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Tightness in subset bounds for coherent configurations

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Abstract Association schemes have many applications to the study of designs, codes and geometries, and are well-studied. Coherent configurations are a natural generalization of association schemes, however, analogous applications have yet to be fully explored. Recently, Hobart [Michigan Math. J. 58 (2009), 231–239] generalized the linear programming bound for association schemes, showing that a subset Y of a coherent configuration determines positive semidefinite matrices, which can be used to constrain certain properties of the subset. The bounds are tight when one of these matrices is singular, and in this paper we show that this gives information on the relations between Y and any other subset. We apply this result to sets of nonincident points and lines in coherent configurations determined by projective planes (where the points of the subset correspond to a maximal arc) and partial geometries.

Keywords Coherent configuration · Delsarte bound · Finite geometry · Association scheme

1 Introduction and definitions

Association schemes are a useful tool in design theory, finite geometry and coding theory. One important technique is the linear programming bound described by Delsarte [4], which has been applied to the Hamming and Johnson schemes to obtain new bounds on codes and designs. New applications of this bound continue to be found (see for example [13]).

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Coherent configurations are a natural generalization of association schemes. Some combinatorial objects are better described by coherent configurations, and some association schemes admit a natural fission into a larger coherent configuration. In 2005, Schrijver [12] split the Hamming scheme into a coherent configuration, and used the group representations of this configuration to obtain better bounds on codes. Hobart [10] generalized the work of Schrijver and Delsarte, giving a bound based on subsets of a coherent configuration. We will refer to this bound as the semidefinite programming bound, since it can be formulated as a semidefinite programming problem. Recently, the authors used this to get new bounds on the independence number of orthogonal polarity graphs of projective planes of even order (see [11]).

When Delsarte's bound is tight, it implies additional structural information on the subset in question. In Section 2 of this paper, we consider the question of what it means for the semidefinite programming bound to be tight, and show that it implies structural information about the subset. In the process, we give a simplified proof of the semidefinite programming bound. Sections 3 and 4 demonstrate applications to sets of nonincident points and lines in projective planes and partial geometries. In the final section, we show that an inequality of Haemers for partial geometries follows from the semidefinite programming bound, and that the structural implications of equality given by Haemers are a special case of the results in Section 4.

Coherent configurations were originally defined by D. G. Higman, and [8] is a useful reference for basic results. We start by giving the definitions and listing some of these results.

A *coherent configuration* (c.c.) is a pair $(X, \mathcal{R})$ such that $\mathcal{R} = \{R_1, \ldots, R_d\}$ is a set of relations on X, satisfying the following.

1. $\mathcal{R}$ is a partition of $X \times X$.
2. If $R_i \cap \mathrm{diag}(X \times X) \neq \emptyset$ then $R_i \subset \mathrm{diag}(X \times X)$.
3. For each $R_i \in \mathcal{R}$, $R_i^T \in \mathcal{R}$.
4. For $R_i, R_j, R_k \in \mathcal{R}$ and $x, y \in X$ with $(x, y) \in R_k$, the number of z such that $(x, z) \in R_i$ and $(z, y) \in R_j$ is a constant p_{ij}^k, independent of the choice of x and y.

The partition of the identity relation given by the second condition yields a partition of X into sets X_α, $\alpha \in \Omega$, called the *fibers* of the c.c. Each relation R_i is contained in some $X_\alpha \times X_\beta$. A coherent configuration with one fiber is called an *association scheme*.

Each relation R_i defines a corresponding $|X| \times |X|$ matrix A_i, with rows and columns indexed by X and where $(A_i)_{xy} = 1$ or 0 depending on whether (x, y) is in R_i or not. Clearly $A_i A_j = \sum_k p_{ij}^k A_k$. The matrices $\{A_i\}$ generate a semisimple matrix algebra $\mathcal{A}$, which is also closed under entrywise multiplication. If the algebra $\mathcal{A}$ is commutative, it implies that there is only one fiber. In this case the configuration is a commutative association scheme, and these are well-studied (for example, see [3]).

We now introduce the representation theory of coherent configurations, see [8] for proofs and more information on this topic. Let $\{\Delta_1, \ldots, \Delta_m\}$ be the set of absolutely irreducible representations of $\mathcal{A}$. We choose the representations so that $\Delta_s(A^*) = (\Delta_s(A))^*$. Denote the degree of the representation Δ_s by e_s, and the multiplicity in the standard module by h_s. Since $\mathcal{A}$ is semisimple, it decomposes into a direct sum of algebras $\oplus\mathcal{E}_s$ where the algebra $\mathcal{E}_s$ is isomorphic to $M_{e_s}(\mathbb{C})$.

We can find a basis $\{\mathcal{E}_{ij}^s\}$ for each algebra $\mathcal{E}_s$, satisfying

$$\mathcal{E}_{ij}^s\mathcal{E}_{kl}^t = \delta_{st}\delta_{jk}\mathcal{E}_{il}^s \quad (\mathcal{E}_{ji}^s)^* = \mathcal{E}_{ij}^s \quad \Delta_s(\mathcal{E}_{ij}^t) = \delta_{st}E_{ij} \tag{1}$$

where E_{ij} is the $e_s \times e_s$ matrix with a 1 in the i, j entry and 0 everywhere else. The collection of all such matrices $\mathcal{E}_{ij}^s$, as s, i, j vary, form a second basis of $\mathcal{A}$. In general, this basis is not unique. In fact, it is only unique when all the irreducible representations have degree 1, i.e. when the c.c. is a commutative association scheme.

We will describe the representations using the following notation. Let $a_{ij}^s(A_k)$ be the i, j entry of the matrix $\Delta_s(A_k)$ and $m_i = |R_i|$. Then, as is shown in [8],

$$A_k = \sum_{i,j,s} a_{ij}^s(A_k)\mathcal{E}_{ij}^s \tag{2}$$

and

$$\mathcal{E}_{ij}^s = h_s \sum_k \frac{1}{m_k}\overline{a_{ij}^s(A_k)}A_k. \tag{3}$$

In the next section, the representations and the basis $\{\mathcal{E}_{ij}^s\}$ will be used to derive information on subsets in coherent configurations.

2 The semidefinite programming bound for coherent configurations

Throughout this section, let $\mathcal{C} = (X, \mathcal{R})$ be a coherent configuration, with parameters and other notation as in Section 1.

Let Y be a subset of X and use y to denote the characteristic vector of Y, that is, the vector indexed by X with x entry equal to 1 if $x \in Y$ and 0 otherwise. Define $b_i = |(Y \times Y) \cap R_i|$, the number of ordered pairs of elements of Y that are in relation R_i, and $m_i = |R_i|$. In [10], Hobart defined

$$D(Y) = \sum_i \frac{b_i}{m_i}A_i$$

and proved the following theorem; the positive semidefiniteness of the matrices gives what we refer to as the semidefinite programming bound.

Theorem 1
(a) We can express $D(Y)$ in terms of the idempotents $\mathcal{E}_{ij}^s$ by $D(Y) = \sum_{i,j,s} \frac{y^*\mathcal{E}_{ji}^s y}{h_s}\mathcal{E}_{ij}^s$.
(b) The matrices $D(Y)$ and $\Delta_s(D(Y))$ are positive semidefinite.

If the coherent configuration is a commutative association scheme, the semidefinite programming bound reduces to the linear programming bound of Delsarte (see [4], p. 26), and we can get information on the so-called outer distribution of a subset when the linear programming bound is tight. Our aim is to give similar information for coherent configurations.

The question is then what tightness means in the case of a positive semidefinite matrix. We interpret it to mean that 0 is an eigenvalue of the matrix $\Delta_s(D(Y))$ (and hence of $D(Y)$). We start by finding a different way of looking at $\Delta_s(D(Y))$, using the matrix

$$\mathcal{F}_s = \sum_{i,j} \mathcal{E}^s_{ij}.$$

Note that by (1), $\mathcal{F}_s^2 = e_s \mathcal{F}_s$ and $\mathcal{F}_s^* = \mathcal{F}_s$, so $\mathcal{F}_s$ is positive semidefinite.

By Theorem 1(a), we know that $\Delta_s(D(Y)) = \sum_{i,j} \frac{y^* \mathcal{E}^s_{ji} y}{h_s} E_{ij}$. Since $y^* \mathcal{E}^s_{ji} y$ is a scalar, $y^* \mathcal{E}^s_{ji} y = (y^* \mathcal{E}^s_{ji} y)^T = (y^* \mathcal{E}^{s*}_{ij} y)^T = \overline{y^* \mathcal{E}^s_{ij} y}$. Using this fact and (1), we compute the i, j entry.

$$\Delta_s(D(Y))_{i,j} = \frac{y^* \mathcal{E}^s_{ji} y}{h_s} = \frac{\overline{y^* \mathcal{E}^s_{ij} y}}{h_s} = \frac{1}{h_s} \overline{y^* \mathcal{E}^s_{ii} \mathcal{F}_s \mathcal{E}^s_{jj} y} = \frac{1}{h_s} \overline{(\mathcal{E}^s_{ii} y)^* \mathcal{F}_s (\mathcal{E}^s_{jj} y)} \quad (4)$$

Motivated by this equation, we define the following matrix. For fixed s and any vector v, let χ_v be the matrix with ith column $\mathcal{E}^s_{ii} v$. Now for the subset Y of X with characteristic vector y, define

$$N_s(Y) = \chi_y^* \mathcal{F}_s \chi_y.$$

Since $\mathcal{F}_s$ is positive semidefinite, we have the following result.

Theorem 2 *$N_s(Y)$ is positive semidefinite.*

By (4), $N_s(Y) = \overline{h_s \Big(\Delta_s(D(Y)) \Big)}$. Hence the matrices $N_s(Y)$ are essentially those of [10], and this gives a simpler proof of Theorem 1(b).

Suppose that this statement is tight; that is, suppose $N_s(Y)$ has eigenvalue 0. Then there is a nonzero vector w such that $N_s(Y)w = 0$. It follows that $w^* N_s(Y) w = 0$. Using the definition of $N_s(Y)$ and the fact that $\mathcal{F}_s^* = \mathcal{F}_s$, this implies that

$$\frac{1}{e_s} w^* \chi_y^* \mathcal{F}_s^* \mathcal{F}_s \chi_y w = 0.$$

Hence,

$$\mathcal{F}_s \chi_y w = 0. \quad (5)$$

Let Z be a subset of X with characteristic vector z. As above, let χ_z be the matrix whose ith column is $\mathcal{E}^s_{ii} z$. Equation (5) implies the following lemma.

Lemma 1 *Let $Y, Z \subseteq X$, and suppose w is a vector such that $N_s(Y)w = 0$. Then $\chi_z^* \mathcal{F}_s \chi_y w = 0$.*

This equation gives information about the relationship between the two subsets of Y and Z, in a similar manner to results for two subsets of an association scheme (see [4], or [3] Section 2.5).

Theorem 3 *Let Y and Z be subsets of X, and let $w = (w_j)$ be a vector such that $N_s(Y)w = 0$. Let $c_i = |(Z \times Y) \cap R_i|$, the number of elements of $Z \times Y$ in relation R_i.*

Then

$$\overline{\Delta_s \left(\sum_k \frac{c_k}{m_k} A_k \right)} w = 0. \tag{6}$$

In terms of entries, this means that for every i,

$$\sum_{j,k} \frac{c_k}{m_k} \overline{a^s_{ij}(A_k)} w_j = 0. \tag{7}$$

Proof The i, j entry of $\chi_z^* \mathcal{F}_s \chi_y$ is

$$(\chi_z^* \mathcal{F}_s \chi_y)_{i,j} = (\mathcal{E}^s_{ii} z)^* \mathcal{F}_s (\mathcal{E}^s_{jj} y) = z^* \mathcal{E}^s_{ij} y.$$

By equation (3), this equals

$$h_s \sum_k \frac{1}{m_k} \overline{a^s_{ij}(A_k)} \, (z^* A_k y) = h_s \sum_k \frac{c_k}{m_k} \overline{a^s_{ij}(A_k)}.$$

This together with Lemma 1 gives (7). Equation (6) now follows, since (7) together with (1) and (2) shows that all the entries of the vector are 0. □

We now give some applications of this theorem to finite geometries.

3 Projective planes

The first example uses a coherent configuration constructed from a projective plane. The resulting characterization is not novel; it can also be deduced from the incidence graph of the plane and in particular follows from [6], Theorem 3.1.1. We include it as an illustration of using the semidefinite programming bound to recognize geometric structures such as maximal arcs.

Let $\Pi = (\mathcal{P}, \mathcal{L})$ be a projective plane of order q. Let $X = \mathcal{P} \cup \mathcal{L}$ and define relations on X as follows.

$R_1 =$ identity on $\mathcal{P}$ $\qquad$ $R_3 =$ identity on $\mathcal{L}$
$R_2 =$ collinear points $\qquad$ $R_4 =$ intersecting lines

$R_5 = \{(p, L) : p \text{ lies on } L\}$ $\qquad$ $R_7 = R_5^T$
$R_6 = \{(p, L) : p \text{ does not lie on } L\}$ $\qquad$ $R_8 = R_6^T$

It is easy to show that $\mathcal{C}_\Pi = (X, \{R_i\})$ is a c.c. with fibers $\mathcal{P}$ and $\mathcal{L}$ and type $\begin{pmatrix} 2 & 2 \\ 2 & 2 \end{pmatrix}$. Table 1 gives the representations of $\mathcal{C}_\Pi$, together with the values

Table 1 Irreducible representations for $\mathcal{C}_\Pi$

i	m_i	$\Delta_1(A_i)$	$\Delta_2(A_i)$
1	v	E_{11}	E_{11}
2	$v(v-1)$	$(q^2+q)E_{11}$	$-E_{11}$
3	v	E_{22}	E_{22}
4	$v(v-1)$	$(q^2+q)E_{22}$	$-E_{22}$
5	$v(q+1)$	$(q+1)E_{12}$	$\sqrt{q}E_{12}$
6	vq^2	q^2E_{12}	$-\sqrt{q}E_{12}$
7	$v(q+1)$	$(q+1)E_{21}$	$\sqrt{q}E_{21}$
8	vq^2	q^2E_{21}	$-\sqrt{q}E_{21}$

of $m_i = |R_i|$; these are easily calculated using the methods of [8]. Note that $v = |\mathcal{P}| = |\mathcal{L}| = q^2+q+1$.

Suppose we have a set of n_p points and n_l lines such that none of the points is incident with any of the lines. What bounds can we give for such sets?

Theorem 4 *Let Π be a projective plane of order q, and suppose Y is a set of $n_p \neq 0$ points and $n_l \neq 0$ lines which are all nonincident. Then*

$$(n_p+q)(n_l+q) \leq q(q+1)^2. \tag{8}$$

If equality holds then each line not in Y contains $\frac{n_p+q}{q+1}$ points of Y, and dually each point not in Y lies on $\frac{n_l+q}{q+1}$ lines of Y.

Proof By Theorem 2, $N_2(Y)$ is positive semidefinite. We know that $b_1 = n_p$, $b_2 = n_p(n_p-1)$, $b_3 = n_l$, $b_4 = n_l(n_l-1)$, $b_5 = b_7 = 0$ and $b_6 = b_8 = n_pn_l$, and hence the matrix

$$\frac{1}{h_2}N_2(Y) = \overline{\Delta_2(D(Y))} = \begin{pmatrix} \frac{b_1}{m_1} - \frac{b_2}{m_2} & \frac{b_5}{m_5}\sqrt{q} - \frac{b_6}{m_6}\sqrt{q} \\ \frac{b_7}{m_7}\sqrt{q} - \frac{b_8}{m_8}\sqrt{q} & \frac{b_3}{m_3} - \frac{b_4}{m_4} \end{pmatrix}$$
$$= \begin{pmatrix} \frac{n_p(v-n_p)}{v(v-1)} & -\frac{n_pn_l}{vq^2}\sqrt{q} \\ -\frac{n_pn_l}{vq^2}\sqrt{q} & \frac{n_l(v-n_l)}{v(v-1)} \end{pmatrix}$$

is positive semidefinite.

Taking the determinant, we get

$$\frac{n_pn_l(v-n_p)(v-n_l)}{v^2(v-1)^2} - \frac{n_p^2n_l^2q}{v^2q^4} \geq 0,$$

and hence $(v-n_p)(v-n_l)q^3 - n_pn_l(v-1)^2 \geq 0$. Since $v = q^2+q+1$, this simplifies to

$$(n_p+q)(n_l+q) \leq q(q+1)^2.$$

In the case of equality, we can solve for n_p in terms of n_l and q. Then the matrix $N_2(Y)$ is singular, and the vector $\begin{pmatrix} n_l+q \\ \sqrt{q}(q+1) \end{pmatrix}$ gives a basis for the null space. We use Theorem 3 to get more information about the set Y.

Suppose $Z = \{m\}$, where m is a line not in Y. Let c be the number of points of Y on m. For this set, $c_4 = n_l$, $c_7 = c$, $c_8 = n_p - c$ and $c_i = 0$ for all other i. Using $s = 2$ and $i = 2$, we have the following equation from (7).

$$\frac{n_l}{v(v-1)}(-1)\sqrt{q}(q+1) + \frac{c}{v(q+1)}\sqrt{q}(n_l+q) + \frac{n_p - c}{vq^2}(-\sqrt{q})(n_l+q) = 0$$

We can use $v = q^2+q+1$ and the assumption that$(n_p+q)(n_l+q) = q(q+1)^2$ to eliminate v and n_l. Then, solving for c, we get

$$c = \frac{n_p + q}{q+1}.$$

If instead $Z = \{P\}$, a point not in Y, the analysis is similar, interchanging n_p and n_l. Note that the vector $\begin{pmatrix} \sqrt{q}(q+1) \\ n_p + q \end{pmatrix}$ also forms a basis for the null space of $N_2(Y)$. □

When equality holds in (8), the points of Y form a (n_p, c)-arc in the projective plane where $c = (n_p + q)/(q + 1)$; that is, a set of n_p points such that every line meets the arc in at most c points (see [2]). The arc is maximal, since $n_p = c(q + 1) - q$. It is also clear that every line meets the arc in either 0 or c points (which is also equivalent to the arc being maximal).

For a Desarguesian projective plane, nontrivial maximal arcs do not exist in planes of odd order (see [1]). The question is open for non-Desarguesian planes; unfortunately our methods do not give any further information about the structure of such an arc. For even order planes, many examples of such maximal arcs are known (for example, see [7]).

4 Partial geometries

A *partial geometry* with parameters (s, t, α) is an incidence structure $(\mathcal{P}, \mathcal{L})$ such that two points are on at most one line, every point is on $t+1$ lines, every line contains $s + 1$ points, and given a point P which does not lie on a line m, there are exactly α points on m collinear with P. See for example [5] for more information about such structures. An inequality for sets of nonincident points and lines was found in [10] using a c.c.; we will use Theorem 3 to investigate the case that the inequality is tight. We start by defining the coherent configuration.

Let $\Gamma = (\mathcal{P}, \mathcal{L})$ be a partial geometry of order (s, t, α) such that $0 < \alpha < \min(s + 1, t + 1)$. Let $X = \mathcal{P} \cup \mathcal{L}$ and define relations on X as follows.

$R_1 =$ identity on $\mathcal{P}$	$R_4 =$ identity on $\mathcal{L}$
$R_2 =$ collinear points	$R_5 =$ intersecting lines
$R_3 =$ noncollinear points	$R_6 =$ nonintersecting lines
$R_7 = \{(p, L) : p \text{ lies on } L\}$	$R_9 = R_7^T$
$R_8 = \{(p, L) : p \text{ does not lie on } L\}$	$R_{10} = R_8^T$

Table 2 Irreducible representations for $\mathcal{C}_\Gamma$

i	m_i	$\Delta_1(A_i)$	$\Delta_2(A_i)$	$\Delta_3(A_i)$	$\Delta_4(A_i)$
1	$\lvert\mathcal{P}\rvert$	E_{11}	E_{11}	1	0
2	$\lvert\mathcal{P}\rvert s(t+1)$	$s(t+1)E_{11}$	$(s-\alpha)E_{11}$	$-t-1$	0
3	$\lvert\mathcal{P}\rvert\frac{st(s+1-\alpha)}{\alpha}$	$\frac{st(s+1-\alpha)}{\alpha}E_{11}$	$(-s-1+\alpha)E_{11}$	t	0
4	$\lvert\mathcal{L}\rvert$	E_{22}	E_{22}	0	1
5	$\lvert\mathcal{L}\rvert t(s+1)$	$t(s+1)E_{22}$	$(t-\alpha)E_{22}$	0	$-s-1$
6	$\lvert\mathcal{L}\rvert\frac{st(t+1-\alpha)}{\alpha}$	$\frac{st(t+1-\alpha)}{\alpha}E_{22}$	$(-t-1+\alpha)E_{22}$	0	s
7	$\lvert\mathcal{P}\rvert(t+1)$	$\sqrt{(s+1)(t+1)}E_{12}$	$\sqrt{s+t+1-\alpha}E_{12}$	0	0
8	$\lvert\mathcal{P}\rvert\frac{st(t+1)}{\alpha}$	$\frac{st\sqrt{(s+1)(t+1)}}{\alpha}E_{12}$	$-\sqrt{s+t+1-\alpha}E_{12}$	0	0
9	$\lvert\mathcal{L}\rvert(s+1)$	$\sqrt{(s+1)(t+1)}E_{21}$	$\sqrt{s+t+1-\alpha}E_{21}$	0	0
10	$\lvert\mathcal{L}\rvert\frac{st(s+1)}{\alpha}$	$\frac{st\sqrt{(s+1)(t+1)}}{\alpha}E_{21}$	$-\sqrt{s+t+1-\alpha}E_{21}$	0	0

It is easy to show that $\mathcal{C}_\Gamma = (X, \{R_i\})$ is a c.c. with fibers $\mathcal{P}$ and $\mathcal{L}$ and type $\begin{pmatrix} 3 & 2 \\ 2 & 3 \end{pmatrix}$; this is a special case of the c.c.'s considered by Higman in [9]. Table 2 gives the irreducible representations for $\mathcal{C}_\Gamma$ together with the values of $m_i = |R_i|$. Note that $|\mathcal{P}| = (s+1)(st+\alpha)/\alpha$ and $|\mathcal{L}| = (t+1)(st+\alpha)/\alpha$.

Suppose again that Y is a set of n_p points and n_l lines such that none of the points is incident with any of the lines; such sets were previously considered in [10].

Once again apply Theorem 2, this time using $N_2(Y)$, $N_3(Y)$, and $N_4(Y)$. We have that $b_1 = n_p$, $b_2 + b_3 = n_p(n_p - 1)$, $b_4 = n_l$, $b_5 + b_6 = n_l(n_l - 1)$, $b_7 = b_9 = 0$ and $b_8 = b_{10} = n_p n_l$. Then

$$\frac{b_1}{m_1} + \frac{-(t+1)b_2}{m_2} + \frac{tb_3}{m_3} \geq 0. \tag{9}$$

$$\frac{b_4}{m_4} + \frac{-(s+1)b_5}{m_5} + \frac{sb_6}{m_6} \geq 0 \tag{10}$$

and the matrix

$$\begin{aligned}
\frac{1}{h_2}N_2(Y) &= \overline{\Delta_2(D(Y))} \\
&= \begin{pmatrix} \frac{b_1}{m_1} + \frac{(s-\alpha)b_2}{m_2} - \frac{(s+1-\alpha)b_3}{m_3} & \frac{b_7\sqrt{s+t+1-\alpha}}{m_7} - \frac{b_8\sqrt{s+t+1-\alpha}}{m_8} \\ \frac{b_9\sqrt{s+t+1-\alpha}}{m_9} - \frac{b_{10}\sqrt{s+t+1-\alpha}}{m_{10}} & \frac{b_4}{m_4} + \frac{(t-\alpha)b_5}{m_5} - \frac{(t+1-\alpha)b_6}{m_6} \end{pmatrix} \\
&= \begin{pmatrix} \frac{n_p}{|\mathcal{P}|} + \frac{(s-\alpha)b_2}{|\mathcal{P}|s(t+1)} - \frac{\alpha(s+1-\alpha)b_3}{|\mathcal{P}|st(s+1-\alpha)} & -\frac{\alpha n_p n_l\sqrt{s+t+1-\alpha}}{|\mathcal{P}|st(t+1)} \\ -\frac{\alpha n_p n_l\sqrt{s+t+1-\alpha}}{|\mathcal{L}|st(s+1)} & \frac{n_l}{|\mathcal{L}|} + \frac{(t-\alpha)b_5}{|\mathcal{L}|t(s+1)} - \frac{\alpha(t+1-\alpha)b_6}{|\mathcal{L}|st(t+1-\alpha)} \end{pmatrix}
\end{aligned}$$

is positive semidefinite. Equivalently, the diagonal entries and determinant of this matrix are nonnegative. For completeness, we record these results as explicit inequalities, using $b_3 = n_p(n_p - 1) - b_2$ and $b_6 = n_l(n_l - 1) - b_5$ and multiplying by $|\mathcal{P}|$, $|\mathcal{L}|$ to simplify.

$$n_p + \frac{(s-\alpha)b_2}{s(t+1)} - \frac{\alpha(n_p(n_p-1) - b_2)}{st} \geq 0 \tag{11}$$

$$n_l + \frac{(t-\alpha)b_5}{t(s+1)} - \frac{\alpha(n_l(n_l-1)-b_5)}{st} \geq 0 \tag{12}$$

$$\left(n_p + \frac{(s-\alpha)b_2}{s(t+1)} - \frac{\alpha(n_p(n_p-1)-b_2)}{st}\right)\left(n_l + \frac{(t-\alpha)b_5}{t(s+1)} - \frac{\alpha(n_l(n_l-1)-b_5)}{st}\right) - \frac{\alpha^2(s+t+1-\alpha)n_p^2 n_l^2}{s^2t^2(s+1)(t+1)} \geq 0 \tag{13}$$

Note that (13) appears in [10], Theorem 4.1. We examine the case where this bound is tight.

Theorem 5 *Let Y be a subset of $n_p \neq 0$ points and $n_l \neq 0$ lines such that none of the points is incident with any of the lines, where the bound (13) is met with equality. Let P be a point of the partial geometry, and let c_2 be the number of points of Y (not including P) collinear with P and c_7 the number of lines of Y incident with P. Then*

$$\alpha n_p n_l c_2 + u c_7 = \begin{cases} \alpha n_l b_2 & \text{if } P \in Y \\ \alpha n_l(b_2 + n_p(t+1)) & \text{if } P \notin Y \end{cases} \tag{14}$$

where $u = -\alpha(t+1)n_p^2 + (t+1)(st+\alpha)n_p + (st+\alpha)b_2$ is positive.

Similarly, let m be a line of the partial geometry and and let c_5 be the number of lines of Y (not including m) meeting m and c_9 the number of points of Y incident with m. Then

$$\alpha n_p n_l c_5 + u' c_9 = \begin{cases} \alpha n_p b_5 & \text{if } m \in Y \\ \alpha n_p(b_5 + n_l(s+1)) & \text{if } m \notin Y \end{cases} \tag{15}$$

where $u' = -\alpha(s+1)n_l^2 + (s+1)(st+\alpha)n_l + (st+\alpha)b_5$ is positive.

Proof We first show that u and u' are positive. Note that u is equal to $|\mathcal{P}|st(t+1)$ times the $1,1$ entry of $\frac{1}{h_2}N_2(Y)$ and u' is $|\mathcal{L}|st(t+1)$ times the $2,2$ entry of $\frac{1}{h_2}N_2(Y)$. By Theorem 2, $u, u' \geq 0$.

Now, (13) can be written as

$$\left(\frac{u}{st(t+1)}\right)\left(\frac{u'}{st(s+1)}\right) - \frac{\alpha^2(s+t+1-\alpha)n_p^2 n_l^2}{s^2t^2(s+1)(t+1)} \geq 0.$$

If either u or u' is 0, then this together with $n_p, n_l \neq 0$ gives

$$\frac{\alpha^2(s+t+1-\alpha)n_p^2 n_l^2}{s^2t^2(s+1)(t+1)} < 0,$$

a contradiction. Hence this shows that u and u' are both positive.

If inequality (13) is tight, then the matrix $\frac{1}{h_2}N_2(Y)$ is singular. Considering the first row of $\frac{1}{h_2}N_2(Y)$, we see that the vector $\begin{pmatrix}\alpha n_p n_l \sqrt{s+t+1-\alpha} \\ u\end{pmatrix}$ forms a basis for the null space of $N_2(Y)$. We now use Theorem 3 with Δ_2 to get more information about the set Y.

Suppose $Z = \{P\}$, where P is a point of the partial geometry. Note that $c_1 = 1$ or 0 depending on whether or not P is in Y, $c_3 = n_p - c_1 - c_2$, $c_8 = n_l - c_7$ and $c_i = 0$ for all other i. Using (7) with $i = 1$ and $s = 2$ and simplifying, we can write the resulting equation as (14).

Since $u > 0$, equation (14) also shows that the number of lines c_7 of Y containing P is determined by the number of points c_2 of Y collinear with P.

If we let $Z = \{m\}$ where m is a line, a similar argument can be used to obtain (15). Using the second row of $\frac{1}{h_2}N_2(Y)$, we see that the vector $\begin{pmatrix} u' \\ \alpha n_p n_l \sqrt{s+t+1-\alpha}\end{pmatrix}$ also forms a basis for the null space of $N_2(Y)$, and use (7) with $i = 2$, $s = 2$. Since $u' > 0$, (15) shows that c_9 is determined by c_5.

□

5 A subset bound of Haemers

We now compare the bounds and results of the previous section to a bound of Haemers. As in that section, let $\mathcal{C}_\Gamma$ be the coherent configuration of a partial geometry Γ with $0 < \alpha < \min(s+1, t+1)$, and Y a set of nonincident points and lines. In [6], Haemers used eigenvalue interlacing on the incidence matrix of the partial geometry to derive the following bound.

Theorem 6 *([6], Corollary 3.1.3) For any set Y of $n_p \neq 0$ points and $n_l \neq 0$ lines where the points and lines are nonincident in a partial geometry with parameters (s, t, α), we have the inequality $r \leq 0$, where*

$$r = (\alpha n_p + (s+t+1-\alpha)(s+1))(\alpha n_l + (s+t+1-\alpha)(t+1)) - (s+t+1-\alpha)(s+1)^2(t+1)^2.$$

We now show that this is a special case of Theorem 5, by deriving it from the bounds in the previous section.

By (9) and (10), we have the two inequalities

$$b_2 \leq \frac{n_p(s(s+1-\alpha) + \alpha(n_p - 1))}{s+1}, \text{ and} \tag{16}$$

$$b_5 \leq \frac{n_l(t(t+1-\alpha) + \alpha(n_l - 1))}{t+1}. \tag{17}$$

The bound (13) is weakest when b_2, b_5 are as large as possible. Hence from (16), (17), and (13), we obtain

$$\frac{n_p n_l(\alpha + st)(s+t+1-\alpha)r}{(s+1)^2 s^2 (t+1)^2 t^2} \leq 0,$$

which is equivalent to Theorem 6.

Applying a result of Haemers ([6], Theorem 3.1.1(ii)) shows that in the case of equality, all of the values of c_i for a point are determined by whether the point is in the set Y or not, and similarly for lines. We can get the same information and more from the coherent configuration.

We have that $r = 0$ (that is, the bound is tight) if and only if equality holds in (16), (17), and (13). This also means inequalities (9) and (10) are tight, allowing us to apply Theorem 3 to the representations Δ_3 and Δ_4. For this subset Y and any subset Z, this gives

$$\frac{c_1}{m_1} + \frac{-(t+1)c_2}{m_2} + \frac{tc_3}{m_3} = 0, \text{and}$$

$$\frac{c_4}{m_4} + \frac{-(s+1)c_5}{m_5} + \frac{sc_6}{m_6} = 0.$$

In the case where Z consists of a single point P, these equations together with Theorem 5 allow us to obtain:

$$c_2 = c_1(s-\alpha) + \frac{\alpha n_p}{s+1} \tag{18}$$

$$c_7 = \frac{\alpha n_l(s+1)(1-c_1)}{(s+1)(st+\alpha) - \alpha n_p} \tag{19}$$

The value of c_1 is 1 or 0, depending on whether P is in the set Y or not. For a given c_1 we can obtain exact values for all c_i for each case. This recovers the same information that eigenvalue interlacing gives. Dually, all c_i for a line m depend only on whether or not m is in Y. These values of c_i must all be integral, giving us necessary divisibility conditions.

These divisibility conditions can be satisfied. For example, consider the generalized quadrangle consisting of the points and lines of the elliptic quadric $Q^-(5,q)$; this is a partial geometry with $s = q$, $t = q^2$, and $\alpha = 1$. The quadric contains copies of the quadric $Q(4,q)$; fix one of these and call it Q. Take Y to be all the lines in Q, and all of the points of $Q^-(5,q)$ not in Q. Then $n_p = q^4 - q^2$, $n_l = q^3 + q^2 + q + 1$, Theorem 6 holds with equality, and c_2 and c_7 are given by (18) and (19).

Theorem 5 also allows us to consider the case when (13) is tight but one or both of (9) and (10) is not, and still obtain information on the outer distribution of Y, whereas interlacing requires all three to be tight.

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