Finite Difference Approximation for Stochastic Optimal Stopping Problems with Delays [*]

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Abstract

This paper considers the computational issue of the optimal stopping problem for the stochastic functional differential equation treated in [4]. The finite difference method developed by Barles and Souganidis [2] is used to obtain a numerical approximation for the viscosity solution of the infinite dimensional Hamilton-Jacobi-Bellman variational inequality (HJBVI) associated with the optimal stopping problem. The convergence results are then established.

KEYWORDS: optimal stopping, stochastic control, stochastic functional differential equations, finite difference.

AMS 2000 subject classifications: primary: 60A09, 60H30; secondary 60G44, 90A16.

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1 Introduction

Optimal control and optimal stopping problems over a finite or an infinite time horizon for Itô's diffusion processes arise in many areas of science, engineering, and finance (see e.g. Fleming and Soner [17], Øksendal [41], Shiryaev [46], Karazas and Shreve [25] and references contained therein). Usually the system is described by stochastic differential equations to reflect the randomness. The objective is to find the optimal control or the optimal stopping time to maximize the payoff or to minimize the cost. The value functions of these problems are normally expressed as viscosity or generalized solutions of Hamilton-Jacobi-Bellman (HJB) equations or Hamilton-Jacobi-Bellman variational inequalities (HJBVIs) that involve second order parabolic or elliptic partial differential equations in finite dimensional Euclidean spaces (see e.g. Lions [36] and [37]).

Stochastic optimal control theory has been used widely in the area of portfolio management in which the stock price process is usually of the form

$$dS(t) = S(t)[\mu dt + \sigma dW(t)], \tag{1}$$

where $W(t)$ is a standard Brownian motion and the interest rate r is usually assumed to be a constant (see e.g. Fleming and Soner [17], Merton [38]). The HJB equations can be derived by virtue of dynamic programming principles. The value functions and the optimal control policy can be derived by solving the HJB equations analytically or numerically. While the HJB equations can be solved analytically only in few cases, the finite difference method can be used to solve the equation numerically in most cases. In addition, the Markov chain approximation method (see Kushner and Dupuis [32]) can also be used to solve this kind of problems. Other techniques to solve different type HJB equations can be found in various literatures, such as [23], [47], etc.

In the real world, the interest rate and the volatility of the stock price usually change randomly from time to time. There have been many researchers trying to accommodate those feature in their models. For example, in [16], [44] and [45] some portfolio management problems involving stochastic interest rate process were considered. In [15], [18], stochastic volatility was introduced in the option pricing and portfolio management problems. In [49], a stochastic interest rate model with stochastic volatility was investigated. There have been more literatures than we can list here. We encourage the reader to see [18], [17], [24] and the references cited therein for more literatures in this area.

On the other hand, in many real world applications (see Kolmanovskii and Shaikhet [30]), some physical systems can only be modeled by stochastic dynamical systems whose evolutions depend on the past history of the states. Stochastic systems describe by (1) apparently can not be used in those models. In stead, stochastic functional differential equations have to be used (see Mohammed [39], [40] for an introduction of these models). The linear-quadratic regulatory problem involving stochastic delay equations was first studied in Kolmanovskii and Maizenberg [29], and some stochastic control problems with delays have been the considered recently in Elsanousi [12], Larrssen [33, 34], Elsanousi et al [14], Øksendal and Sulem [43], Larrssen and Risebro [35], and Bauer and Rieder [4].

The controlled or uncontrolled stochastic delay equations considered by the aforementioned researchers are described by the following special class of equation that contain discrete and averaged delays:

$$\begin{aligned} dX(s) &= \alpha\Big(s, X(s), X(s-r), \int_{-r}^{0} e^{\lambda\theta} X(s+\theta)d\theta, u(s)\Big)ds \qquad (2) \\ &+ \beta\Big(s, X(s), X(s-r), \int_{-r}^{0} e^{\lambda\theta} X(s+\theta)d\theta, u(s)\Big)dW(s), \quad s \in [t,T]. \end{aligned}$$

In the above equation, $W(\cdot) = \{W(s), s \geq 0\}$ is an m-dimensional standard Brownian motion defined on a certain filtered probability space $(\Omega, \mathcal{F}, P, \mathbf{F})$, $u(\cdot) = \{u(s), s \in [t,T]\}$ is a control process taking values in the control set U in an Euclidean space, α and β are $\mathbf{R}^n$ and $\mathbf{R}^{n\times m}$-valued functions defined on

$$[0,T] \times \mathbf{R}^n \times \mathbf{R}^n \times \mathbf{R}^n \times U,$$

and $\lambda > 0$ is a given constant.

There are some literatures based on the above equation or its variants. Larssen [33, 34] obtained an HJB equation for an optimal control problem. Elsanousi et al [14] considered a singular control problem and obtained some explicit solutions. Øksendal and Sulem [43] derived the maximum principle for the optimal stochastic control. When the dynamics of the control problem with delay exhibit a special structure, Larssen and Risebro [35] and Bauer and Rieder [4] showed that the value function actually lives in a finite-dimensional space and the original problem can be reduced to a classical stochastic control problem without delay. Elsanousi and Larssen [13] treated an optimal control problem and its applications to consumption for (2) under partial observation.

In the authors' recent paper [8], a stochastic control system with delays in more general form is considered. The system is described by the following stochastic functional differential equation:

$$dX(s) = f(s, X_s, u(s))ds + g(s, X_s, u(s))dW(s), \quad \forall s \in [t, T], \tag{3}$$

where $X_s : [-r, 0] \to \mathbf{R}^n$ is defined by $X_s(\theta) = X(s + \theta), \theta \in [-r, 0]$ and r is a constant standing for the duration of the delay. Apparently, (3) contains (2) as a special case. In [8], we have derived the infinite dimensional HJB equation for the value function, and it is then showed that the value function is the unique viscosity solution of the HJB equation. In the authors' another paper [7], the finite difference method has been used to approximate the solution of the HJB equation.

Not only the theory of stochastic control has been widely used in the area of financial mathematics and financial engineering, the theory of optimal stopping for stochastic systems has also been widely used in those areas. One of the most important application is the valuation of the American options, in which the optimal time to sell the underlying asset needs to be determined. The price process of the underlying asset is usually modeled by a stochastic differential equation assuming that the information are available immediately to all investors. The value function of these problems are normally expressed as a generalized solution of a variational inequality that involves a second order parabolic partial differential equation. For general theory about optimal stopping, please see [46] and the references therein. There have been many literatures that consider the optimal stopping and its application in finance. For example, in [48], a linear complementarity problem arising from pricing American options was considered and the numerical scheme for solving the associated PDE was given. In [50], augmented Lagrangian Method (ALM) was used to value the price to the American options.

More recently, many researcher have considered the effects of the delayed information in pricing the financial derivatives, including the effects of delay in optimal stopping problem (see e.g. [1], [9], [10], [19], [20], [28], [26], [27], [42]). In most of the papers mentioned above, the system is usually given by

$$\begin{aligned} dX(s) &= \alpha\Big(s, X(s), X(s-r), \int_{-r}^{0} e^{\lambda\theta} X(s+\theta)d\theta\Big)ds \qquad (4) \\ &+ \beta\Big(s, X(s), X(s-r), \int_{-r}^{0} e^{\lambda\theta} X(s+\theta)d\theta\Big)dW(s), \quad s \in [t, T]. \end{aligned}$$

In the above equation, $W(\cdot) = \{W(s), s \geq 0\}$ is an m-dimensional standard Brownian motion defined on a certain filtered probability space $(\Omega, \mathcal{F}, P, \mathbf{F})$,

$u(\cdot) = \{u(s), s \in [t,T]\}$ is a control process taking values in the control set U in an Euclidean space, α and β are $\mathbf{R}^n$ and $\mathbf{R}^{n\times m}$-valued functions defined on or

$$[0,T] \times \mathbf{R}^n \times \mathbf{R}^n \times \mathbf{R}^n,$$

and $\lambda > 0$ is a given constant. We also mention that optimal stopping problems were studied in Elsanousi's unpublished dissertation [12] for such special type of equations. For systems described by (4), under certain conditions, the HJBVI for the value function can be derived in finite dimensional spaces.

Infinite dimensional HJBVIs for optimal stopping problems and their applications to pricing of American option have been studied very recently by a few researchers. They either considered stochastic delay equations of special form (4)(see e.g. Gapeev and Reiss [20] and [21]) or stochastic equations in Hilbert spaces (see e.g. Gatarek and Świech [19] and Barbu and Marinelli [2]).

In the authors' recent paper [6], we investigate an optimal stopping problem over a finite time horizon for a general system of stochastic functional differential equations described below:

$$dX(s) = f(s, X_s)dt + g(s, X_s)dW(s), \quad s \in [t,T], \tag{5}$$

where $T > 0$ and $t \in [0,T]$, respectively, denote the terminal time and an initial time of the optimal stopping problem. Again, $W(\cdot) = \{W(s), s \geq 0\}$ is a standard m-dimensional Brownian motion, and the drift $f(s, X_s)$ and the diffusion coefficient $g(s, X_s)$ (taking values in $\mathbf{R}^n$ and $\mathbf{R}^{n\times m}$, respectively) depend explicitly on the segment X_s of the state process $X(\cdot) = \{X(s), s \in [t-r, T]\}$ over the time interval $[s-r, s]$, where $X_s : [-r, 0] \to \mathbf{R}^n$ is defined by $X_s(\theta) = X(s+\theta), \theta \in [-r, 0]$. It is clear that this equation also includes (4) as a special case and many other equations that can not be modelled in the form of (4). The consideration of such a system enable us to model many real world problems that have aftereffects (see e.g. Kolmanovsky and Shaikhet [30] and references contained therein for application examples). In [6], we derive an infinite dimensional HJB variational inequality (HJBVI) for the value function. It is then showed that the value function is the unique viscosity solution of the HJBVI. The proof of uniqueness involves embedding the function space $\mathbf{C} = C([-r,0]; \mathbf{R}^n)$ into the Hilbert space $L^2([-r,0]; \mathbf{R}^n)$ and extending the concept of viscosity solution for controlled Itô's diffusion process developed by Crandall et al [11], and Lions [36] and [37] to an infinite dimensional setting. The segmented solution process $\{X_s, s \in [t,T]\}$

is a strong Markov process in the Banach space $\mathbf{C}$ whose norm is not differentiable and is therefore more difficult to handle than any Hilbert space considered in [19] and [2].

This paper gives a numerical method for the optimal stopping problems studied in [6]. The method we used here is the finite difference method which was introduced in Barles and Souganidis [3]. Similar method was used to deal with the general stochastic control problems with delay in the authors' recent work [7]. Recently, Kushner [31] gives some numerical approximation results for general stochastic control problems for a stochastic functional differential equation using Markov Chain approximation method, which is entirely different from ours in the techniques and control problem considered.

The rest of this paper is organized as follows. In Section 2, the formulation of the optimal stopping problem and the associated HJBVI are given. In addition, the existence and unique results for the viscosity solution of the HJBVI obtained in [4] are re-stated in Section 2. In Section 3, the numerical approximation of the viscosity solution of the infinite dimensional HJB variational inequality, along the line of finite difference method developed by Barles and Souganidis [20] is obtained and the convergence results are proved. A computational algorithm is given in Section 4.

2 Problem Formulation

Let $r > 0$ be a fixed constant, and let $\mathbf{J} = [-r, 0]$ denote the duration of the bounded memory (delay) of the stochastic functional differential equations considered in this paper. For the sake of simplicity, we denote $C(\mathbf{J}; \mathbb{R}^n)$, the space of continuous functions $\phi : \mathbf{J} \to \mathbb{R}^n$, by $\mathbf{C}$. Note that $\mathbf{C}$ is a real separable Banach space under the supremum-norm defined by

$$\|\phi\| = \sup_{t \in \mathbf{J}} |\phi(t)|, \quad \phi \in \mathbf{C}$$

where $|\cdot|$ is the Euclidean norm in $\mathbb{R}^n$.

We denote the inner product in $L^2(\mathbf{J}, \mathbb{R}^n)$ by $(\cdot \mid \cdot)$ and the inner product in $\mathbb{R}^n$ by $\langle \cdot, \cdot \rangle$. Given ϕ and ψ in $\mathbf{C}$, the inner product $(\cdot \mid \cdot)$ and the norm $\|\cdot\|_2$ for $L^2(\mathbf{J}, \mathbb{R}^n)$ are defined by

$$(\phi|\psi) = \int_{-r}^{0} \langle \phi(s), \psi(s) \rangle ds, \qquad \text{and } \|\phi\|_2 = (\phi|\phi)^{\frac{1}{2}}.$$

Note that the space $\mathbf{C}$ can be continuously embedded into $L^2(\mathbf{J}; \mathbb{R}^n)$.

Convention 2.1 *For the rest of the paper, we use the following conventional notation for functional differential equations (see Hale [22]):*

If $\psi \in C([-r,\infty);\mathbb{R}^n)$ and $t \in \mathbb{R}_+$, let $\psi_t \in \mathbf{C}$ be defined by

$$\psi_t(\theta) = \psi(t+\theta), \quad \forall \theta \in \mathbf{J}.$$

Let $\{W(t), t \geq 0\}$ be an m-dimensional standard Brownian motion defined on a complete filtered probability space $(\Omega, \mathcal{F}, P; \mathbf{F})$, where $\mathbf{F} = \{\mathcal{F}(t), t \geq 0\}$ is the P-augmentation of the natural filtration $\{\mathcal{F}^W(t), t \geq 0\}$ generated by the Brownian motion $\{W(t), t \geq 0\}$, *i.e.,* if $t \geq 0$,

$$\mathcal{F}^W(t) = \sigma\{W(s), 0 \leq s \leq t\}$$

and

$$\mathcal{F}(t) = \mathcal{F}^W(t) \vee \{A \subset \Omega | \exists B \in \mathcal{F} \text{ such that } A \subset B \text{ and } P(B) = 0\}$$

where the operator $\vee$ denotes that $\mathcal{F}(t)$ is the smallest σ-algebra such that $\mathcal{F}^W(t) \subset \mathcal{F}(t)$ and $\{A \subset \Omega | \exists B \in \mathcal{F} \text{ such that } A \subset B \text{ and } P(B) = 0\} \subset \mathcal{F}(t)$.

Let $T > 0$ and $t \in [0,T]$. Consider the following system of stochastic functional differential equations:

$$dX(s) = f(s, X_s)ds + g(s, X_s)dW(s), \quad s \in [t,T]; \tag{6}$$

with the initial function $X_t = \psi_t$, where ψ_t is a given $\mathbf{C}$-valued random variable that is $\mathcal{F}(t)$-measurable. Here, $f : [0,T] \times \mathbf{C} \to \mathbb{R}^n$ and $g : [0,T] \times \mathbf{C} \to \mathbb{R}^{n \times m}$ are given deterministic functions.

Let $L^2(\Omega, \mathbf{C})$ be the space of $\mathbf{C}$-valued random variables $\Xi : \Omega \to \mathbf{C}$ such that

$$\|\Xi\|_{L^2} = \left\{ \int_\Omega \|\Xi(\omega)\|^2 dP(\omega) \right\}^{\frac{1}{2}} < \infty.$$

Let $L^2(\Omega, \mathbf{C}; \mathcal{F}(t))$ be those $\Xi \in L^2(\Omega, \mathbf{C})$ which are $\mathcal{F}(t)$-measurable.

Definition 2.2 *A process $\{X(s; t, \psi_t), s \in [t-r, T]\}$ is said to be a (strong) solution of (6) on the interval $[t-r, T]$ and through the initial datum $(t, \psi_t) \in \mathbb{R}_+ \times L^2(\Omega, \mathbf{C}; \mathcal{F}(t))$ if it satisfies the following conditions:*

1. *$X_t(\theta; t, \psi_t) = \psi_t(\theta), \quad \forall\theta \in [-r, 0]$;*

2. *$X(s; t, \psi_t)$ is $\mathcal{F}(s)$-measurable for each $s \in [t, T]$;*

3. *The process $\{X(s; t, \psi_t), s \in [t, T]\}$ is continuous and it satisfies the following stochastic integral equation P-a.s.*

$$X(s) = \psi_t(0) + \int_t^s f(\lambda, X_\lambda)d\lambda + \int_t^s g(\lambda, X_\lambda)dW(\lambda), \; s \in [t, T]. \quad (7)$$

In addition, the solution process $\{X(s; t, \psi_t), s \in [t-r, T]\}$ is said to be (strongly) unique if $\{\tilde{X}(s; t, \psi_t), s \in [t-r, T]\}$ is also a solution of (1) on $[t-r, T]$ and through the same initial datum (t, ψ_t), then

$$P\{X(s; t, \psi_t) = \tilde{X}(s; t, \psi_t), \forall s \in [t, T]\} = 1.$$

In this paper, we assume that the functions $f : [0, T] \times \mathbf{C} \to \mathbb{R}^n$, and $g : [0, T] \times \mathbf{C} \to \mathbb{R}^{n \times m}$ are continuous functions and satisfy the following conditions (Assumption 2.3 & 2.4) to ensure the existence and uniqueness of a global (strong) solution $\{X(s; t, \psi_t), s \in [t-r, T]\}$ for each $(t, \psi_t) \in [0, T] \times L^2(\Omega, C; \mathcal{F}(t))$. (See Mohammed [39, 40].)

Assumption 2.3 *There exists a constant $K > 0$ such that*

$$|f(t, \varphi) - f(t, \phi)| + |g(t, \varphi) - g(t, \phi)| \leq K\|\varphi - \phi\|, \; \forall(t, \varphi), (t, \phi) \in [0, T] \times \mathbf{C}.$$

Assumption 2.4 *There exists a constant $K > 0$ such that*

$$|f(t, \phi)| + |g(t, \phi)| \leq K(1 + \|\phi\|), \; \forall(t, \phi) \in [0, T] \times \mathbf{C}.$$

Let $\{X(s; t, \psi_t), s \in [t, T]\}$ be the solution of (6) through the initial datum $(t, \psi_t) \in [0, T] \times \mathbf{C}$. We consider the corresponding $\mathbf{C}$-valued process $\{X_s(t, \psi_t), s \in [t, T]\}$ defined by

$$X_s(\theta; t, \psi_t) = X(s + \theta; t, \psi_t), \; \theta \in \mathbf{J}. \quad (8)$$

Let $\mathbf{G}(t) = \{\mathcal{G}(t, s), s \in [t, T]\}$ be the filtration defined by

$$\mathcal{G}(t, s) = \sigma(X(\lambda; t, \psi_t), \lambda \in [t, s]).$$

Note that, it can be shown that for each $s \in [t, T]$,

$$\mathcal{G}(t, s) = \sigma(X_\lambda(t, \psi_t), \lambda \in [t, s]).$$

This is due to the sample path continuity of the process $\{X(s;t,\psi_t), s \in [t,T]\}$.

One can then establish the following Markov property (see Mohammed [39], [40]):

Theorem 2.5 *Let Assumptions 2.3 and 2.4 hold. Then the corresponding* $\mathbf{C}$*-valued solution process of (6) describes a* $\mathbf{C}$*-valued Markov process in the following sense:*

For any $(t,\psi_t) \in [0,T] \times L^2(\Omega, \mathbf{C})$*, the Markovian property*

$$P\{X_s(t,\psi_t) \in B | \mathcal{G}(t,\alpha)\} = P\{X_s(t,\psi_t) \in B | X_\alpha(t,\psi_t)\} \equiv p(\alpha, X_\alpha(t,\psi_t), s, B)$$

holds a.s. for $t \leq \alpha \leq s$ *and* $B \in \mathcal{B}(\mathbf{C})$*, where* $\mathcal{B}(\mathbf{C})$ *is the Borel* σ*-algebra of subsets of* $\mathbf{C}$*.*

In the above, the function $p : [t,T] \times \mathbf{C} \times [t,T] \times \mathcal{B}(\mathbf{C}) \to [0,1]$ denotes the transition probabilities of the $\mathbf{C}$-valued Markov process $\{X_s(t,\psi_t), s \in [t,T]\}$.

Let $\mathcal{T}_t^T$ be the collection of all $\mathbf{G}(t)$-stopping times $\tau : \Omega \to [0,\infty]$ such that $t \leq \tau \leq T$ a.s. We write $\mathcal{T}_t^T = \mathcal{T}$ when $t = 0$ and $T = \infty$. For each $\tau \in \mathcal{T}_t^T$, let the sub-σ-algebra $\mathcal{G}(t,\tau)$ of $\mathcal{F}$ be defined by

$$\mathcal{G}(t,\tau) = \{A \in \mathcal{F} \mid A \cap \{t \leq \tau \leq s\} \in \mathcal{G}(t,s) \ \forall s \in [t,T]\}.$$

With a little bit more effort, one can also show that the corresponding $\mathbf{C}$-valued process of (6) is also a strong Markov process in $\mathbf{C}$. That is,

$$P\{X_s(t,\psi_t) \in B | \mathcal{G}(t,\tau)\} = P\{X_s(t,\psi_t) \in B | X_\tau(t,\psi_t)\} \equiv p(\tau, X_\tau(t,\psi_t), s, B)$$

holds a.s. for all $\tau \in \mathcal{T}_t^T$ and $B \in \mathcal{B}(\mathbf{C})$.

If the drift coefficient f and the diffusion coefficient g are time-independent, *i.e.*, $f(t,\phi) \equiv f(\phi)$ and $g(t,\phi) \equiv g(\phi)$, then (6) reduces to the following autonomous system:

$$dX(s) = f(X_s)ds + g(X_s)dW(s). \tag{9}$$

In this case, we usually assume the initial datum $(t,\psi_t) = (0,\psi)$ and denote the solution process of (9) through $(0,\psi)$ and on the interval $[-r,T]$ by

$\{X(s;\psi), s \in [-r,T]\}$. Then the corresponding $\mathbf{C}$-valued process $\{X_s(\psi), s \in [-r,T]\}$ of (9) is a strong Markov process with time-homogeneous probability transition probabilities $p(\psi, s, B) \equiv p(0, \psi, s, B) = p(t, \psi, t+s, B)$ for all $s, t \geq 0$, $\psi \in \mathbf{C}$, and $B \in \mathcal{B}(\mathbf{C})$.

Assume that L and Ψ are two Lipschitz continuous real-valued functions on $[0,T] \times \mathbf{C}$ with at most linear growth in $L^2(\mathbf{J};\mathbb{R}^n)$. In other words, there exist constants K_1, K_2 such that

$$|L(t,\phi)| \leq K_1(1 + \|\phi\|_2), \qquad \text{and} \quad |\Psi(t,\phi)| \leq K_2(1 + \|\phi\|_2),$$

for all $(t,\phi) \in [0,T] \times \mathbf{C}$.

Our objective is to find an optimal stopping time $\tau^* \in \mathcal{T}_t^T$ that maximizes the following expected cost functional:

$$J(\tau;t,\psi) = \mathbf{E}\left[\int_t^\tau e^{-\rho(s-t)} L(s, X_s)ds + e^{-\rho(\tau-t)}\Psi(\tau, X_\tau)\right], \tag{10}$$

where $\rho > 0$ denotes a discount factor. In this case, the value function $V : [0,T] \times \mathbf{C} \to \mathbb{R}$ is defined to be

$$V(t,\psi) = \sup_{\tau \in \mathcal{T}_t^T} J(\tau;t,\psi). \tag{11}$$

Before we give the HJBVI satisfied by $V(t,\psi)$, we need to introduce some spaces and operators. Define $\mathbf{C}^*$ and $\mathbf{C}^\dagger$ as the space of bounded linear functionals $\Phi : \mathbf{C} \to \mathbb{R}$ and bounded bilinear functionals $\tilde{\Phi} : \mathbf{C} \times \mathbf{C} \to \mathbb{R}$, of the space $\mathbf{C}$, respectively. They are equipped with the dual norms which will be, respectively, denoted by $\|\cdot\|^*$ and $\|\cdot\|^\dagger$.

Let $\mathbf{B} = \{v\mathbf{1}_{\{0\}}, v \in \mathbb{R}^n\}$, where $\mathbf{1}_{\{0\}} : [-r,0] \to \mathbb{R}$ is defined by

$$\mathbf{1}_{\{0\}}(\theta) = \begin{cases} 0 & \text{for } \theta \in [-r,0), \\ 1 & \text{for } \theta = 0. \end{cases}$$

We form the direct sum

$$\mathbf{C} \oplus \mathbf{B} = \{\phi + v\mathbf{1}_{\{0\}} \mid \phi \in \mathbf{C}, v \in \mathbb{R}^n\}$$

and equip it with the norm $\|\cdot\|$ defined by

$$\|\phi + v\mathbf{1}_{\{0\}}\| = \sup_{\theta \in [-r,0]} |\phi(\theta)| + |v|, \quad \phi \in \mathbf{C}, v \in \mathbb{R}^n.$$

Note that for each sufficiently smooth function $\Phi : \mathbf{C} \to \mathbb{R}$, its first order Fréchet derivative (with respect to $\phi \in \mathbf{C}$), $D\Phi(\varphi) \in \mathbf{C}^*$, has a unique and continuous linear extension $\overline{D\Phi(\varphi)} \in (\mathbf{C} \oplus \mathbf{B})^*$. Similarly, its second order Fréchet derivative, $D^2\Phi(\varphi) \in \mathbf{C}^\dagger$, has the unique and continuous linear extension $\overline{D^2\Phi(\varphi)} \in (\mathbf{C} \oplus \mathbf{B})^\dagger$, where $(\mathbf{C} \oplus \mathbf{B})^*$and $(\mathbf{C} \oplus \mathbf{B})^\dagger$ are spaces of bounded linear and bilinear functionals of $\mathbf{C} \oplus \mathbf{B}$, respectively. (See Lemma (3.1) and Lemma (3.2) on pp 79-83 of Mohammed [39] for details).

For a Borel measurable function $\Phi : \mathbf{C} \to \mathbb{R}$, we also define

$$\mathcal{S}(\Phi)(\phi) = \lim_{h \to 0+} \frac{1}{h} \left[\Phi(\tilde{\phi}_h) - \Phi(\phi) \right]$$

for all $\phi \in \mathbf{C}$, where $\tilde{\phi} : [-r, T] \to \mathbb{R}^n$ is an extension of ϕ defined by

$$\tilde{\phi}(t) = \begin{cases} \phi(t) & \text{if } t \in [-r, 0), \\ \phi(0) & \text{if } t \geq 0, \end{cases}$$

and again $\tilde{\phi}_t \in \mathbf{C}$ is defined by

$$\tilde{\phi}_t(\theta) = \tilde{\phi}(t + \theta), \quad \theta \in [-r, 0].$$

Let $\hat{\mathcal{D}}(\mathcal{S})$, the domain of the operator $\mathcal{S}$, be the set of $\Phi : \mathbf{C} \to \mathbb{R}$ such that the above limit exists for each $\phi \in \mathbf{C}$.

Define $C^{1,2}_{lip}([0, T] \times \mathbf{C})$ as the space of functions $\Phi : [0, T] \times \mathbf{C} \to \mathbf{R}$ such that $\frac{\partial \Phi}{\partial t} : [0, T] \times \mathbf{C} \to \mathbb{R}$ and $D^2\Phi : [0, T] \times \mathbf{C} \to \mathbf{C}^\dagger$ exist and are continuous and satisfy the following Lipschitz condition:

$$\|D^2\Phi(t, \phi) - D^2\Phi(t, \varphi)\|^\dagger \leq K\|\phi - \varphi\|, \quad \forall t \in [0, T], \ \forall \phi, \varphi \in \mathbf{C},$$

where $K > 0$ is a constant.

Let $\mathcal{D}(\mathcal{S})$ be the collection of those $\Phi : [0, T] \times \mathbf{C} \to \mathbb{R}$ such that $\Phi(t, \cdot) \in \hat{\mathcal{D}}(\mathcal{S})$ for each $t \in [0, T]$.

The following result has been proved in [6].

Theorem 2.6 *The value function V is the unique viscosity solution of the HJBVI*

$$\max \left\{ \Psi(t, \psi) - V(t, \psi), \frac{\partial V}{\partial t}(t, \psi) + \mathcal{A}V(t, \psi) + L(t, \psi) - \rho V(t, \psi) \right\} = 0, \quad \textit{for all} \quad (t, \psi) \in [0, T) \times \mathbf{C}. \tag{12}$$

and $V(T,\psi) = \Psi(T,\psi)$ *for all* $\psi \in \mathbf{C}$*, where* $\mathcal{A}$ *is given by*

$$\begin{aligned} \mathcal{A}V(t,\psi) &= \mathcal{S}(V)(t,\psi) + \overline{DV(t,\psi)}(f(t,\psi)\mathbf{1}_{\{0\}}) \\ &\quad + \frac{1}{2}\sum_{i=1}^{m} \overline{D^2V(t,\psi)}(g(t,\psi)\mathbf{e}_i\mathbf{1}_{\{0\}}, g(t,\psi)\mathbf{e}_i\mathbf{1}_{\{0\}}), \qquad (13) \end{aligned}$$

where $\mathbf{e}_i$ *is the* i*-th unit vector of the standard basis in* $\mathbb{R}^m$*.*

3 Finite Difference Approximation

In this section, we consider an explicit finite difference scheme and show that it converges to the unique viscosity solution of equation (12). We will use a method introduced by Barles and Souganidis [3].

To obtain the existence results of the finite difference equation which will be given later, we will use the Banach fixed point theorem. Therefore, it will be more convenient to consider the bounded functionals. For this reason, we will use a truncated optimal stopping problem as the follows.

Given a positive integer M, we consider the following truncated optimal stopping problem with value function $V_M : [0,T] \times \mathbf{C} \to \mathbb{R}$

$$\begin{aligned} V_M(t,\psi) &= \sup_{\tau \in \mathcal{T}_t^T} \mathbf{E}\Big[\int_t^\tau e^{-\rho(s-t)}(L(s,X_s) \wedge M)ds \\ &\qquad + e^{-\rho(\tau-t)}(\Psi(\tau,X_\tau) \wedge M)\Big], \qquad (14) \end{aligned}$$

where $a \wedge b$ is defined by $a \wedge b = \min\{a,b\}$ for all $a,b \in \mathbb{R}$.

The corresponding HJBVI is given by

$$\begin{aligned} \min\Big\{ & V_M(t,\psi) - (\Psi(t,\psi) \wedge M), \rho V_M(t,\psi) - \frac{\partial V_M}{\partial t}(t,\psi) - \mathcal{A}V_M(t,\psi) \\ & -(L(t,\psi) \wedge M)\Big\} = 0, \qquad \forall (t,\psi) \in [0,T] \times \mathbf{C}, \quad (15) \end{aligned}$$

and $V_M(T,\psi) = \min(\Psi(T,\psi), M), \ \forall \psi \in \mathbf{C}$.

Similarly as in [6], it can be showed that the value function V_M is the unique viscosity solution of the equation (15).

Moreover, it is easy to see that $V_M \to V$ as $M \to \infty$. In view of these, we need only find the numerical solution for V_M.

Let ε with $0 < \varepsilon < 1$ be the stepsize for variables ψ and η, where $0 < \eta < 1$ is the stepsize for t. We consider the finite difference operators Δ_η, Δ_ε and Δ_ε^2 defined by

$$
\begin{aligned}
\Delta_\eta W(t,\psi) &\equiv \frac{W(t+\eta,\psi) - W(t,\psi)}{\eta}, \\
\Delta_\varepsilon W(t,\psi)(h + v\mathbf{1}_{\{0\}}) &\equiv \frac{W(t,\psi+\varepsilon(h+v\mathbf{1}_{\{0\}})) - W(t,\psi)}{\varepsilon}, \\
\Delta_\varepsilon^2 W(t,\psi)(h+v\mathbf{1}_{\{0\}}, k+w\mathbf{1}_{\{0\}}) &\equiv \frac{W(t,\psi+\varepsilon(h+v\mathbf{1}_{\{0\}})) - W(t,\psi)}{\varepsilon^2} \\
&+ \frac{W(t,\psi-\varepsilon(k+w\mathbf{1}_{\{0\}})) - W(t,\psi)}{\varepsilon^2}.
\end{aligned}
$$

where $h, k \in \mathbf{C}$ and $v, w \in \mathbb{R}^n$. Recall that

$$
\mathcal{S}(\Phi)(\phi) = \lim_{\varepsilon \to 0+} \frac{1}{\varepsilon}\left[\Phi(\tilde{\phi}_\varepsilon) - \Phi(\phi)\right].
$$

Therefore, we can define

$$
\mathcal{S}_\varepsilon(\Phi)(\phi) \equiv \frac{1}{\varepsilon}\left[\Phi(\tilde{\phi}_\varepsilon) - \Phi(\phi)\right].
$$

It is clear that $\mathcal{S}_\varepsilon(\Phi)$ is an approximation of $\mathcal{S}(\Phi)$ as ε approaches 0. In other words, we have

$$
\lim_{\varepsilon \to 0} \mathcal{S}_\varepsilon(\Phi)(\phi) = \mathcal{S}(\Phi)(\phi). \tag{16}
$$

Next we will show that $\Delta_\varepsilon W(t,\psi)$ and $\Delta_\varepsilon^2 W(t,\psi)$ are the approximations of $\overline{DW(t,\psi)}$ and $\overline{D^2 W(t,\psi)}$ respectively.

Lemma 3.1 *For any $W : [0,T] \times \mathbf{C} \to \mathbb{R}$, $W \in \mathcal{C}^{1,2}([0,T] \times \mathbf{C})$ such that W can be smoothly extended on $[0,T] \times (\mathbf{C} \oplus \mathbf{B})$, we have*

$$
\lim_{\varepsilon \to 0} \Delta_\varepsilon W(t,\psi)(h+v\mathbf{1}_{\{0\}}) = \overline{DW(t,\psi)}(h+v\mathbf{1}_{\{0\}}), \tag{17}
$$

and

$$
\begin{aligned}
&\lim_{\varepsilon \to 0} \Delta_\varepsilon^2 W(t,\psi)(h+v\mathbf{1}_{\{0\}}, k+w\mathbf{1}_{\{0\}}) \\
= \; &\overline{D^2 W(t,\psi)}(h+v\mathbf{1}_{\{0\}}, k+w\mathbf{1}_{\{0\}}).
\end{aligned} \tag{18}
$$

Proof. Note that the function W can be extended from $[0,T]\times\mathbf{C}$ to $[0,T]\times(\mathbf{C}\oplus\mathbf{B})$. Let us denote by $\widetilde{W}$ the smooth extension of W to $[0,T]\times(\mathbf{C}\oplus\mathbf{B})$. It is clear that

$$\lim_{\varepsilon\to 0}\Delta_\varepsilon W(t,\psi)(h+v\mathbf{1}_{\{0\}}) = d_G\widetilde{W}(t,\psi)(h+v\mathbf{1}_{\{0\}})$$

where $d_G\widetilde{W}$ denote the Gâteaux derivative of $\widetilde{W}$ with respect to its second variable. From the fact that $\widetilde{W}$ is smooth, we can obtain that the Gâteaux derivative and the Fréchet derivative of $\widetilde{W}$ coincide with each other. Moreover, they are the continuous extension of the DW, the Fréchet derivative of W. On the other hand, the uniqueness of the linear continuous extension give us the following:

$$\begin{aligned}\lim_{\varepsilon\to 0}\Delta_\varepsilon W(t,\psi)(h+v\mathbf{1}_{\{0\}}) &= \lim_{\varepsilon\to 0}\Delta_\varepsilon \widetilde{W}(t,\psi)(h+v\mathbf{1}_{\{0\}}) \\ &= \overline{DW(t,\psi)}(h+v\mathbf{1}_{\{0\}}). \qquad (19)\end{aligned}$$

Similar argument can be used for (18). □

For $\varepsilon,\eta>0$, the corresponding discrete version of equation (15) is given by

$$\begin{aligned}\min\Bigg\{ & V_M(t,\psi)-(\Psi(t,\psi)\wedge M), \\ & \rho V_M(t,\psi) - \frac{V_M(t+\eta,\psi)-V_M(t,\psi)}{\eta} - \frac{V_M(t,\tilde{\psi}_\varepsilon)-V_M(t,\psi)}{\varepsilon} \\ & -\frac{V_M(t,\psi+\varepsilon(f(t,\psi)\mathbf{1}_{\{0\}}))-V_M(t,\psi)}{\varepsilon} \\ & -\frac{1}{2}\sum_{i=1}^m\Bigg(\frac{V_M(t,\psi+\varepsilon(g(t,\psi)\mathbf{e}_i\mathbf{1}_{\{0\}}))-V_M(t,\psi)}{\varepsilon^2} \\ & \qquad +\frac{V_M(t,\psi-\varepsilon(g(t,\psi)\mathbf{e}_i\mathbf{1}_{\{0\}}))-V_M(t,\psi)}{\varepsilon^2}\Bigg) \\ & -(L(t,\psi)\wedge M)\Bigg\} = 0. \qquad (20)\end{aligned}$$

Rearranging terms, we obtain

$$\begin{aligned}\min\Bigg[& V_M(t,\psi)-(\Psi(t,\psi)\wedge M), \\ & (\frac{2}{\varepsilon}+\frac{1}{\eta}+\frac{m}{\varepsilon^2}+\rho)V_M(t,\psi)-\frac{1}{\varepsilon}V_M(t,\tilde{\psi}_\varepsilon)\end{aligned}$$

$$
\begin{aligned}
&-\frac{V_M(t,\psi+\varepsilon(f(t,\psi)\mathbf{1}_{\{0\}}))}{\varepsilon}-\frac{V_M(t+\eta,\psi)}{\eta}\\
&-\frac{1}{2}\sum_{i=1}^{m}\frac{V_M(t,\psi+\varepsilon(g(t,\psi)\mathbf{e}_i\mathbf{1}_{\{0\}}))+V_M(t,\psi-\varepsilon(g(t,\psi)\mathbf{e}_i\mathbf{1}_{\{0\}}))}{\varepsilon^2}\\
&-(L(t,\psi)\wedge M)\Bigg]=0. \qquad (21)
\end{aligned}
$$

Let $\mathcal{C}([0,T]\times(\mathbf{C}\oplus\mathbf{B}))_b$ denote the space of bounded continuous functions W from $[0,T]\times(\mathbf{C}\oplus\mathbf{B})$ to $\mathbb{R}$. Define a mapping $\mathcal{S}_M:(0,1)^2\times[0,T]\times\mathbf{C}\times\mathbb{R}\times\mathcal{C}([0,T]\times(\mathbf{C}\oplus\mathbf{B}))_b\to\mathbb{R}$ as the following

$$
\begin{aligned}
&\mathcal{S}_M(\varepsilon,\eta,t,\psi,x,W)\\
\equiv\;&\varepsilon\min\Bigg[x-(\Psi(t,\psi)\wedge M),\\
&\quad(\frac{2}{\varepsilon}+\frac{1}{\eta}+\frac{m}{\varepsilon^2}+\rho)x-\frac{1}{\varepsilon}W(t,\tilde{\psi}_\varepsilon)\\
&\quad-\frac{W(t,\psi+\varepsilon(f(t,\psi)\mathbf{1}_{\{0\}}))}{\varepsilon}-\frac{W(t+\eta,\psi)}{\eta}\\
&\quad-\frac{1}{2}\sum_{i=1}^{m}\frac{W(t,\psi+\varepsilon(g(t,\psi)\mathbf{e}_i\mathbf{1}_{\{0\}}))+W(t,\psi-\varepsilon(g(t,\psi)\mathbf{e}_i\mathbf{1}_{\{0\}}))}{\varepsilon^2}\\
&\quad-(L(t,\psi)\wedge M)\Bigg]. \qquad (22)
\end{aligned}
$$

Then, (20) is equivalent to

$$\mathcal{S}_M(\varepsilon,\eta,t,\psi,V_M(t,\psi),V_M)=0.$$

Moreover, note that the coefficients of all the terms that involve W in $\mathcal{S}_M$ are negative. This implies that $\mathcal{S}_M$ is monotone, i.e., for all $W_1,W_2\in\mathcal{C}([0,T]\times(\mathbf{C}\oplus\mathbf{B}))_b$, $\varepsilon,\eta\in(0,1)$, $t\in[0,T]$, $\psi\in\mathbf{C}$, and $x\in\mathbb{R}$, we have

$$\mathcal{S}_M(\varepsilon,\eta,t,\psi,x,W_1)\le\mathcal{S}_M(\varepsilon,\eta,t,\psi,x,W_2)\text{ whenever }W_1\ge W_2. \qquad (23)$$

Definition 3.2 *The scheme $\mathcal{S}_M$ is said to be consistent if, for every $t\in[0,T]$, $\psi\in\mathbf{C}\oplus\mathbf{B}$, and for every test function $W\in\mathcal{C}^{1,2}([0,T]\times(\mathbf{C}\oplus\mathbf{B}))_b$,*

$$
\begin{aligned}
&\min\Bigg\{W(t,\psi)-(\Psi(t,\psi)\wedge M),\\
&\qquad\rho W(t,\psi)-\frac{\partial W}{\partial t}(t,\psi)-\mathcal{A}W(t,\psi)-(L(t,\psi)\wedge M)\Bigg\}\\
=&\lim_{(\tau,\phi)\to(t,\psi),\,\varepsilon,\eta\downarrow0,\,\xi\to0}\frac{\mathcal{S}_M(\varepsilon,\eta,\tau,\phi,W(\tau,\phi)+\xi,W+\xi)}{\varepsilon}.
\end{aligned}
$$

We have the following result:

Lemma 3.3 *The scheme $\mathcal{S}_M$ defined by* (22) *is consistent.*

Proof. Let $W \in \mathcal{C}^{1,2}([0,T] \times (\mathbf{C} \oplus \mathbf{B}))_b \cap \mathcal{D}(\mathcal{S})$. We write

$$
\begin{aligned}
&\frac{\mathcal{S}_M(\varepsilon, \eta, \tau, \phi, W(\tau,\phi) + \xi, W + \xi)}{\varepsilon} \\
= \quad & \min \Bigg[(W(\tau,\phi) + \xi) - (\Psi(\tau,\phi) \wedge M), \\
& \quad (\frac{2}{\varepsilon} + \frac{1}{\eta} + \frac{m}{\varepsilon^2} + \rho)(W(\tau,\phi) + \xi) - \frac{W(\tau, \phi + \varepsilon(f(\tau,\phi)\mathbf{1}_{\{0\}})) + \xi}{\varepsilon} \\
& \quad - \frac{W(\tau + \eta, \phi) + \xi}{\eta} - \frac{1}{\varepsilon}(W(\tau, \tilde{\phi}_\varepsilon) + \xi) \\
& \quad - \frac{1}{2}\sum_{i=1}^m \frac{W(\tau, \phi + \varepsilon(g(\tau,\phi)\mathbf{e}_i\mathbf{1}_{\{0\}})) + 2\xi + W(\tau, \phi - \varepsilon(g(\tau,\phi)\mathbf{e}_i\mathbf{1}_{\{0\}}))}{\varepsilon^2} \\
& \quad - (L(\tau,\phi) \wedge M) \Bigg] \\
= \quad & \min \Bigg\{ (W(\tau,\phi) + \xi) - (\Psi(\tau,\phi) \wedge M), \\
& \quad \rho(W(\tau,\phi) + \xi) - \frac{W(\tau + \eta, \phi) - W(\tau,\phi)}{\eta} - \frac{W(\tau, \tilde{\phi}_\varepsilon) - W_M(\tau,\phi)}{\varepsilon} \\
& \quad - \frac{W(\tau, \phi + \varepsilon(f(\tau,\phi)\mathbf{1}_{\{0\}})) - W(\tau,\phi)}{\varepsilon} \\
& \quad - \frac{1}{2}\sum_{i=1}^m \Bigg(\frac{W(\tau, \phi + \varepsilon(g(\tau,\phi)\mathbf{e}_i\mathbf{1}_{\{0\}})) - W(\tau,\phi)}{\varepsilon^2} \\
& \qquad + \frac{W(\tau, \phi - \varepsilon(g(\tau,\phi)\mathbf{e}_i\mathbf{1}_{\{0\}})) - W(\tau,\phi)}{\varepsilon^2} \Bigg) \\
& \quad - (L(\tau,\phi) \wedge M) \Bigg\} \\
= \quad & \min \Bigg\{ (W(\tau,\phi) + \xi) - (\Psi(\tau,\phi) \wedge M), \\
& \qquad \rho(W(\tau,\phi) + \xi) - \Delta_\eta W(\tau,\phi) - \mathcal{S}_\varepsilon(W(\tau,\cdot))(\phi) \\
& \qquad - \Delta_\varepsilon W(\tau,\phi)(f(\tau,\phi)\mathbf{1}_{\{0\}}) \\
& \qquad - \frac{1}{2}\sum_{i=1}^m \Delta_\varepsilon^2 W(\tau,\phi)(g(\tau,\phi)\mathbf{e}_i\mathbf{1}_{\{0\}}, g(\tau,\phi)\mathbf{e}_i\mathbf{1}_{\{0\}})
\end{aligned}
$$

$$- (L(\tau,\phi)\wedge M)\Big\}. \tag{24}$$

By virtue of (16), Lemma 3.1, and sending $\xi \to 0$, $\tau \to t$, $\phi \to \psi$, $\varepsilon, \eta \to 0$ in (24), we deduce

$$\begin{aligned}
&\min\Big\{W(t,\psi) - \Psi(t,\psi)\wedge M, \\
&\qquad \rho W(t,\psi) - \frac{\partial W}{\partial t}(t,\psi) - \mathcal{A}W(t,\psi) - (L(t,\psi)\wedge M)\Big\} \\
= &\lim_{(\tau,\phi)\to(t,\psi),\, \varepsilon,\eta\downarrow 0,\, \xi\to 0} \frac{\mathcal{S}_M(\varepsilon,\eta,\tau,\phi,W(\tau,\phi)+\xi, W+\xi)}{\varepsilon}.
\end{aligned}$$

This completes the proof. $\square$

Next, we will show that the equation

$$\mathcal{S}_M(\varepsilon,\eta,t,\psi,W(t,\psi),W) = 0 \tag{25}$$

has a solution. Using (21), we see that the equation $\mathcal{S}_M(\varepsilon,\eta,t,\psi,W(t,\psi),W) = 0$ is equivalent to the equation

$$\begin{aligned}
&W(t,\psi) \\
= \;&\max\Bigg[(\Psi(t,\psi)\wedge M), \frac{1}{\frac{2}{\varepsilon}+\frac{1}{\eta}+\frac{m}{\varepsilon^2}+\rho}\Bigg(\frac{W(t,\psi+\varepsilon(f(t,\psi)\mathbf{1}_{\{0\}}))}{\varepsilon} \\
&-\frac{1}{2}\sum_{i=1}^{m}\frac{W(t,\psi+\varepsilon(g(t,\psi,)\mathbf{e}_i\mathbf{1}_{\{0\}})) + W(t,\psi-\varepsilon(g(t,\psi)\mathbf{e}_i\mathbf{1}_{\{0\}}))}{\varepsilon^2} \\
&-\frac{1}{\varepsilon}W(t,\tilde{\psi}_\varepsilon) - \frac{W(t+\eta,\psi)}{\eta} - (L(t,\psi)\wedge M)\Bigg)\Bigg].
\end{aligned} \tag{26}$$

We define an operator $\mathcal{T}_{\varepsilon,\eta}$ on $C_b([0,T]\times(\mathbf{C}\oplus\mathbf{B}))$ as follows,

$$\begin{aligned}
&\mathcal{T}_{\varepsilon,\eta}W(t,\psi) \\
\equiv \;&\max\Bigg[(\Psi(t,\psi)\wedge M), \frac{1}{\frac{2}{\varepsilon}+\frac{1}{\eta}+\frac{m}{\varepsilon^2}+\rho}\Bigg(\frac{W(t,\psi+\varepsilon(f(t,\psi)\mathbf{1}_{\{0\}}))}{\varepsilon} \\
&-\frac{1}{2}\sum_{i=1}^{m}\frac{W(t,\psi+\varepsilon(g(t,\psi,)\mathbf{e}_i\mathbf{1}_{\{0\}})) + W(t,\psi-\varepsilon(g(t,\psi)\mathbf{e}_i\mathbf{1}_{\{0\}}))}{\varepsilon^2} \\
&-\frac{1}{\varepsilon}W(t,\tilde{\psi}_\varepsilon) - \frac{W(t+\eta,\psi)}{\eta} - (L(t,\psi)\wedge M)\Bigg)\Bigg]
\end{aligned} \tag{27}$$

If we can show that $\mathcal{T}_{\varepsilon,\eta}$ is a contraction map, then we can obtain the existence result for (25).

Lemma 3.4 *For each $\varepsilon > 0$ and $\eta > 0$, $\mathcal{T}_{\varepsilon,\eta}$ is a contraction map.*

Proof. To prove that $\mathcal{T}_{\varepsilon,\eta}$ is a contraction map, we need to show that there exists a constant $0 < \beta < 1$ such that

$$\|\mathcal{T}_{\varepsilon,\eta}W_1 - \mathcal{T}_{\varepsilon,\eta}W_2\| \leq \beta\|W_1 - W_2\| \text{ for all } W_1, W_2 \in \mathcal{C}([0,T] \times (\mathbf{C} \oplus \mathbf{B}))_b,$$

where $\|\cdot\|$ is the supremum norm. Let us define $c_{\varepsilon,\eta}$ by

$$c_{\varepsilon,\eta} \equiv \frac{2}{\varepsilon} + \frac{1}{\eta} + \frac{m}{\varepsilon^2} + \rho.$$

Now we have

$$\begin{aligned}
&|\mathcal{T}_{\varepsilon,\eta}W_1(t,\psi) - \mathcal{T}_{\varepsilon,\eta}W_2(t,\psi)| \\
&\leq \max\left[\frac{1}{c_{\varepsilon,\eta}}\left|\left(\frac{1}{\varepsilon}W_1(t,\tilde{\psi}_\varepsilon) + \frac{W_1(t,\psi+\varepsilon(f(t,\psi,u)\mathbf{1}_{\{0\}}))}{\varepsilon} + \frac{W_1(t+\eta,\psi)}{\eta}\right.\right.\right. \\
&\left. + \frac{1}{2}\sum_{i=1}^m \frac{W_1(t,\psi+\varepsilon(g(t,\psi,u)\mathbf{e}_i\mathbf{1}_{\{0\}})) + W_1(t,\psi-\varepsilon(g(t,\psi,u)\mathbf{e}_i\mathbf{1}_{\{0\}}))}{\varepsilon^2}\right) \\
&-\left(\frac{1}{\varepsilon}W_2(t,\tilde{\psi}_\varepsilon) + \frac{W_2(t,\psi+\varepsilon(f(t,\psi,u)\mathbf{1}_{\{0\}}))}{\varepsilon} + \frac{W_2(t+\eta,\psi)}{\eta}\right. \\
&\left.\left.\left. + \frac{1}{2}\sum_{i=1}^m \frac{W_2(t,\psi+\varepsilon(g(t,\psi,u)\mathbf{e}_i\mathbf{1}_{\{0\}})) + W_2(t,\psi-\varepsilon(g(t,\psi,u)\mathbf{e}_i\mathbf{1}_{\{0\}}))}{\varepsilon^2}\right)\right|\right].
\end{aligned}$$

Noting that $\|\cdot\|$ denotes the supremum norm, the above inequality implies that, for all t, ψ,

$$\left|\mathcal{T}_{\varepsilon,\eta}W_1(t,\psi) - \mathcal{T}_{\varepsilon,\eta}W_2(t,\psi)\right| \leq \left[\frac{\frac{2}{\varepsilon}+\frac{1}{\eta}+\frac{m}{\varepsilon^2}}{c_{\varepsilon,\eta}}\right]\|W_1 - W_2\|.$$

In addition, note that

$$\frac{\frac{2}{\varepsilon}+\frac{1}{\eta}+\frac{m}{\varepsilon^2}}{c_{\varepsilon,\eta}} = \frac{\frac{2}{\varepsilon}+\frac{1}{\eta}+\frac{m}{\varepsilon^2}}{\frac{2}{\varepsilon}+\frac{1}{\eta}+\frac{m}{\varepsilon^2}+\rho} < 1.$$

Take

$$\beta \equiv \frac{\frac{2}{\varepsilon}+\frac{1}{\eta}+\frac{m}{\varepsilon^2}}{c_{\varepsilon,\eta}}.$$

Then we have that $0 < \beta < 1$ and

$$\|\mathcal{T}_{\varepsilon,\eta}W_1 - \mathcal{T}_{\varepsilon,\eta}W_2\| \leq \beta\|W_1 - W_2\|.$$

This completes the proof. □

Definition 3.5 The scheme $\mathcal{S}_M$ is said to be **stable** if for every $\varepsilon, \eta \in (0,1)$, there exists a bounded solution $W_{\varepsilon,\eta} \in \mathcal{C}([0,T] \times (\mathbf{C} \oplus \mathbf{B}))_b$ to the equation

$$\mathcal{S}_M(\varepsilon, \eta, t, \psi, W(t,\psi), W) = 0, \tag{28}$$

with the bound independent of ε and η.

We have the following result:

Lemma 3.6 *The scheme $\mathcal{S}_M$ defined by* (22) *is stable.*

Proof. By the Banach fixed point theorem, the strict contraction $\mathcal{T}_{\varepsilon,\eta}$ defined by (27) has a unique fixed point that we denote by $W^M_{\varepsilon,\eta}$. From the definition of $\mathcal{T}_{\varepsilon,\eta}$, it is easy to see that $W^M_{\varepsilon,\eta}$ is actually the solution of the equation

$$\mathcal{S}_M(\varepsilon, \eta, t, \psi, W(t,\psi), W) = 0,$$

where the scheme $\mathcal{S}_M$ is defined by (22). Next we need to show that the solution $W^M_{\varepsilon,\eta}$ has a bound which is independent of ε, η.

Given any function $W_0 \in \mathcal{C}([0,T] \times (\mathbf{C} \oplus \mathbf{B}))_b$, we construct a sequence as follows, $W_{n+1} = \mathcal{T}_{\varepsilon,\eta} W_n$ for $n \geq 0$. It is clear that

$$\lim_{n\to\infty} W_n = W^M_{\varepsilon,\eta}.$$

Moreover, we have

$$\begin{aligned}
& W_{n+1}(t,\psi) \\
= \quad & \max\Bigg[(\Psi(t,\psi) \wedge M), \frac{1}{\frac{2}{\varepsilon} + \frac{1}{\eta} + \frac{m}{\varepsilon^2} + \rho}\Bigg(\frac{W_n(t, \psi + \varepsilon(f(t,\psi)\mathbf{1}_{\{0\}}))}{\varepsilon} \\
& - \frac{1}{2}\sum_{i=1}^m \frac{W_n(t, \psi + \varepsilon(g(t,\psi,)\mathbf{e}_i\mathbf{1}_{\{0\}})) + W_n(t, \psi - \varepsilon(g(t,\psi)\mathbf{e}_i\mathbf{1}_{\{0\}}))}{\varepsilon^2} \\
& - \frac{1}{\varepsilon}W_n(t, \tilde{\psi}_\varepsilon) - \frac{W_n(t+\eta, \psi)}{\eta} - (L(t,\psi) \wedge M)\Bigg)\Bigg]
\end{aligned} \tag{29}$$

By virtue of

$$0 < \frac{c_{\varepsilon,\eta} - \rho}{c_{\varepsilon,\eta}} < 1,$$

we can get that

$$\|W_{n+1}\| \leq \max\left[M, \frac{c_{\varepsilon,\eta} - \rho}{c_{\varepsilon,\eta}}\|W_n\| + \frac{1}{c_{\varepsilon,\eta}}M\right]. \tag{30}$$

From (30), we deduce that

$$\|W_{n+1}\| \le \max\left[M, \left(\frac{c_{\varepsilon,\eta}-\rho}{c_{\varepsilon,\eta}}\right)^{n+1}\|W_0\| + \frac{M}{c_{\varepsilon,\eta}}\sum_{i=0}^{n}\left(\frac{c_{\varepsilon,\eta}-\rho}{c_{\varepsilon,\eta}}\right)^{i}\right].$$

Taking the limit as $n \to \infty$, we obtain

$$\|W^M_{\varepsilon,\eta}\| \le \frac{M}{c_{\varepsilon,\eta}} \cdot \frac{1}{1-\frac{c_{\varepsilon,\eta}-\rho}{c_{\varepsilon,\eta}}} = \frac{M}{\rho}.$$

This implies the stability of the scheme $\mathcal{S}_M$. $\Box$

Given the results of Lemma 3.3 and Lemma 3.6, now we are ready to show the main result of this paper:

Theorem 3.7 *Let $W^M_{\varepsilon,\eta}$ denote the solution to* (28). *Then, as $(\varepsilon, \eta) \to 0$, the sequence $W^M_{\varepsilon,\eta}$ converges uniformly on $[0,T] \times \mathbf{C}$ to the unique viscosity solution V_M of* (15).

Proof. Define

$$\begin{aligned} W^*_M(t,\psi) &= \limsup_{\tau\to t,\phi\to\psi,\varepsilon\downarrow 0,\eta\downarrow 0} W^M_{\varepsilon,\eta}(\tau,\phi), \\ W_{*M}(t,\psi) &= \liminf_{\tau\to t,\phi\to\psi,\varepsilon\downarrow 0,\eta\downarrow 0} W^M_{\varepsilon,\eta}(\tau,\phi). \end{aligned} \tag{31}$$

We claim that W^*_M and W_{*M} are subsolution and supersolutions of (15), respectively. To prove this claim, we only consider the case for W^*_M. The argument for that of W_{*M} is similar.

To prove that W^*_M is the subsolution of (15), we need to show:

$$\begin{aligned} &\min\Big\{\Gamma(t,\psi) - (\Psi(t,\psi)\wedge M), \rho\Gamma(t,\psi) - \frac{\partial\Gamma}{\partial t}(t,\psi) - \mathcal{S}(\Gamma)(t,\psi) \\ &- \overline{D\Gamma(t,\psi)}(f(t,\psi)\mathbf{1}_{\{0\}}) - \frac{1}{2}\sum_{i=1}^{m}\overline{D^2\Gamma(t,\psi)}(g(t,\psi)\mathbf{e}_i\mathbf{1}_{\{0\}}, g(t,\psi)\mathbf{e}_i\mathbf{1}_{\{0\}}) \\ &-(L(t,\psi)\wedge M)\Big\} \le 0, \end{aligned}$$

for any test function $\Gamma \in C^{1,2}_{lip}([0,T] \times (\mathbf{C} \oplus \mathbf{B})) \cap \mathcal{D}(\mathcal{S})$ such that (t,ψ) is a strictly local maximum of $W^*_M(\tau,\phi) - \Gamma(\tau,\phi)$. Without loss of generality, here we assume that $W^*_M \le \Gamma$ and $W^*_M(t,\psi) = \Gamma(t,\psi)$ in a neighborhood $B((t,\psi), l)$ of (t,ψ).

Moreover, by virtue of Lemma 3.6, we know that our scheme is stable. Thus we can also assume that $\Gamma \geq 2\sup_{\varepsilon,\eta} \|W^M_{\varepsilon,\eta}\|$ outside of the ball $B((t,\psi),l)$ where $l > 0$ satisfies

$$W^*_M(\tau,\phi) - \Gamma(\tau,\phi) \leq 0 = W^*_M(t,\psi) - \Phi(t,\psi) \text{ for } \ (\tau,\phi) \in B((t,\psi),l).$$

This implies that there exist sequences $\varepsilon_n > 0$, $\eta_n > 0$, and $(\tau_n,\phi_n) \in [0,T] \times (\mathbf{C} \oplus \mathbf{B})$ such that as $n \to \infty$,

$$\begin{array}{l} \varepsilon_n \to 0, \ \eta_n \to 0, \ \tau_n \to t, \ \phi_n \to \psi, \ W^M_{\varepsilon_n,\eta_n}(\tau_n,\phi_n) \to W^*_M(t,\psi), \\ \text{and } \ (\tau_n,\phi_n) \ \text{ is a global maximum } W^M_{\varepsilon_n,\eta_n} - \Gamma, \end{array} \tag{32}$$

where $W^M_{\varepsilon_n,\eta_n}$ is the solution of the equation

$$\mathcal{S}_M(\varepsilon_n,\eta_n,\tau_n,\phi_n,W(\tau_n,\phi_n),W) = 0.$$

Denote $\alpha_n = W^M_{\varepsilon_n,\eta_n}(\tau_n,\phi_n) - \Gamma(\tau_n,\phi_n)$. Obviously $\alpha_n \to 0$ and

$$W^M_{\varepsilon_n,\eta_n}(\tau,\phi) \leq \Gamma(\tau,\phi) + \alpha_n \ \text{ for all } \ (\tau,\phi) \in [0,T] \times (\mathbf{C} \oplus \mathbf{B}). \tag{33}$$

We know that

$$\mathcal{S}_M(\varepsilon_n,\eta_n,\tau_n,\phi_n,W^M_{\varepsilon_n,\eta_n}(\tau_n,\phi_n),W^M_{\varepsilon_n,\eta_n}) = 0.$$

The monotonicity of $\mathcal{S}_M$ and (33) implies

$$\begin{aligned} & \mathcal{S}_M(\varepsilon_n,\eta_n,\tau_n,\phi_n,\Gamma(\tau_n,\phi_n)+\alpha_n,\Gamma+\alpha_n) \\ \leq \ & \mathcal{S}_M(\varepsilon_n,\eta_n,\tau_n,\phi_n,W^M_{\varepsilon_n,\eta_n}(\tau_n,\phi_n),W^M_{\varepsilon_n,\eta_n}) \\ = \ & 0. \end{aligned} \tag{34}$$

Therefore,

$$\lim_{n\to\infty} \frac{\mathcal{S}_M(\varepsilon_n,\eta_n,\tau_n,\phi_n,\Gamma(\tau_n,\phi_n)+\alpha_n,\Gamma+\alpha_n)}{\varepsilon_n} \leq 0,$$

Using Lemma 3.3 we obtain,

$$\begin{aligned} & \min\Big\{\Gamma^*_M(t,\psi) - (\Psi(t,\psi) \wedge M), \\ & \rho\Gamma^*_M(t,\psi) - \frac{\partial \Gamma}{\partial t}(t,\psi) - \mathcal{S}(\Gamma)(t,\psi) - \overline{D\Gamma(t,\psi)}(f(t,\psi)\mathbf{1}_{\{0\}}) \\ & -\frac{1}{2}\sum_{i=1}^m \overline{D^2\Gamma(t,\psi)}(g(t,\psi)\mathbf{e}_i\mathbf{1}_{\{0\}}, g(t,\psi)\mathbf{e}_i\mathbf{1}_{\{0\}}) - (L(t,\psi) \wedge M)\Big\} \\ & = \lim_{n\to\infty} \frac{\mathcal{S}_M(\varepsilon_n,\eta_n,\tau_n,\phi_n,\Gamma(\tau_n,\phi_n)+\alpha_n,\Gamma+\alpha_n)}{\varepsilon_n} \leq 0. \end{aligned} \tag{35}$$

This proves that W_M^* is a viscosity subsolution of (15).

Similarly we can prove that W_{*M} is a viscosity supersolution. By virtue of the Comparison Principle (Theorem 4.7 in [6]), we can get that

$$W_{*M}(t,\psi) \geq W_M^*(t,\psi), \quad \forall (t,\psi) \in [0,T] \times \mathbf{C}. \tag{36}$$

On the other hand, by the definitions of W_{*M}, W_M^*, it is easy to see that

$$W_{*M}(t,\psi) \leq W_M^*(t,\psi), \quad \forall (t,\psi) \in [0,T] \times \mathbf{C}.$$

Combined with (36), the above implies

$$W_{*M}(t,\psi) = W_M^*(t,\psi), \quad \forall (t,\psi) \in [0,T] \times \mathbf{C}.$$

Since W_{*M} is a viscosity supersolution and W_M^* is a viscosity subsolution, they are also viscosity solutions of (15). Now, using the uniqueness of the viscosity solution (15), we see that $V_M = W_M^* = W_{*M}$. Therefore, we conclude that the sequence $(W_{\varepsilon,\eta}^M)_{\varepsilon,\eta}$ converges locally uniformly to V_M as desired. □

4 The Computational Algorithm

Based on the results obtained in the last section, we can construct the computational algorithm to obtain a numerical solution. For example, one algorithm can be like the following:

Step 0. Choose any function $W^{(0)} \in \mathcal{C}([0,T] \times \mathbf{C} \oplus \mathbf{B})_b$;

Step 1. Pick the starting values for $\varepsilon(1), \eta(1)$. For example, we can choose $\varepsilon(1) = 10^{-2}, \eta(1) = 10^{-3}$;

Step 2. For the given $\varepsilon, \eta > 0$, compute the function

$$W_{\varepsilon(1),\eta(1)}^{(1)} \in \mathcal{C}([0,T] \times \mathbf{C} \oplus \mathbf{B})_b$$

by the following formula

$$W_{\varepsilon(1),\eta(1)}^{(1)} = \mathcal{T}_{\varepsilon(1),\eta(1)} W^{(0)},$$

where $\mathcal{T}_{\varepsilon(1),\eta(1)}$, which is defined on $C_b([0,T] \times \mathbf{C} \oplus \mathbf{B})$, is given by (27);

Step 3. Repeat Step 2 for $i = 2, 3, \cdots$ using

$$W^{(i)}_{\varepsilon(1),\eta(1)} = \mathcal{T}_{\varepsilon(1),\eta(1)} W^{(i-1)}_{\varepsilon(1),\eta(1)},$$

where $\mathcal{T}_{\varepsilon(1),\eta(1)}$, which is defined on $C_b([0,T] \times \mathbf{C} \oplus \mathbf{B})$, is given by (27). Stop the iteration when

$$\|W^{i+1}_{\varepsilon(1),\eta(1)}(t,\psi) - W^{i}_{\varepsilon(1),\eta(1)}(t,\psi)\| \leq \delta_1,$$

where δ_1 is a preselected number which is small enough to achieve the accuracy we want. Denote the final solution by $W_{\varepsilon(1),\eta(1)}(t,\psi)$.

Step 4. Choose two sequences of $\varepsilon(k)$ and $\eta(k)$, such that

$$\lim_{k\to\infty} \varepsilon(k) = \lim_{k\to\infty} \eta(k) = 0.$$

For example, we may choose $\varepsilon(k) = \eta(k) = 10^{-(2+k)}$. Then repeat Step 2 and Step 3 for each $\varepsilon(k), \eta(k)$ until

$$\|W_{\varepsilon(k+1),\eta(k+1)}(t,\psi) - W_{\varepsilon(k),\eta(k)}(t,\psi)\| \leq \delta_2,$$

where δ_2 is chosen to obtain the expected accuracy.

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