Output feedback model predictive control for LPV systems using parameter-dependent Lyapunov function

S.M. Lee a, Ju H. Park b,*

a Platform Verification Division, BcN Business Unit, KT Co. Ltd., Daejeon, Republic of Korea
b Department of Electrical Engineering, Yeungnam University, 214-1 Daedong, Kyongsan 712-749, Republic of Korea

Abstract

In this paper, we consider a robust dynamic output feedback model predictive controller (MPC) design for linear parameter varying (LPV) systems. According to the proposed MPC algorithm, the control law is computed based on linear matrix inequality (LMI) at each sampling time by solving convex optimization problem. Also, a new parameter-dependent Lyapunov function is proposed to get a less conservative condition for stability of the system. The effectiveness of the result proposed is verified via a numerical example.

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1. Introduction

Model predictive control (MPC) technique has received much attention due to its many advantages. MPC can easily handle constrained systems and time-varying systems and provides good tracking performance. However, one of the drawbacks of MPC is its difficulty in incorporating plant model uncertainties. For this reason, robust MPC has received much attention in this research area [1–6]. In order to analyze the system, some researchers used a polytopic uncertain model whose system matrices are represented an affine function of uncertain parameters [1,2]. Recently, Cuzzola et al. [2] proposed a new robust MPC method for systems with polytopic description by using parameter-dependent Lyapunov function [4]. Especially, Lu and Arkun [8] proposed quasi-min–max MPC algorithms for LPV system whose parameters are measured in real-time. But in the case that the system matrices are expressed as rational function of parameters, the uncertainty of the polytopic model propagate over the prediction horizon. Moreover, a polytopic type parameter-dependent terminal weighting matrix such as that of Cuzzola et al. [2] for the MPC synthesis of rational parameter-dependent system cause to degradation of the performance. Therefore, we consider a linear fractional representation (LFR) to represent systems with more general uncertainties than polytopic ones [5].

* Corresponding author.
E-mail address: jessie@ynu.ac.kr (J.H. Park).

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In this paper, we propose a one horizon robust MPC method for LPV systems with LFR based on a new parameter-dependent Lyapunov function. This Lyapunov function matrix has generalized Lur’e type Lyapunov function structure [7]. At the prediction horizon, we also relax the stability condition for input constraint by using a parameter-dependent controller gain. Simulation result demonstrates the benefits of the proposed model predictive control methodology for the LPV systems.

Throughout the paper, \( \text{Tr} \) denotes the trace of given matrix. \( \text{diag} \{ \cdot \cdot \} \) represents the diagonal matrix. For real symmetric matrices \( X \) and \( Y \), the notation \( X \preceq Y \) (respectively, \( X \succeq Y \)) means that the matrix \( X - Y \) is positive semi-definite (respectively, positive definite).

2. Problem statement and main results

Consider the following linear parameter varying discrete-time systems:

\[
\begin{align*}
x(k+1) &= Ax(k) + B_1p(k) + B_2u(k), \\
p(k) &= C_1x(k) + D_{11}p(k), \\
y(k) &= C_2x(k) + D_{21}p(k) + D_{22}u(k), \\
p(k) &= A(k)p(k),
\end{align*}
\]

where \( x(k) \in \mathbb{R}^n \) is the state of the plant, \( u(k) \in \mathbb{R}^m \) is the control input, \( p(k) \) and \( q(k) \) are additional variable accounting for the uncertainty, and \( y(k) \in \mathbb{R}^p \) is the output of the linear block, respectively. \( A(k) \) is a time-varying matrix-valued parameter evolving in a polytopic set \( \Delta \), defined as

\[
\Delta = \{ A(k) | A(k) = \text{diag}[\delta_1(k)I, \delta_2(k)I, \ldots, \delta_l(k)I], \quad ||\delta_i(k)|| \leq 1, \quad i = 1, 2, \ldots, l, \quad \text{for all } k \in [0, \infty) \},
\]

and we assume that parameters \( \delta_i(k), \quad i = 1, \ldots, l \) are measured in real-time. Without loss of generality, to significantly simplify the synthesis, we assume that \( D_{22} \) is zero.

The objective of this paper is to find a control law with following representation for the system (1) at each time \( k \):

\[
\begin{align*}
x_c(k+1) &= A_c x_c(k) + B_c z(k), \\
u(k) &= C_c x_c(k),
\end{align*}
\]

where \( x_c(k) \in \mathbb{R}^n \) and \( A_c, B_c, C_c \) are matrices of appropriate dimensions to be determined later.

The problem is redefined to the determination of the matrices \( A_c, B_c, C_c \) so that the closed-loop system is stable. Then, the augmented system is represented by

\[
\begin{align*}
x(k+1) &= \tilde{A}x(k) + \tilde{B}p(k), \\
u(k) &= Kx(k), \\
q(k) &= \tilde{C}(x)(k) + \tilde{D}p(k),
\end{align*}
\]

where

\[
\begin{align*}
\tilde{A} &= \begin{bmatrix} A & B_2C_c \\ B_c & A_c \end{bmatrix} \in \mathbb{R}^{2n \times 2n}, \\
\tilde{B} &= \begin{bmatrix} B_1 \\ B_cD_{21} \end{bmatrix} \in \mathbb{R}^{2n \times m}, \\
\tilde{C} &= \begin{bmatrix} C_1 & 0 \end{bmatrix} \in \mathbb{R}^{p \times 2n}, \\
\tilde{D} &= D_{11}, \quad K = \begin{bmatrix} 0 & C_c \end{bmatrix}.
\end{align*}
\]

In order to guarantee stability of the system, we consider an infinite horizon MPC problem [4]. The control law, at each step \( k \), can be computed by minimizing the objective function given by

\[
J_\infty(k) = \sum_{i=0}^{\infty} [x(k+i|k)]^T \tilde{Q}[x(k+i|k) + u(k+i|k)]^T \tilde{R}[u(k+i|k)],
\]
where $R > 0$, $\bar{x}(k+i|k)$ denotes the augmented state predicted based on the measurements at time $k$, $\bar{x}(k|k) = \bar{x}(k)$ and
\[
\mathbf{Q} = \begin{bmatrix} Q & 0 \\ 0 & 0 \end{bmatrix}
\tag{6}
\]
with $Q \succeq 0$.

Let us consider a quadratic function:
\[
V(\bar{x}(k|k)) = \bar{x}(k|k)^T P(\Delta(k)) \bar{x}(k|k), \quad P(\Delta(k)) > 0,
\tag{7}
\]
where
\[
P(\Delta(k)) = \left[ \begin{array}{c} I \\ \Delta_a(k) \end{array} \right]^T P_a(k) \left[ \begin{array}{c} I \\ \Delta_a(k) \end{array} \right],
\]
\[
\Delta_a(k) = (I - \Delta(k)D_{11})^{-1} \Delta(k)C_1
\tag{8}
\]
for all $\Delta(k) \in \Delta$. It is the parameter-dependent representation and we suppose the parameter can be measured.

For the cost monotonicity, let us suppose $V$ satisfies the following inequality for all $x(k+i|k), u(k+i|k), i \geq 0$:
\[
V(k+i+1|k) - V(k+i|k) < -[x(k+i|k)^T \mathbf{Q} + u(k+i|k)^T R u(k+i|k)].
\tag{9}
\]

By summing (10) from $i = 0$ to $i = \infty$, we obtain
\[
-V(x(k|k)) \leq -J_\infty(k),
\tag{10}
\]
that means the quadratic function, $V$, can be an upper bound for the objective function (5). Thus our goal is redefined to the minimization of (7).

For simplicity, let denote that
\[
x = \bar{x}(k+j|k), \quad p = \Delta_a(k) \bar{x}(k+j|k), \quad w = \Delta_a(k+1) \bar{x}(k+j+1|k), \quad A = \tau I (\tau > 0).
\tag{11}
\]

Now, we have our main result.

**Theorem 1.** Consider the system (1) at time instant $k$. The dynamic output feedback control law (3) that minimize $J_\infty(k)$ can be solved by the following semi-definite programming:

\[
\begin{array}{c}
\min_{X,Y,Z,F,G,L,T} \quad \text{Tr}(P_a) \\
\end{array}
\begin{bmatrix}
-X & * & * & * & * & * & * & * & * & * \\
-I & -Y & * & * & * & * & * & * & * & * \\
0 & 0 & -Z^{-1} & * & * & * & * & * & * & * \\
0 & 0 & 0 & -A & * & * & * & * & * & * \\
G & YA + FC_2 & YB_1 + FD_{21} & 0 & -Y & * & * & * & * & * \\
AX + B_2L & A & 0 & 0 & -I & -X & * & * & * & * \\
0 & 0 & 0 & Z^{-1}A & 0 & 0 & -Z^{-1} & * & * & * \\
C_1AX + C_1B_2L & C_1A & C_1B_1 & D_{11}A & 0 & 0 & 0 & -A & * & * \\
Q^{1/2}X & Q^{1/2} & 0 & 0 & 0 & 0 & 0 & 0 & -I & * \\
R^{1/2}L & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -I
\end{bmatrix} < 0.
\tag{12}
\]

**Proof.** The minimization of the upper bound on the worst-case value of (5) can be solved by
\[
\begin{array}{c}
\min \quad \text{Tr}(P_a) \\
\end{array}
\begin{bmatrix}
-X & * & * & * & * & * & * & * & * & * \\
-I & -Y & * & * & * & * & * & * & * & * \\
0 & 0 & -Z^{-1} & * & * & * & * & * & * & * \\
0 & 0 & 0 & -A & * & * & * & * & * & * \\
G & YA + FC_2 & YB_1 + FD_{21} & 0 & -Y & * & * & * & * & * \\
AX + B_2L & A & 0 & 0 & -I & -X & * & * & * & * \\
0 & 0 & 0 & Z^{-1}A & 0 & 0 & -Z^{-1} & * & * & * \\
C_1AX + C_1B_2L & C_1A & C_1B_1 & D_{11}A & 0 & 0 & 0 & -A & * & * \\
Q^{1/2}X & Q^{1/2} & 0 & 0 & 0 & 0 & 0 & 0 & -I & * \\
R^{1/2}L & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -I
\end{bmatrix} < 0.
\tag{13}
\]

subject to the cost monotonicity condition (9).
Since \( p(k + j + 1|k) = \Delta(0(k + j + 1))q(k + j + 1|k) = \Delta(0(k + j + 1))\{C_1x(k + j + 1|k) + D_{11}p(k + j + 1|k)\}, \)

it can be written as

\[
p(k + j + 1|k) = A_a(k + j + 1)x(k + j + 1|k).
\]

Thus, the inequality (9) can be rewritten as follows by using the parameter-dependent matrix \( P(\theta(k + j)) \):

\[
\begin{bmatrix}
\bar{x} \\
\bar{p} \\
\bar{w}
\end{bmatrix}^T \left\{ \begin{bmatrix}
\bar{A}^T & 0 \\
\bar{B}^T & 0 \\
0 & I
\end{bmatrix} A_a \begin{bmatrix}
0 & 0 & 1 \\
0 & 0 & 1
\end{bmatrix} A_a^T \begin{bmatrix}
0 & 0 & 1 \\
0 & 0 & 1
\end{bmatrix} \right\} \begin{bmatrix}
\bar{x} \\
\bar{p} \\
\bar{w}
\end{bmatrix} \leq \begin{bmatrix}
Q + K^T R K & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & 0
\end{bmatrix} \begin{bmatrix}
\bar{x} \\
\bar{p} \\
\bar{w}
\end{bmatrix}.
\]

This inequality holds for all nonzero vector \([\bar{x}; \bar{p}; \bar{w}]\) satisfying

\[
w^T w = (\bar{A} \bar{x} + \bar{B} \bar{p})^T A_a^T (k + 1) A_a (k + 1) (\bar{A} \bar{x} + \bar{B} \bar{p}) = (\bar{C} \bar{A} \bar{x} + \bar{C} \bar{B} \bar{p} + \bar{D} w)^T A_a^T A(\bar{C} \bar{A} \bar{x} + \bar{C} \bar{B} \bar{p} + \bar{D} w) \leq (\bar{C} \bar{A} \bar{x} + \bar{C} \bar{B} \bar{p} + \bar{D} w)^T (\bar{C} \bar{A} \bar{x} + \bar{C} \bar{B} \bar{p} + \bar{D} w).
\]

Using S-procedure [9], it is easy to see that inequality (16) and constraint (17) are satisfied if

\[
\begin{bmatrix}
A^T P_a A - P_a + W & A^T P_a B \\
B^T P_a A & B P_a B - A^{-1}
\end{bmatrix} + \begin{bmatrix}
C^T D^T \end{bmatrix} A^{-1} [C \ D] < 0,
\]

where

\[
A = \begin{bmatrix}
\bar{A} & \bar{B} \\
0 & 0
\end{bmatrix}, \quad B = \begin{bmatrix}
0 \\
1
\end{bmatrix}, \quad C = \begin{bmatrix}
C \bar{A} & C \bar{B}
\end{bmatrix},
\]

\[
D = \bar{D}, \quad W = \begin{bmatrix}
\bar{Q} + K^T R K & 0 \\
0 & 0
\end{bmatrix}.
\]

By Congruence transformation with \(\text{diag}\{I, A, I, I\}\) after Schur’s complement [9], Eq. (18) is equivalent to the following inequality:

\[
\begin{bmatrix}
-P_a + W & * & * & * \\
0 & -A^{-1} & * & * \\
A & BA & -Q_a & * \\
C & DA & 0 & -A
\end{bmatrix} < 0,
\]

where \(Q_a = P_a^{-1}\).

Let us partition matrices \(Q_a\) and \(P_a\) in the form:

\[
Q_a = \begin{bmatrix}
X & U \\
U^T & \hat{X} \\
0 & 0 & Z
\end{bmatrix}, \quad P_a = \begin{bmatrix}
Y & V \\
V^T & \hat{Y} \\
0 & 0 & Z^{-1}
\end{bmatrix}
\]

and define the matrices:

\[
T_1 = \begin{bmatrix}
X & I & 0 \\
U^T & 0 & 0 \\
0 & 0 & I
\end{bmatrix}, \quad T_2 = \begin{bmatrix}
Y & I & 0 \\
V^T & 0 & 0 \\
0 & 0 & I
\end{bmatrix}, \quad T = \begin{bmatrix}
T_1 & 0 & 0 \\
0 & A & 0 \\
0 & 0 & T_2
\end{bmatrix}.
\]

By pre-multiplying and post-multiplying Eq. (20) by \(T_1\) and its transpose, respectively, we obtain the following:
Here, let us define
\[
F = VB_c, \quad L = C_cU^T, \quad G = YAX + FC_2X + YB_2L + VA_CU^T.
\] (24)

By substituting (24) into (23), we obtain the LMI (13).

By solving Eq. (24), we get the variables \(X, Y, Z, F, L\) and \(G\). Therefore, the parameters of the controller (3) is given by
\[
V = (I - XY)(U^T)^{-1}, \quad B_c = V^{-1}F, \quad C_c = L(U^T)^{-1}, \quad A_c = V^{-1}G(U^T)^{-1},
\] (25)

where \(G = G + YAX - FC_2X - YB_2L\). This completes the proof. \(\square\)
Example 1. Let us consider angular positioning system described in Fig. 1. It consists of the linear block described by

\[
x(k + 1) = \begin{bmatrix}
\theta(k + 1) \\
\dot{\theta}(k + 1)
\end{bmatrix} = \begin{bmatrix}
1 & 0.1 \\
0 & 1 - 0.1x(k)
\end{bmatrix} x(k) + \begin{bmatrix}
0 \\
0.08
\end{bmatrix} u(k) = A(k)x(k) + B(k)u(k),
\]

where \(x(k)\) is time-varying parameter.

Now, let us consider the following system which is interpreted as a structured uncertain model:

\[
A = \begin{bmatrix}
1 & 0.1 \\
0 & 0.495
\end{bmatrix}, \quad B_1 = \begin{bmatrix}
0 \\
0.08
\end{bmatrix}, \quad B_2 = \begin{bmatrix}
0 \\
-0.1
\end{bmatrix},
\]

\[
C_1 = \begin{bmatrix}
0 & 0.495
\end{bmatrix}, \quad C_2 = \begin{bmatrix}
1 & 0
\end{bmatrix}, \quad D_{11} = 0, \quad D_{21} = 0, \quad Q = \begin{bmatrix}
20 & 0 \\
0 & 20
\end{bmatrix}, \quad R = 0.1
\]

with the initial condition \(x(0) = [0.05 \quad 0]^T, x_c(0) = [0 \quad 0]^T\).

By solving the optimization problem given in Theorem 1 via MATLAB, we have the following simulation result. Fig. 2 shows the output and the control trajectory of the constrained MPC. The dashed line is the result of a MPC using a common Lyapunov function presented in [1] and the solid line shows the case of using the parameter-dependent Lyapunov function presented in this paper. Compared to the case of common Lyapunov function, the improved result can be obtain by the parameter-dependent Lyapunov function proposed in the work.

3. Conclusions

We have constructed a dynamic output feedback controller for LPV systems. The MPC algorithm have been adopted to control the LPV system. Also, we have proposed the parameter-dependent Lyapunov function. LMI technique has been used to solve the optimization problem. A numerical example has been represented performance of the parameter-dependent Lyapunov function and the result has shown rather improved closed-loop response than the case of a common Lyapunov function.

References