Edge-connectivity augmentation of graphs over symmetric parity families

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8 April 2009
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2. $T$-cuts
3. Symmetric parity families
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   1. Definitions
   2. Cut equivalent trees
   3. Edge-connectivity augmentation

2. $T$-cuts

3. Symmetric parity families
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   2. Cut equivalent trees
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3. Symmetric parity families
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Global edge-connectivity

Given a graph $G = (V, E)$ and an integer $k$, $G$ is called $k$-edge-connected if each cut contains at least $k$ edges.
Definitions

Global edge-connectivity

Given a graph $G = (V, E)$ and an integer $k$, $G$ is called \textit{k-edge-connected} if each cut contains at least $k$ edges.

Local edge-connectivity

Given a graph $G = (V, E)$ and $u, v \in V$, the \textit{local edge-connectivity} $\lambda_G(u, v)$ is defined as the minimum cardinality of a cut separating $u$ and $v$. 
**Theorem (Gomory-Hu)**

For every graph $G = (V, E)$, we can find, in polynomial time, a tree $H = (V, E')$ and $c : E' \rightarrow \mathbb{Z}$ such that for all $u, v \in V$

1. the local edge-connectivity $\lambda_G(u, v)$ is equal to the minimum value $c(e)$ of the edges $e$ of the unique $(u, v)$-path in $H$,
2. if $e$ achieves this minimum, then a minimum cut of $G$ separating $u$ and $v$ is given by the two connected components of $H - e$. 
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Cut equivalent tree $H = (V, E')$
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Graph $G = (V, E)$

Cut equivalent tree $H = (V, E')$
Global edge-connectivity augmentation of a graph

Given a graph $G = (V, E)$ and an integer $k \geq 2$, what is the minimum number of new edges whose addition results in a $k$-edge-connected graph?

1. Minimax theorem (Watanabe, Nakamura)
2. Polynomially solvable (Cai, Sun)

Graph $G, k = 4$
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Opt $\geq \lceil \frac{5}{2} \rceil = 3
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Graph $G + F$ is 4-edge-connected and $|F| = 3$
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$$\text{Opt} = \left\lceil \frac{1}{2} \text{maximum deficiency of a subpartition of } V \right\rceil$$
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**General method**

**Frank’s algorithm**

1. **Minimal extension,**
   - (i) Add a new vertex $s$,
   - (ii) Add a minimum number of new edges incident to $s$ to satisfy the edge-connectivity requirements,
   - (iii) If the degree of $s$ is odd, then add an arbitrary edge incident to $s$.

2. **Complete splitting off.**

$$G = (V, E)$$

$$G' \text{ k-e-c in } V$$

$$G'' \text{ k-e-c}$$
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Definition

A function $p$ on $2^V$ is called **skew-supermodular** if at least one of following inequalities hold for all $X, Y \subseteq V$:

\[
p(X) + p(Y) \leq p(X \cap Y) + p(X \cup Y),
\]

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p(X) + p(Y) \leq p(X - Y) + p(Y - X).
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Minimal extension

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Theorem (Frank)

Let $p : 2^V \rightarrow \mathbb{Z} \cup \{-\infty\}$ be a symmetric skew-supersmodular function.

1. The minimum number of edges in an extension ($d(X) \geq p(X)$ for all $X \subseteq V$) is equal to the maximum $p$-value of a subpartition of $V$.
2. An optimal extension can be found in polynomial time in the special cases mentioned in this talk.
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For global edge-connectivity augmentation $p(X) := k - d_G(X)$. 

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Z. Szigeti (G-SCOP, Grenoble)
Complete splitting off

Definitions

\[ G' \xrightarrow{\text{Splitting off}} G_{uv} \xrightarrow{\text{Complete Splitting off}} G'' \]

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Z. Szigeti (G-SCOP, Grenoble)
Complete splitting off

Definitions

Theorem (Mader)

Let $G' = (V + s, E)$ be a graph so that $d(s)$ is even and no cut edge is incident to $s$.

1. Then there exists a complete splitting off at $s$ that preserves the local edge-connectivity between all pairs of vertices in $V$.
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Negative Result
A graph $H$ covers a function $p : 2^{V(H)} \to \mathbb{Z} \cup \{-\infty\}$ if each cut $\delta_H(X)$ contains at least $p(X)$ edges.
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**Minimum Cover of a Symmetric Skew-Supermodular Function by a Graph**

*Instance*: $p : 2^V \rightarrow \mathbb{Z}$ symmetric skew-supermodular, $\gamma \in \mathbb{Z}^+$.

*Question*: Does there exist a graph on $V$ with at most $\gamma$ edges that covers $p$?
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A graph $H$ covers a function $p : 2^V(H) \rightarrow \mathbb{Z} \cup \{-\infty\}$ if each cut $\delta_H(X)$ contains at least $p(X)$ edges.

Minimum Cover of a Symmetric Skew-Supermodular Function by a Graph

Instance: $p : 2^V \rightarrow \mathbb{Z}$ symmetric skew-supermodular, $\gamma \in \mathbb{Z}^+$.  
Question: Does there exist a graph on $V$ with at most $\gamma$ edges that covers $p$?

Theorem (Z. Király, Z. Nutov)

The above problem is NP-complete.
**Definitions**

Given a connected graph $G = (V, E)$ and $T \subseteq V$ with $|T|$ even.

1. A subset $X$ of $V$ is called $T$-odd if $|X \cap T|$ is odd.
2. A cut $\delta(X)$ is called $T$-cut if $X$ is $T$-odd.
3. A subset $F$ of $E$ is called $T$-join if $T = \{v \in V : d_F(v) \text{ is odd}\}$.

**Examples:**

(a) $T = \{u, v\}$: a $(u, v)$-path is a $T$-join.
(b) $T = V$: a perfect matching is a $T$-join.
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Properties

1. If $X, Y$ are $T$-odd, then either $X \cap Y, X \cup Y$ or $X - Y, Y - X$ are $T$-odd.
2. A $T$-join and a $T$-cut always have an edge in common.
$T$-cut, $T$-join

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How to find a minimum $T$-join?

**Theorem (Edmonds-Johnson)**

A minimum $T$-join can be found in polynomial time using

1. shortest paths algorithm (Dijkstra) and
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Graph $G$ and vertex set $T$
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A minimum $T$-join can be found in polynomial time using
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Graph $G$ and minimum $T$-join
How to find a minimum $T$-cut?

**Theorem (Padberg-Rao)**

A minimum $T$-cut can be found in polynomial time using a cut equivalent tree $H$ and:

1. taking the set $J(H)$ edges $e$ of $H$ for which the two connected components of $H - e$ are $T$-odd,
2. taking the minimum value $c(e^*)$ of an edge of $J(H)$,
3. taking the cut defined by the two connected components of $H - e^*$.

Graph $G$ and vertex set $T$
How to find a minimum \( T \)-cut?

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Graph \( G \) and vertex set \( T \)

Cut equivalent tree \( H \)
How to find a minimum $T$-cut?

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![Graph $G$ and vertex set $T$](image1)

![Cut equivalent tree $H$](image2)
How to find a minimum $T$-cut?

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Cut equivalent tree $H$
How to find a minimum $T$-cut?

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Graph $G$ and vertex set $T$

Cut equivalent tree $H$ and edge set $J(H)$
How to find a minimum $T$-cut?

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Cut equivalent tree $H$
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Minimum $T$-cut in $G$

Cut equivalent tree $H$
Lemma

For any $T$-cut $\delta(X)$ there exist $x \in X, y \notin X$ such that $\lambda_G(x, y) \geq c(e^*)$. 
Proof

Lemma

For any $T$-cut $\delta(X)$ there exist $x \in X, y \notin X$ such that $\lambda_G(x, y) \geq c(e^*)$.

Proof : $J(H)$ is a $T$-join so there exists $xy \in J(H) \cap \delta_H(X)$ and $\lambda_G(x, y) = c(xy) \geq c(e^*)$. 
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Correctness of Padberg-Rao’s algorithm

Let $\delta(X)$ be a minimum $T$-cut and $\delta(Y)$ the $T$-cut defined by $e^*$. By the lemma, there exist $x \in X, y \notin X$ such that

$$c(e^*) = d(Y) \geq d(X) \geq \lambda_G(x, y) \geq c(e^*).$$
How to augment a minimum $T$-cut?

**Theorem (Z.Sz.)**

Given a connected graph $G = (V, E)$, $T \subseteq V$ and $k \in \mathbb{Z}$, the minimum number of edges whose addition results in a graph so that each $T$-cut is of size at least $k$ is equal to $\lceil \frac{1}{2} \text{ maximum p-value of a subpartition of } V \rceil$. An optimal augmentation can be found in polynomial time using

1. Frank’s minimal extension and
2. Mader’s complete splitting off.

**Proof**

1. works because $p(X) := k - d_G(X)$ if $X$ is $T$-odd and $-\infty$ otherwise is symmetric skew-supermodular
   - (i) $k - d_G(X)$ satisfies both inequalities,
   - (ii) $X, Y$ are $T$-odd $\implies$ either $X \cap Y, X \cup Y$ or $X - Y, Y - X$ are $T$-odd.

2. works because for all $T$-odd sets, $d_{G'}(X) \geq k$ and, by the above lemma, $k \leq \lambda_{G'}(x, y) = \lambda_{G''}(x, y) \leq d_{G''}(X)$.
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Graph $G$, vertex set $T$ and $k = 4$
How to augment a minimum $T$-cut?

**Theorem (Z. Sz.)**

Given a connected graph $G = (V, E)$, $T \subseteq V$ and $k \in \mathbb{Z}$, the minimum number of edges whose addition results in a graph so that each $T$-cut is of size at least $k$ is equal to $\left\lceil \frac{1}{2} \right. \text{maximum p-value of a subpartition of } V \right\rceil$. An optimal augmentation can be found in polynomial time using

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Minimum $T$-cut in $G + F$ is 4
Definition: symmetric parity family

A family $\mathcal{F}$ of subsets of $V$ is called symmetric parity family if

1. $\emptyset, V \notin \mathcal{F}$,
2. if $A \in \mathcal{F}$, then $V - A \in \mathcal{F}$,
3. if $A, B \notin \mathcal{F}$ and $A \cap B = \emptyset$, then $A \cup B \notin \mathcal{F}$. 
Definition: symmetric parity family

A family $\mathcal{F}$ of subsets of $V$ is called **symmetric parity family** if

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A family $F$ of subsets of $V$ is called symmetric parity family if

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3. if $A, B \notin F$ and $A \cap B = \emptyset$, then $A \cup B \notin F$. 
Definition: symmetric parity family

A family $F$ of subsets of $V$ is called symmetric parity family if

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Z. Szigeti (G-SCOP, Grenoble)
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Examples

The most important examples are:

1. $\mathcal{F} := 2^V - \{\emptyset, V\}$
2. $\mathcal{F} := \{X \subset V : X \text{ is } T\text{-odd}\}$ where $T \subseteq V$ with $|T|$ even.
Definition: symmetric parity family

A family $\mathcal{F}$ of subsets of $V$ is called **symmetric parity family** if

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Definition: symmetric parity family

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Examples

The most important examples are:

1. $\mathcal{F} := 2^V - \{\emptyset, V\}$
2. $\mathcal{F} := \{X \subset V : X$ is $T$-odd$\}$ where $T \subseteq V$ with $|T|$ even.
How to find a minimum $\mathcal{F}$-cut?

**Theorem (Goemans-Ramakrishnan)**

Given a connected graph $G$ and a symmetric parity family $\mathcal{F}$, a minimum $\mathcal{F}$-cut, that is a minimum cut over $\mathcal{F}$, can be found in polynomial time using a cut equivalent tree $H$ and:

1. taking the set $J(H)$ edges $e$ of $H$ for which the two connected components of $H - e$ are in $\mathcal{F}$,
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Proof

Lemma

For any $X \in \mathcal{F}$ there exist $x \in X$, $y \notin X$ such that $\lambda_G(x, y) \geq c(e^*)$. 
Proof

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For any $X \in \mathcal{F}$ there exist $x \in X$, $y \notin X$ such that $\lambda_{G}(x, y) \geq c(e^*)$.

**Proof:** Exercise: there exists an edge $xy \in \delta_{J(H)}(X)$. 

Z. Szigeti (G-SCOP, Grenoble)
Proof

Lemma

For any \( X \in \mathcal{F} \) there exist \( x \in X, y \notin X \) such that \( \lambda_G(x, y) \geq c(e^*) \).

Proof: Exercise: there exists an edge \( xy \in \delta_{J(H)}(X) \).

Correctness of Goemans-Ramakrishnan’s algorithm

The same proof works as for Padberg-Rao’s algorithm.
How to augment a minimum $\mathcal{F}$-cut?

**Theorem (Z.Sz.)**

Given a connected graph $G$, a symmetric parity family $\mathcal{F}$ and $k \in \mathbb{Z}$, the minimum number of edges whose addition results in a graph so that each $\mathcal{F}$-cut is of size at least $k$ is equal to $\lceil \frac{1}{2} \text{ maximum } p\text{-value of a subpartition of } V \rceil$. An optimal augmentation can be found in polynomial time using

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**Proof**

1. works because $p(X) := k - d_G(X)$ if $X \in \mathcal{F}$ and $-\infty$ otherwise is symmetric skew-supermodular
   
   (i) $k - d_G(X)$ satisfies both inequalities,
   (ii) If $X, Y \in \mathcal{F}$, then either $X \cap Y, X \cup Y \in \mathcal{F}$ or $X - Y, Y - X \in \mathcal{F}$.

2. works because for all $X \in \mathcal{F}$, $d_{G'}(X) \geq k$ and, by the above lemma,
   
   $k \leq \lambda_{G'}(x, y) = \lambda_{G''}(x, y) \leq d_{G''}(X)$. 

Conclusion

1. Special cases:
   1. Global edge-connectivity augmentation (Watanabe, Nakamura)
   2. Minimum $T$-cut augmentation

2. A new polynomial special case of the NP-complete problem
   Minimum Cover of a Symmetric Skew-Supermodular Function by a Graph
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