  $G \in \Delta_2$  so that  $\sup\{\int F(f)d\mu : f \in \mathcal{K}\} < \infty$  then  $\mathcal{K}$  is a bounded subset of  $L^p$  for some  $p > 1$ .

Indeed, if  $G \in \Delta_2$  then there is  $q > 1$  so that  $L^q \subset L_G^*$ . Let  $T : L^q \rightarrow L_G^*$  denote the natural inclusion map. Then if  $\frac{1}{p} + \frac{1}{q} = 1$  the adjoint operator  $T^* : L_F^* \rightarrow L^p$  is also a natural inclusion map. Since  $T$  is continuous so is  $T^*$ . Hence  $\mathcal{K}$  bounded in  $L_F^*$ , implies that  $\mathcal{K}$  is also bounded in  $L^p$ .

## 2.2 Orlicz Spaces and the weak Banach-Saks property

In this section we deal with a special class of non-reflexive Orlicz spaces, namely those spaces whose generating  $N$ -function  $F$  satisfies  $\Delta_2$  and the function  $G$  complementary to  $F$  satisfies  $\lim_{t \rightarrow \infty} \frac{G(ct)}{G(t)} = \infty$  for some  $c > 0$ . In particular we will show that these spaces satisfy the weak Banach-Saks property. This class of spaces has been examined by D. Leung [13]. At this point we should mention that V. A. Akimovich has shown in [1] that every reflexive Orlicz space over a probability is isomorphic to a uniformly convex Orlicz space. Combining

this result with Kakutani's result in [10] that states that uniformly convex spaces have the Banach-Saks property, one can immediately conclude that reflexive Orlicz spaces have the Banach-Saks property.

**Lemma 2.6** *Let  $\mathcal{K} \subset L_F^*$  where  $F \in \Delta_2$ . Suppose that  $\mathcal{K}$  fails to have equi-absolutely continuous norms. Then there is an  $\varepsilon_0 > 0$ , a sequence  $(f_n) \subset \mathcal{K}$  and a sequence  $(E_n)$  of pairwise disjoint measurable sets, so that  $\|\chi_{E_n} f_n\|_F > \varepsilon_0$  for all positive integers  $n$ .*

*Proof* : Since  $\mathcal{K}$  does not have equi-absolutely continuous norms, there is an  $\eta_0 > 0$  and sequences  $(k_n) \subset \mathcal{K}$ ,  $(A_n) \subset \Sigma$ , with  $\mu(A_n) < \frac{1}{2^n}$ , so that  $\|\chi_{A_n} k_n\|_F > \eta_0$  for all positive integers  $n$ . For each  $n$  let  $B_n = \bigcup_{j=n}^{\infty} A_j$ . Then  $B_n \supset B_{n+1}$ . Furthermore

$$\mu(B_n) = \mu\left(\bigcup_{j=n}^{\infty} A_j\right) \leq \sum_{j=n}^{\infty} \mu(A_j) \leq \sum_{j=n}^{\infty} \frac{1}{2^j} \rightarrow 0$$

as  $n \rightarrow \infty$ , with  $\|\chi_{B_n} k_n\|_F \geq \|\chi_{A_n} k_n\|_F > \eta_0$  for all positive integers  $n$ . Since  $F \in \Delta_2$  we have that each  $f \in L_F^*$  has absolutely continuous norm. So if  $n_1 = 1$  then there is  $n_2 > n_1$  so that  $\|\chi_{B_{n_1} \setminus B_{n_2}} k_{n_1}\|_F > \frac{\eta_0}{2}$  (After all  $\mu(B_n) \searrow 0$ ). Let  $E_1 = B_{n_1} \setminus B_{n_2}$  and let  $f_1 = k_{n_1}$ . Now choose  $n_3 > n_2$  so that  $\|\chi_{B_{n_2} \setminus B_{n_3}} k_{n_2}\|_F > \frac{\eta_0}{2}$ . Let  $E_2 = B_{n_2} \setminus B_{n_3}$  and let  $f_2 = k_{n_2}$ . Continue on. The result is now established if we take  $\varepsilon_0 = \frac{\eta_0}{2}$ . ■

We next present a ‘‘Rosenthal's Lemma’’ type of result. (cf. [5, page 82].)

**Lemma 2.7** *Let  $X$  be a Banach space. Suppose that  $(x_n) \subset X$  is weakly null and  $(x_n^*) \subset X^*$  is weak\* null. Then for each  $\varepsilon > 0$  there is a subsequence  $(n_k)$  of the positive integers, so that, for each positive integer  $k$  we have*

$$\sum_{j \neq k} |\langle x_{n_j}^*, x_{n_k} \rangle| < \varepsilon .$$

*Proof* : Let  $\varepsilon > 0$ . Let  $n_1 = 1$ . Since  $x_n^* \rightarrow 0$  weak\* there is an infinite subset  $A_1$  of the positive integers so that  $\sum_{j \in A_1} |\langle x_j^*, x_{n_1} \rangle| < \frac{\varepsilon}{2}$ . Since  $x_n \rightarrow 0$  weakly and since  $A_1$  is infinite, we can find  $n_2 > n_1$  with  $n_2 \in A_1$ , so that  $|\langle x_{n_1}^*, x_{n_2} \rangle| < \frac{\varepsilon}{2}$ . Similarly there is an

infinite subset  $A_2$  of  $A_1$  so that  $\sum_{j \in A_2} |\langle x_j^*, x_{n_2} \rangle| < \frac{\varepsilon}{2}$ . Again choose  $n_3 > n_2$  with  $n_3 \in A_2$  so that  $|\langle x_{n_1}^*, x_{n_3} \rangle| < \frac{\varepsilon}{4}$  and  $|\langle x_{n_2}^*, x_{n_3} \rangle| < \frac{\varepsilon}{4}$ . There is an infinite subset  $A_3$  of  $A_2$  so that  $\sum_{j \in A_3} |\langle x_j^*, x_{n_3} \rangle| < \frac{\varepsilon}{2}$ . Choose  $n_4 > n_3$  with  $n_4 \in A_3$  so that  $|\langle x_{n_i}^*, x_{n_4} \rangle| < \frac{\varepsilon}{6}$  for  $i = 1 \dots 3$ . Continue inductively to construct a sequence of infinite subsets of the positive integers,  $A_1 \supset A_2 \cdots \supset A_k \supset \cdots$  and a sequence  $n_1 < n_2 < \cdots$  of positive integers with

- (i)  $n_{k+1} \in A_k$  for all  $k$ .
- (ii)  $\sum_{j \in A_k} |\langle x_j^*, x_{n_{k+1}} \rangle| < \frac{\varepsilon}{2}$  for all  $k$ .
- (iii)  $|\langle x_{n_j}^*, x_{n_{k+1}} \rangle| < \frac{\varepsilon}{2k}$  for all  $k$  and for  $j = 1, 2, \dots, k$ .

Now for fixed positive integer  $k$  we have

$$\begin{aligned} \sum_{j \neq k} |\langle x_{n_j}^*, x_{n_k} \rangle| &= \sum_{j=1}^{k-1} |\langle x_{n_j}^*, x_{n_k} \rangle| + \sum_{j=k+1}^{\infty} |\langle x_{n_j}^*, x_{n_k} \rangle| \\ &< \frac{\varepsilon}{2(k-1)}(k-1) + \sum_{j \in A_k} |\langle x_{n_j}^*, x_{n_k} \rangle| \\ &< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon. \end{aligned}$$

And so we are done. ■

Now we are ready for the main result of this section.

**Theorem 2.8** *Suppose that  $F \in \Delta_2$  and that its complement  $G$  satisfies*

$$\lim_{t \rightarrow \infty} \frac{G(ct)}{G(t)} = \infty \text{ for some } c > 0.$$

*Then any weakly null sequence in  $L_F^*$  has equi-absolutely continuous norms.*

*Proof :* Suppose not. Then there is a weakly null sequence  $(f_n) \subset L_F^*$  that fails to have equi-absolutely continuous norms. Using Lemma (2.6) we may assume that there is an  $\varepsilon_0 > 0$  and a sequence  $(E_n)$  of pairwise disjoint measurable sets so that  $\|\chi_{E_n} f_n\|_F > \varepsilon_0$  for all positive integers  $n$ . Now choose a sequence  $(g_n) \subset L_G$  so that each  $g_n$  is supported on  $E_n$

with  $\int G(g_n) d\mu \leq 1$  and so that  $|\int g_n f_n d\mu| > \varepsilon_0$ . For a fixed  $f \in L_F^*$  Hölder's Inequality yields

$$|\int f g_n d\mu| = |\int \chi_{E_n} f g_n d\mu| \leq \|\chi_{E_n} f\|_F \cdot \|g_n\|_G.$$

But since  $(E_n)$  are pairwise disjoint and  $\mu$  is finite we have that  $\mu(E_n) \rightarrow 0$ . Furthermore since  $F \in \Delta_2$  and  $f \in L_F^*$ ,  $f$  has absolutely continuous norm. Thus  $\|\chi_{E_n} f\|_F \rightarrow 0$ . As  $(g_n)$  is norm bounded, we can conclude that  $\|\chi_{E_n} f\|_F \cdot \|g_n\|_G \rightarrow 0$  and so  $\int f g_n d\mu \rightarrow 0$ . Hence  $(g_n)$  is weak\* null. By Lemma (2.7) there is a subsequence  $(n_k)$  of the positive integers so that for each  $k$  we have  $\sum_{j \neq k} |\int g_{n_j} f_{n_k} d\mu| < \frac{\varepsilon_0}{2}$ .

We now claim that  $\int G(\frac{g_n}{c}) d\mu \rightarrow 0$ . Fix  $\varepsilon > 0$ . Since  $\lim_{t \rightarrow \infty} \frac{G(ct)}{G(t)} = \infty$  then  $\lim_{t \rightarrow \infty} \frac{G(t/c)}{G(t)} = 0$ . Choose  $t_0 > 0$  so that  $\frac{G(t/c)}{G(t)} < \frac{\varepsilon}{2}$  whenever  $t \geq t_0$ . Since  $\mu(E_n) \rightarrow 0$ , there is a positive integer  $N$  so that  $\mu(E_n) < \frac{\varepsilon}{2G(t_0/c)}$  whenever  $n \geq N$ . Hence if  $n \geq N$  we have

$$\begin{aligned} \int G(g_n/c) d\mu &= \int_{[|g_n| < t_0]} G(g_n/c) d\mu + \int_{[|g_n| \geq t_0]} G(g_n/c) d\mu \\ &\leq G(t_0/c) \mu(E_n) + \int \frac{\varepsilon}{2} G(g_n) d\mu \\ &< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon. \end{aligned}$$

So the claim is established.

Now choose a subsequence  $(n_{k_m})$  of  $(n_k)$  so that

$$\sum_{m=1}^{\infty} \int G(\frac{g_{n_{k_m}}}{c}) d\mu < \infty.$$

Let  $g = \sum_{m=1}^{\infty} g_{n_{k_m}}$ . Then  $g$  is well defined and  $g \in L_G^*$ , since  $\int G(g/c) d\mu < \infty$ . Since  $(f_n)$  is weakly null, we must have  $\int g f_{n_{k_m}} d\mu \rightarrow 0$  as  $m \rightarrow \infty$ . But for each positive integer  $m$  we have

$$|\int g f_{n_{k_m}} d\mu| = |\int (\sum_{j=1}^{\infty} g_{n_{k_j}}) f_{n_{k_m}} d\mu|$$

$$\begin{aligned}
&\geq \left| \int g_{n_{k_m}} f_{n_{k_m}} d\mu \right| - \sum_{j \neq m} \left| \int g_{n_{k_j}} f_{n_{k_m}} d\mu \right| \\
&\geq \left| \int g_{n_{k_m}} f_{n_{k_m}} d\mu \right| - \sum_{j \neq k_m} \left| \int g_{n_j} f_{n_{k_m}} d\mu \right| \\
&> \varepsilon_0 - \frac{\varepsilon_0}{2} = \frac{\varepsilon_0}{2},
\end{aligned}$$

which is a contradiction. ■

As a corollary to the theorem above, we get the following result that resembles the Dunford-Pettis theorem for  $L^1$ .

**Corollary 2.9** *Let  $F \in \Delta_2$  and suppose that its complement  $G$  satisfies*

$$\lim_{t \rightarrow \infty} \frac{G(ct)}{G(t)} = \infty \text{ for some } c > 0.$$

*Then a bounded set  $\mathcal{K} \subset L_F^*$  is relatively weakly compact if and only if  $\mathcal{K}$  has equi-absolutely continuous norms.*

*Proof:* Suppose that  $\mathcal{K} \subset L_F^*$  is relatively weakly compact. If  $\mathcal{K}$  fails to have equi-absolutely continuous norms then there is an  $\varepsilon_0 > 0$ , a sequence  $(f_n) \subset \mathcal{K}$  and a sequence  $(E_n)$  of measurable sets with  $\mu(E_n) \rightarrow 0$  so that  $\|\chi_{E_n} f_n\|_F > \varepsilon_0$ , for each positive integer  $n$ . By the Eberlein-Smulian theorem, there is an  $f \in L_F^*$  and a subsequence  $(f_{n_k})$  of  $(f_n)$  so that  $f_{n_k} \rightarrow f$  weakly in  $L_F^*$ . So by Theorem (2.8),  $(f_{n_k} - f)$  has equi-absolutely continuous norms. Thus  $\|\chi_{E_{n_k}}(f_{n_k} - f)\|_F \rightarrow 0$  as  $k \rightarrow \infty$ . As  $F \in \Delta_2$  and  $f \in L_F^*$ ,  $f$  has absolutely continuous norm. Hence  $\|\chi_{E_{n_k}} f\|_F \rightarrow 0$  as  $k \rightarrow \infty$ . But

$$\varepsilon_0 < \|\chi_{E_{n_k}} f_{n_k}\|_F \leq \|\chi_{E_{n_k}} f\|_F + \|\chi_{E_{n_k}}(f_{n_k} - f)\|_F$$

which is a contradiction.

The converse is just Theorem (2.3). ■

**Corollary 2.10** *Under the hypothesis of Corollary (2.9),  $L_F^*$  has the weak Banach-Saks property.*

*Proof* : It follows directly from Corollary (2.9) and Theorem (2.3). ■

Recall that an N-function  $G$  satisfies the  $\Delta_3$  condition if there is  $c > 0$  so that  $tG(t) \leq G(ct)$  for large values of  $t$ . If  $G \in \Delta_3$  then its complement  $F \in \Delta_2$  [12, pages 29–30]. Furthermore it is clear that  $\lim_{t \rightarrow \infty} \frac{G(ct)}{G(t)} = \infty$ . Also note that if  $G \in \Delta^2$  then  $G \in \Delta_3$ .

In [12, page 30] the following question is posed: Given an N-function  $F \in \Delta'$  is it possible to find an N-function  $H$ , equivalent to  $F$  so that for some  $K > 0$

$$H(xy) \leq K \cdot H(x) \cdot H(y) \quad \forall x, y \in \mathbb{R} ?$$

The following theorem answers this question in the negative.

**Theorem 2.11** *Suppose that  $G \in \Delta^2$  and let  $F$  denote the complement of  $G$ . Then there is no N-function  $H$  equivalent to  $F$  which satisfies the following condition:*

*There is a  $K > 0$  so that  $H(t_1 \cdot t_2) \leq K \cdot H(t_1) \cdot H(t_2)$  for all real  $t_1$  and  $t_2$ .*

*Proof* : Suppose that such an  $H$  existed. Let  $\mu$  denote Lebesgue measure on the interval  $[0, 1]$ . Since  $\mu$  is non-atomic, we can find a sequence  $(E_n)$  of pairwise disjoint measurable sets, each of which has positive measure. For each positive integer  $n$ , let  $h_n = H^{-1}(\frac{1}{\mu(E_n)})\chi_{E_n}$ . Then  $h_n \in L_H^*$  with  $\int H(h_n)d\mu = 1$  for all positive integers  $n$ . It follows from Lemma (2.1) that no subsequence of  $(h_n)$  has equi-absolutely continuous norms.

We now claim that  $(h_n)$  is weakly null. Let  $(h_{n_k})$  be any subsequence of  $(h_n)$ . Then for any positive integer  $N$  we have

$$\begin{aligned} \int H\left(\frac{1}{N} \sum_{k=1}^N h_{n_k}\right)d\mu &\leq K \cdot H\left(\frac{1}{N}\right) \cdot \int H\left(\sum_{k=1}^N h_{n_k}\right)d\mu \\ &= K \cdot H\left(\frac{1}{N}\right) \cdot \sum_{k=1}^N \int H(h_{n_k})d\mu \\ &= K \cdot H\left(\frac{1}{N}\right) \cdot \sum_{k=1}^N \frac{1}{\mu(E_{n_k})} \mu(E_{n_k}) \\ &= K \cdot H\left(\frac{1}{N}\right) \cdot N . \end{aligned}$$

Since  $H$  is an  $N$ -function,  $\lim_{t \rightarrow 0} \frac{H(t)}{t} = 0$ . Thus  $\lim_{N \rightarrow \infty} K \cdot H(\frac{1}{N}) \cdot N = 0$ . But since  $H \in \Delta'$  then  $H \in \Delta_2$ . So  $\| \frac{1}{N} \sum_{k=1}^N h_{n_k} \|_H \rightarrow 0$ . To summarize, every subsequence of  $(h_n)$  has norm null arithmetic means and so  $(h_n)$  is weakly null as we claimed. Now since  $F$  is equivalent to  $H$ , there are constants  $\lambda_1 > 0$  and  $\lambda_2 > 0$  so that

$$\lambda_1 \| f \|_F \leq \| f \|_H \leq \lambda_2 \| f \|_F \quad \text{for all } f \in L_H^*(= L_F^*).$$

By Theorem(2.8),  $(h_n)$  has equi-absolutely continuous  $F$ -norms and thus, by the inequality above,  $(h_n)$  also has equi-absolutely continuous  $H$ -norms.

But this is clearly a contradiction. ■

**Remark:** The same result can be obtained from the work of T. Ando in [2]. Specifically it follows directly from [2, Theorem 1], that *given  $F \in \Delta_2$ , a subset  $\mathcal{K}$  of  $L_F^*$  is relatively weakly compact, if and only if*

$$\lim_{t \rightarrow 0} (\sup \{ \frac{\mathbf{F}(tf)}{t} : f \in \mathcal{K} \}) = 0$$

With this fact in hand, we can easily prove the following theorem.

**Theorem 2.12** *Let  $F$  be an  $N$ -function satisfying the  $\Delta'$  condition for all real  $x, y$ . That is there is  $K > 0$  so that  $F(xy) \leq K \cdot F(x) \cdot F(y)$  for all  $x, y \in \mathbb{R}$ . Then  $L_F^*$  is reflexive.*

*Proof:* Since  $F \in \Delta'$  then  $F \in \Delta_2$ . Furthermore

$$\begin{aligned} \lim_{t \rightarrow 0} (\sup \{ \frac{\mathbf{F}(tf)}{t} : f \in B_{L_F^*} \}) &= \lim_{t \rightarrow 0} (\sup \{ \frac{\int_{\Omega} F(tf(\omega)) d\mu(\omega)}{t} : f \in B_{L_F^*} \}) \\ &\leq \lim_{t \rightarrow 0} (\sup \{ \frac{\int_{\Omega} K \cdot F(t) \cdot F(f(\omega)) d\mu(\omega)}{t} : f \in B_{L_F^*} \}) \\ &= \lim_{t \rightarrow 0} \frac{K \cdot F(t)}{t} = 0. \end{aligned}$$

Thus  $B_{L_F^*}$  is relatively weakly compact and so  $L_F^*$  is reflexive. ■

Now it is easy to see that given any N-function  $F \in \Delta'$  so that its complement  $G \notin \Delta_2$ , then there is no N-function  $H$  equivalent to  $F$  so that,  $H$  satisfies  $\Delta'$  for all real  $x, y$ .

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