International Journal of Mathematical Analysis
Vol. 8, 2014, no. 40, 1973 - 1980
HIKARI Ltd, www.m-hikari.com
http://dx.doi.org/10.12988/ijma.2014.47224

Weakly Connected Dominating Sets in the Lexicographic Product of Graphs

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Abstract

In this paper we characterize the weakly connected dominating sets in the lexicographic product of two connected graphs. From these characterization, we easily determine the weakly connected domination number of the corresponding graph.

Mathematics Subject Classification: 05C69

Keywords: domination, weakly connected, weakly connected domination, lexicographic product

1 Introduction

Let $G = (V(G), E(G))$ be a connected graph and $v \in V(G)$. The neighborhood of v is the set $N_G(v) = N(v) = \{u \in V(G) : uv \in E(G)\}$. If $X \subseteq V(G)$, then

[1]This research is partially funded by the Commission on Higher Education, Philippines

the *open neighborhood* of X is the set $N_G(X) = N(X) = \cup_{v \in X} N_G(v)$. The *closed neighborhood* of X is $N_G[X] = N[X] = X \cup N(X)$.

A subset X of $V(G)$ is a *dominating set* of G if for every $v \in (V(G)\backslash X)$, there exists $x \in X$ such that $xv \in E(G)$, i.e., $N[X] = V(G)$. It is a *total dominating set* if $N(X) = V(G)$. The *domination number* $\gamma(G)$ (resp., *total domination number* $\gamma_t(G)$) of G is the smallest cardinality of a dominating (resp., total dominating) set of G. A dominating set S of $V(G)$ is a *weakly connected dominating set* of G if the subgraph $\langle S\rangle_w = (N_G[S], E_w) = (V(G), E_w)$ weakly induced by S is connected. Here, E_w consists of all the edges in G with at least one vertex in S. A total dominating set S of $V(G)$ is a *weakly connected total dominating set* of G if $\langle S\rangle_w = (V(G), E_w)$ is connected. The *weakly connected domination number* $\gamma_w(G)$ (*weakly connected total domination number* $\gamma_{wt}(G)$) of G is the smallest cardinality of a weakly connected dominating (resp., weakly connected total dominating) set of G. The concept of weakly connected domination has been investigated previously in [1], [2], and [3].

2 Weakly Connected Domination in the Lexicographic Product of Graphs

The *lexicographic product* $G[H]$ of two graphs G and H is the graph with $V(G[H]) = V(G) \times V(H)$ and $(u, u')(v, v') \in E(G[H])$ if and only if either $uv \in E(G)$ or $u = v$ and $u'v' \in E(H)$.

Observe that any non-empty subset C of $V(G) \times V(H)$ (in fact, any set of ordered-pairs) can be written as $C = \cup_{x \in S}(\{x\} \times T_x)$, where $S \subseteq V(G)$ and $T_x \subseteq V(H)$ for each $x \in S$. Henceforth, we shall use this form to denote any subset C of $V(G) \times V(H)$.

Lemma 2.1 *Let G be a connected non-trivial graph. A subset S of $V(G)$ is weakly disconnected (i.e. $\langle S\rangle_w$ is not connected) if and only if the following property is satisfied: (N) There exist $x, y \in S$ with $x \neq y$ such that $N_G[x] \cap N_G[y] = \emptyset$ and for every x-y path $P = [x, a_1, a_2, \cdots, a_k, y]$ in G, there exists $i \in \{1, 2, ..., k-1\}$ with $x_i, x_{i+1} \in V(P)\backslash S$.*

Proof: Suppose $\langle S\rangle_w$ is not connected. Then there exist $a, b \in N_G[S]$ which are not connected by a path in $\langle S\rangle_w$. Assume first that $a, b \in N_G[S]\backslash S$. Let $x, y \in S$ such that $ax, by \in E(G)$ (which are also edges in $\langle S\rangle_w$). Clearly, $xy \notin E(G)$ and $N_G(x) \cap N_G(y) = \emptyset$. Hence, $N_G[x] \cap N_G[y] = \varnothing$. Let $P = [x, a_1, a_2, \cdots, a_k, y]$ be an x-y path. Since $N_G(x) \cap N_G(y) = \emptyset$, $k \geq 2$. Also, by assumption, $V(P)\backslash S \neq \emptyset$. Suppose now that $a_i \in S$ or $a_{i+1} \in S$ for all $i = 1, 2, ..., k-1$. Then $E(P) \subseteq E(\langle S\rangle_w)$ and so P is an $x-y$ path in $\langle S\rangle_w$. Therefore, $[a, x, a_1, ..., a_k, y, b]$ is an $a-b$ path in $\langle S\rangle_w$, contrary to our assumption that $\langle S\rangle_w$ is disconnected.

Accordingly, there exist $i \in \{1, 2, ..., k-1\}$ with $x_i, x_{i+1} \in V(P)\backslash S$. Using a similar argument it can be shown that property (N) is satisfied when $a, b \in S$ and $a \in N_G[S]\backslash S$ and $b \in S$.

For the converse, suppose that there exist $x, y \in S$ with $x \neq y$ such that $N_G[x] \cap N_G[y] = \emptyset$ and for every x-y path $P = [x, a_1, a_2, ...a_k, y]$ in G, there exist $i = 1, 2, ..., k-1$ such that $a_i, a_j \in V(G)\backslash S$. Then x and y are not connected by a path in $\langle S \rangle_w$. Therefore $\langle S \rangle_w$ is disconnected. $\Box$

Lemma 2.2 *Let G and H be connected non-trivial graphs and let $C = \cup_{x \in S}(\{x\} \times T_x) \subseteq V(G[H])$, where $S \subseteq V(G)$ and $T_x \subseteq V(H)$ for all $x \in S$. Then C is weakly connected in $G[H]$ if and only if S is weakly connected in G.*

Proof: Suppose C is weakly connected in $G[H]$. Suppose further that S is not weakly connected in G. By Lemma 2.1, there exist $x, y \in S$ with $x \neq y$ such that $N_G[x] \cap N_G[y] = \emptyset$ and for every x-y path $P = [x, a_1, a_2, ..., a_k, y]$ in G, there exist iß$\{1, 2, ..., k-1\}$ with $x_i, x_{i+1} \in V(P)\backslash S$. Pick $a \in T_x$ and $b \in T_y$. Then $(x, a), (y, b) \in C$, $(x, a) \neq (y, b)$ and $N_{G[H]}[(x, a)] \cap N_{G[H]}[(y, b)] = \emptyset$. Let $P^* = [(x, a), (c_1, a_1), (c_2, a_2), \cdots, (c_k, a_k), (y, b)]$ be an (x, a)-(y, b) path. Since $N_{G[H]}((x, a)) \cap N_{G[H]}((y, b)) = \emptyset$, $k \geq 2$. Also, since $d_G(x, y) \geq 3$, there are at least two distinct c_i's which are different from x and y. Let these vertices be labeled $d_1, d_2, \cdots, d_r (r \geq 2)$ (such that $[d_1, d_2, \cdots, d_r]$ is a path) and the corresponding a_j's as $e_1, e_2, \cdots, e_r$. Consider the $(x, a) - (y, b)$ path $P^{**} = [(x, a), (d_1, e_1), (d_2, e_2), ..., (d_r, e_r), (y, b)]$. Then $[x, d_1, d_2, \cdots, d_r, y]$ is an x-y path. Hence, by assumption, there exists j with $1 \leq j \leq r-1$ such that $d_j, d_{j+1} \in V(P)\backslash S$. It follows that there exists $m \in \{1, 2, \cdots, k-1\}$ such that $(c_m, a_m), (c_{m+1}, a_{m+1}) \in V(P^*)\backslash C$. By Lemma 2.1, C is weakly disconnected in $G[H]$, contrary to our assumption. Therefore S is weakly connected in G.

For the converse, suppose that $\langle S \rangle_w$ is connected. Let $(x, a), (y, b) \in C$ with $(x, a) \neq (y, b)$ and $(x, a)(y, b) \notin E(G)$. Suppose first that $x = y$. Let $v \in V(G)$ such that $xv \in E(G)$. Then $P = [(x, a), (v, c), (y, b)]$ is an (x, a)-(y, b) path in G (also in $\langle C \rangle_w$). Next, suppose that $x \neq y$. Since $\langle S \rangle_w$ is connected and $x, y \in S$, by Lemma 2.1, there exists an x-y path $P = [x, x_1, x_2, \cdots, x_k, y]$ in G such that $x_i \in S$ or $x_{i+1} \in S$ for all $i = 1, 2, ..., k-1$. For each $x_i \in S$, choose $b_i \in T_{x_i}$. If $x_i \notin S$, let $b_i = a \in V(H)$. Clearly, $P^* = [(x, a), (x_1, b_1), (x_2, b_2), ..., (x_k, b_k), (y, b)]$ is an (x, a)-(y, b) path in $G[H]$. By the construction of P^*, $(x_i, b_i) \in C$ or $(x_{i+1}, b_{i+1}) \in C$ for all $i = 1, 2, ..., k-1$. Therefore, $\langle C \rangle_w$ is connected. $\Box$

Theorem 2.3 *Let G and H be non-trivial connected graphs. Then $C = \cup_{x \in S}(\{x\} \times T_x) \subseteq V(G[H])$, where $S \subseteq V(G)$ and $T_x \subseteq V(H)$ for each $x \in S$, is a weakly connected dominating set in $G[H]$ if and only if either*

(i) *S is a weakly connected total dominating set in G or* (*ii*) *S is a weakly connected dominating set in G and T_x is a dominating set in H for every $x \in S \setminus N_G(S)$.*

Proof: Suppose C is a weakly connected dominating set of $G[H]$. Let $u \in V(G)\setminus S$ and pick $b \in V(H)$. Since C is a dominating set in $G[H]$, there exists $(y,c) \in C$ such that $(y,c)(u,b) \in E(G[H])$. This implies that $y \in S$ and $u \in N_G(y)$. This shows that S is a dominating set in G. Moreover, by Lemma 2.2, $\langle S\rangle_w$ is connected. If S is a total dominating set in G, then we are done. So suppose S is not a total dominating set in G. Then $S \setminus N_G(S) \neq \emptyset$. Let $x \in S \setminus N_G(S)$. Suppose there exists $a \in V(H) \setminus N_H[T_x]$. Then $a \notin T_x$ and $ad \notin E(H)$ for all $d \in T_x$. This implies that $(x,a) \notin N_{G[H]}[C]$, contrary to our assumption that C is a dominating set in $G[H]$. Therefore, $N_H[T_x] = V(H)$, i.e., T_x is a dominating set in H.

For the converse, let $C = \cup_{x\in S}(\{x\}\times T_x)$ and $(u,t) \in V(G[H])\setminus C$. Assume first that S is a weakly connected total dominating set of G. Then there exists $x \in S \setminus \{u\}$ such that $u \in N_G(x)$. Choose $d \in T_x$. Then $(x,d) \in C$ and $(u,t)(x,d) \in E(G[H])$. Hence, $(u,t) \in N_{G[H]}[C]$. This shows that C is a dominating set of $G[H]$. By Lemma 2.2, C is a weakly connected dominating set in $G[H]$.

Suppose now that (*ii*) holds. If $u \notin S$, then because S is a dominating set of G, there exists $y \in S$ such that $u \in N_G(y)$. Pick $a \in T_y$. Then $(y,a) \in C$ and $(u,t)(y,a) \in E(G[H])$. Suppose $u \in S$. If $u \in N_G(z)$ for some $z \in S\setminus\{u\}$, then there exists $(z,b) \in C$ such that $(u,t)(z,b) \in E(G[H])$. If $u \notin N_G(z)$ for all $z \in S \setminus \{u\}$, then by assumption, T_u is a dominating set in H. Since $(u,t) \notin C$, $t \notin T_u$. This implies that there exists $s \in T_u$ such that $ts \in E(H)$. It follows that $(u,s) \in C$ and $(u,t)(u,s) \in E(G[H])$. Thus, $(u,t) \in N_{G[H]}(C)$. In both cases, we have shown that $(u,t) \in N_{G[H]}(C)$. Therefore, $N_{G[H]}[C] = V(G[H])$, i.e., C is a dominating set in $G[H]$. Again, by Lemma 2.2, $\langle C\rangle_w$ is connected.

Accordingly, C is a weakly connected dominating set of $G[H]$. □

Corollary 2.4 *Let G and H be non-trivial connected graphs with $\gamma(H) = 1$. Then a subset $C = \cup_{x\in S}(\{x\} \times T_x)$ of $V(G[H])$ is a minimum weakly connected dominating set of $G[H]$ if and only if S is a minimum weakly connected dominating set of G and $|T_x| = 1$ for every $x \in S$, where T_x is a minimum dominating set of H for each $x \in S\setminus N(S)$ if $\gamma_w(G) \neq \gamma_{wt}(G)$ (S is not a total dominating set).*

Proof: Suppose $C = \cup_{x\in S}(\{x\} \times T_x)$ is a minimum weakly connected dominating set of $G[H]$. By Theorem 2.3, S is a weakly connected dominating set in G, where T_x is a dominating set of H for every $x \in S \setminus N_G(S)$ if S is not a total dominating set in G. Suppose $|T_z| \geq 2$ for some $z \in S$. Let $a \in V(H)$ with $deg_H(a) = |V(H)| - 1$. Define $D_x = \{a\}$ for all $x \in S$.

Then $C_1 = \cup_{x\in S}(\{x\} \times D_x)$ is a weakly connected dominating set of $G[H]$ by Theorem 2.3(ii). Moreover,

$$|C_1| = |S| < \sum_{x\in D_1} |T_x| + \sum_{x\in D_2} |T_x| = |C|,$$

where $D_1 = S \cap N_G(S)$ and $D_2 = S \setminus N_G(S)$. This contradicts the fact that C is a minimum weakly connected dominating set of $G[H]$. Therefore, $|T_x| = 1$ for all $x \in S$. Consequently, $|C| = |S|$.

Next, let S_1 be a weakly connected dominating set of G. Set $M_x = \{a\}$, where $deg_H(a) = |V(H)| - 1$, for each $x \in S_1$. Then $C_2 = \cup_{x\in S_1}(\{x\} \times M_x)$ is a weakly connected dominating set by Theorem 2.3(ii). Moreover, $|S| = |C| \leq |C_2| = |S_1|$. This implies that S is a minimum weakly connected dominating set of G.

For the converse, suppose that $C = \cup_{x\in S}(\{x\} \times T_x)$ and S is a minimum weakly connected dominating set of G with $|T_x| = 1$ for all $x \in S$, where T_x is a minimum dominating set of H for each $x \in S\setminus N(S)$ if $\gamma_w(G) \neq \gamma_{wt}(G)$. By Theorem 2.3, C is a weakly connected dominating set of $G[H]$. If $C_1 = \cup_{x\in S_1}(\{x\}\times L_x)$ is a weakly connected dominating set of $G[H]$, then, by Theorem 2.3, S_1 is a weakly connected dominating set of G. Let $D_1 = S_1 \cap N_G(S_1)$ and $D_2 = S_1 \setminus N_G(S_1)$. Then

$$|C| = |S| \leq |S_1| = |D_1| + |D_2| \leq \sum_{x\in D_1} |L_x| + \sum_{x\in D_2} |L_x| = |C_1|.$$

This implies that C is a minimum weakly connected dominating set of $G[H]$. □

The next result follows from Corollary 2.4.

Corollary 2.5 *Let G and H be non-trivial connected graphs with $\gamma(H) = 1$. Then $\gamma_w(G[H]) = \gamma_w(G)$.*

We shall need the following simple lemma.

Lemma 2.6 *Let G be a connected graph and let S be a weakly connected dominating set of G. Then $\gamma_{wt}(G) \leq |S \cap N_G(S)| + 2|S \setminus N_G(S)|$. In particular, $\gamma_{wt}(G) \leq 2\gamma_w(G)$.*

Proof: Let S be a weakly connected dominating set in G. If S is a total dominating set, then we are done. So suppose S is not a total dominating set. Then $S \setminus N_G(S) \neq \emptyset$. For each $y \in S \setminus N_G(S)$, choose $v_y \in V(G)$ such that $yv_y \in E(G)$ and let $T_1 = \{v_y : y \in S \setminus N_G(S)\}$. Then $|T_1| \leq |S \setminus N_G(S)|$.

Further, since S is weakly connected and $T_1 \subseteq N_G(S)$, it follows that $T = S \cup T_1$ is a weakly connected total dominating set of G. Thus,

$$\gamma_{wt}(G) \le |T| = |S \cup T_1| = |S \cap N_G(S)| + |S \setminus N_G(S)| + |T_1|$$

$$\le |S \cap N_G(S)| + 2|S \setminus N_G(S)|.$$

If, in particular, S is a minimum weakly connected dominating set of G, then $\gamma_{wt}(G) \le |T| = |S| + |T_1| \le 2|S| = 2\gamma_w(G)$. This proves the assertion. □

Theorem 2.7 *Let G and H be nontrivial connected graphs with $\gamma(H) = 2$. Then a subset $C = \cup_{x \in S}(\{x\} \times T_x)$ of $V(G[H])$ is a minimum weakly connected dominating set in $G[H]$ if and only if either*

- (i) *S is a minimum weakly connected total dominating set of G and $|T_x| = 1$ for all $x \in S$ or*
- (ii) *S is a weakly connected dominating set of G such that $|S \cap N_G(S)| + 2|S \setminus N_G(S)| = \gamma_{wt}(G)$, $|T_x| = 1$ for each $x \in S \cap N_G(S)$, and T_x is a minimum dominating set in H (hence, $|T_x| = 2$) for every $x \in S \setminus N_G(S)$.*

Proof: Suppose $C = \cup_{x \in S}(\{x\} \times T_x)$ is a minimum weakly connected dominating set of $G[H]$. By Theorem 2.3, S is a weakly connected total dominating set of G or S is a weakly connected dominating set of G and T_x is a dominating set in H for every $x \in S \setminus N_G(S)$. Suppose that S is weakly connected total dominating set. Suppose further that that $|T_z| \ge 2$ for some $z \in S$. Let $a \in T_z$ and define $T_z^* = \{a\}$. Then $C^* = [\cup_{x \in S \setminus \{z\}}(\{x\} \times T_x)] \cup (\{z\} \times T_z^*)$ is a weakly connected dominating set by Theorem 2.3(i). This, however, is impossible because $|C^*| < |C|$. Thus, $|T_x| = 1$ for all $x \in S$ and (i) holds.

Suppose now that S is not a total dominating set in G. Suppose first that $\gamma_{wt}(G) < |S \cap N_G(S)| + 2|S \setminus N_G(S)| \le |C|$. Choose a minimum weakly connected total dominating set R of G and set $S_x = \{v\}$ for every $x \in R$, where $v \in V(H)$. Then $Y = \cup_{x \in R}(\{x\} \times S_x)$ is a weakly connected dominating set by Theorem 2.3(i). It follows that $\gamma_{wt}(G) = |R| = |Y| < |C|$, contrary to our assumption of C. Thus, by Lemma 2.6, $\gamma_{wt}(G) = |S \cap N_G(S)| + 2|S \setminus N_G(S)|$. Next, suppose that there exists $z \in S \cap N_G(S)$ with $|T_z| \ge 2$. Let $a \in T_z$ and define $T_z^* = \{a\}$. Then $C^* = [\cup_{x \in S \setminus \{z\}}(\{x\} \times T_x)] \cup (\{z\} \times T_z^*)$ is a weakly connected dominating set by Theorem 2.3(ii). This is not possible because $|C^*| < |C|$. Therefore, $|T_x| = 1$ for all $x \in S \cap N_G(S)$. Finally, suppose there exists $w \in S \setminus N_G(S)$ such that T_w is not a minimum dominating set in H. Since T_w is not a minimum dominating set in H, $|T_w| > 2$. Let $L_w = \{a, b\}$ be a minimum dominating set in H. Then $C_1 = [\cup_{x \in S \setminus \{w\}}(\{x\} \times T_x)] \cup (\{w\} \times L_w)$ is a weakly connected dominating set by Theorem 2.3(ii). Again, this is not possible because $|C_1| < |C|$. Therefore, T_x is a minimum dominating set in H for every $x \in S \setminus N_G(S)$.

For the converse, let $C = \cup_{x\in S}(\{x\} \times T_x)$ and suppose first that (i) holds. Then $|C| = |S| = \gamma_{wt}(G)$. Also, by Theorem 2.3$(i)$, C is a weakly connected dominating set in $G[H]$. Let $C_1 = \cup_{x\in S_1}(\{x\} \times D_x)$ be a weakly connected dominating set in $G[H]$. By Theorem 2.3, S_1 is a weakly connected dominating set of G. If S_1 is a total dominating set, then $|C| = |S| = \gamma_{wt}(G) \leq |S_1| = |C_1|$. If S_1 is not a total dominating set, then D_x is a dominating set in H for every $x \in S_1 \setminus N_G(S_1)$. Since $\gamma(H) = 2$, $|D_x| \geq 2$ for every $x \in S_1 \setminus N_G(S_1)$. Therefore, by Lemma 2.6,

$$|C| = \gamma_{wt}(G) \leq |D_1| + 2|D_2| \leq |C_1|,$$

where $D_1 = S_1 \cap N_G(S_1)$ and $D_2 = S_1 \setminus N_G(S_1)$. This shows that C is a minimum weakly dominating set in $G[H]$. If (ii) holds, then a similar argument may be used to show that C is a minimum weakly dominating set in $G[H]$. □

Theorem 2.8 *Let G and H be nontrivial connected graphs with $\gamma(H) > 2$. Then a subset $C = \cup_{x\in S}(\{x\} \times T_x)$, where $S \subseteq V(G)$ and $T_x \subseteq V(H)$ $\forall$ $x \in S$, of $V(G[H])$ is a minimum weakly connected dominating set in $G[H]$ if and only if S is a minimum weakly connected total dominating set of G and $|T_x| = 1$ for all $x \in S$.*

Proof: Suppose $C = \cup_{x\in S}(\{x\} \times T_x)$ is a minimum weakly connected dominating set in $G[H]$. By Theorem 2.3, S is a weakly connected domi-ating set. Suppose S is not a total dominating set. Then $S \setminus N_G(S) \neq \emptyset$ and T_x is a dominating set in H for every $x \in S \setminus N_G(S)$, by Theorem 2.3. Since $\gamma(H) > 2$, it follows that $|T_x| > 2$ for every $x \in S \setminus N_G(S)$. Now, by Lemma 2.6, $\gamma_{wt}(G) \leq |S \cap N_G(S)| + 2|S \setminus N_G(S)|$. Since $|C| = \sum_{x\in D_1} |T_x| + \sum_{x\in D_2} |T_x|$, where $D_1 = S \cap N_G(S)$ and $D_2 = S \setminus N_G(S)$ it follows that $\gamma_{wt}(G) < |C|$. Let S_1 be a minimum weakly connected total dominating set of G and set $Q_x = \{a\}$ for every $x \in S$, where $a \in V(H)$. Put $Q = \cup_{x\in S_1}(\{x\} \times Q_x)$. Then Q is a weakly connected dominating set in $G[H]$ by Theorem 2.3 (ii). Moreover, $|Q| = |S_1| = \gamma_{wt}(G)$. Thus, $|Q| < |C|$, contrary to our assumption of C. Therefore, S is a weakly connected total dominating set of G. Using a similar argument, it can be shown that S minimum weakly connected total dominating set of G and $|T_x| = 1$ for all $x \in S$.

For the converse, suppose that $C = \cup_{x\in S}(\{x\} \times T_x)$ and S is a minimum weakly connected total dominating set of G with $|T_x| = 1$ for all $x \in S$. By Theorem 2.3 (i), C is a weakly connected dominating set of $G[H]$. If $C_1 = \cup_{x\in S_1}(\{x\} \times L_x)$ is a weakly connected dominating set of $G[H]$, then, by Theorem 2.3 (i), S_1 is weakly connected dominating set of G. Let $D_1 = S_1 \cap N_G(S_1)$ and $D_2 = S_1 \setminus N_G(S_1)$. Then

$$|D_1| + 2|D_2| \leq \sum_{x\in D_1} |L_x| + \sum_{x\in D_2} |L_x| = |C_1|.$$

Thus, by Lemma 2.6, $\gamma_{wt}(G) = |S| = |C| \leq |C_1|$. This implies that C is a minimum weakly connected dominating set of $G[H]$. □

The following result gives the weakly connected domination number of the composition of two connected graphs.

Corollary 2.9 *Let G and H be nontrivial connected graphs with $\gamma(H) \geq 2$. Then $\gamma_w(G[H]) = \gamma_{wt}(G)$.*

Proof: Let S be a minimum weakly connected total dominating set of G. Pick $a \in V(H)$ and set $T_x = \{a\}$ and $C = \cup_{x \in S}(\{x\} \times T_x)$. By Theorem 2.7 and Theorem 2.8, C is a minimum weakly connected dominating set in $G[H]$. Thus, $\gamma_w(G[H]) = |C| = |S| = \gamma_{wt}(G)$. □

Corollary 2.10 *Let G be a non-trivial connected graph and let $n \geq 2$. Then*

$$\gamma_w(K_n[G]) = \begin{cases} 1 & , \text{ if } \gamma(G) = 1 \\ 2 & , \text{ if } \gamma(G) \geq 2 \end{cases} .$$

Proof: If $\gamma(G) = 1$, then $\gamma_w(K_n[G]) = \gamma_w(K_n) = 1$ by Corollary 2.5. If $\gamma(G) \geq 2$, then $\gamma_w(K_n[G]) = \gamma_{wt}(K_n) = 2$ by Corollary 2.9. □

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Received: July 25, 2014