International Journal of Pure and Applied Mathematics
Volume 115 No. 2 2017, 285-300
ISSN: 1311-8080 (printed version); ISSN: 1314-3395 (on-line version)
url: http://www.ijpam.eu
doi: 10.12732/ijpam.v115i2.7
ijpam.eu

FI-SEMISIMPLE, $FI-t$-SEMISIMPLE AND STRONGLY $FI-t$-SEMISIMPLE MODULES

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Abstract: In this paper, we introduce the notions of FI-semisimple, FI-t-semisimple and strongly FI-t-semisimple modules. This is a generalization of semisimple modules. Many results connected with these concepts are given.

AMS Subject Classification: 16D10, 16D70, 16D90, 16P70
Key Words: FI-semisimple modules, FI-t-semisimple modules, strongly FI-t-semisimple modules, t-semisimple modules

1. Introduction

Through this paper R be a ring with unity and M is a right R-module. A submodule A of an R-module M is said to be essential in M (denoted by $A \leq_{ess} M$), if $A \bigcap W(0)$ for every non-zero submodule W of M. Equivalently $A \leq_{ess} M$ if whenever $A \bigcap W = 0$, then $W = 0$, see [12]. Recall that a submodule N of R-module M is called closed ($N \leq_c M$) if whenever $N \leq_{ess} W \leq M$, then $N = W$, see [10]. In other word, $N \leq_c M$, if N has no proper essential extension in M, see [12]. The singular submodule of an R-module M denoted by $Z(M)$ and

Received: January 3, 2017
Revised: April 3, 2017
Published: July 14, 2017

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defined by $x : inM : ann(x)_R \leq_{ess} R$ where $ann(x)_R = r \in R : xr = 0$. M is called singular submodule if $Z(M) = M$ and nonsingular if $Z(M) = 0$.

Let $Z_2(M)$ be the second singular (or Goldi torsion) of M which is defined by $Z(M/(Z(M))) = (Z_2(M))/(Z(M))$ where $Z(M)$ is the singular submodule of M A module M is called Z_2-torsion if $Z_2(M) = M$ and a ring R is called right Z_2-torsion if $Z_2(R_R) = R_R$, see [11]. The concept of t-essential submodules is introduced as generalization of essential submodules, see [5]. A submodule N of M is said to be t-essential in M (denoted by $(N \leq_{tes} M)$ if for every submodule B of M , $N \bigcap B \leq Z_2(M)$ implies that $B \leq Z_2(M)$. It is clear that every essential submodule is t-essential, but not conversely. However, for a nonsingular R-module M, the two concepts are coincide.

As a generalization of closed submodule Asgari and Haghany in [3] introduced concept of t-closed. A submodule N of M is t-closed in M if $N \leq_{tes} W \leq M$ implies that $N = W$. Every t-close is closed, but not conversely and they are equivalent in nonsingular modules.

A submodule N of M is called fully invariant if $f(N) \leq N$ for every R-endomorphism f of M. Clearly 0 and M are fully invariant submodules of M, see [10]. M is called duo module if every submodule of M is fully invariant. A submodule N of an R-module is called stable if for each homomorphism f:$N \to M$, $f(N) \leq N$. A module is called fully stable if every submodule of M is stable, [1]. Recall that an R-module M is multiplication if for each submodule N of M, there exists ideal I of R such that $N = MI$. Equivalently $M$is a multiplication R-module if for each submodule N of M $N = M(N :_R M)$, where $(N : M) = r \in R : rM \leq N$, see [10]. A module M is called semisimple if every submodule is direct summand. It is known that a module M is semisimple if every submodule N contains a direct summand K of M such that $N \leq_{ess} N$.

This observation lead Asgari et al [4] to introduce the notion [3] of t-semisimple modules as a generalization of semisimple modules. A module M is t-semisimple if for every submodule N of M, there exists a direct summand K such that $K \leq_{tes} N$.

In this paper we present three generalizations of semisimple and t-semisimple modules namely FI-semisimple, $FI{-}t$-semisimple and strongly $FI{-}t$-semisimple.

It is clear that the class of strongly t-semisimple modules contains the class of t-semisimple.

This paper consists of four sections, in Section 2 we present the concept namely FI-semisimple modules. Where an R -module M is called FI-semisimple if for each fully invariant submodule N of M, there exists K direct summand of M such that K essential in N. Many properties about this concept, and many connections between it and other related concepts are introduced.

In Section 3 we study generalization of t-semisimple namely, $FI{-}t$-semisimple module is introduced. An R-module M is called $FI - t-$semisimple if for each fully invariant submodule N of M, there exists $K \leq^{\square} M$ such that $K \leq_{tes} N$. Also, many properties about this concept, are given. Section 4 we present another generalization of t-semisimple namely strongly $FI - t$-semisimple. An R-module M is called strongly $FI - t$-semisimple if for each fully invariant submodule N of M, there exists a fully invariant direct summand K such that $K \leq_{tes} N$. Many properties about this concept are introduced, and many connections between it and other related concepts are presented.

We quote the following for future use

Proposition 1.1 ([3]). *The following statements are equivalent for a submodule A of an R-module:*

(1) A is t-essential in M;

(2) $(A + Z_2(M))/Z_2(M)$ is essential in $M/Z_2(M)$;

(3) $A + Z_2(M)$ is essential in M;

(4) M/A is Z_2-torsion.

Lemma 1.2 ([5]). *Let A_λ be submodule of $M\lambda$ for all λ in a set Λ.*

(1) If Λ is a finite and $A_\lambda \leq_{tes} M_\lambda$, then $\bigcap_\Lambda \mid A_\lambda \leq_{tes} \bigcap_\Lambda |M_\lambda$ for all $\lambda \in \Lambda$.

(2) $\oplus_\Lambda A_\lambda \leq_{tes} \oplus \Lambda M_\lambda$, if and only if $A_\lambda \leq_{tes} M_\lambda$, for all $\lambda \in \Lambda$.

Lemma 1.3 ([14]). *Let R be a ring and let $L \leq K$ be submodules of an R-module M such that L is a fully invariant submodule of K and K is a fully invariant submodule of M. Then L is a fully invariant submodule of M.*

The following results are well known.

Proposition 1.4 ([7]). *Any sum (or intersection) of fully invariant submodules an R-module M is fully invariant submodules M.*

Proposition 1.5 ([14]). *If $M = \oplus_i \in \Lambda$ where X_i is an R-module, for each $i \in \Lambda$ and N is a fully invariant submodule of M, then $N = \oplus_i \in \Lambda x_i \bigcap N$ and $X_i \bigcap N$ is fully invariant submodule of X_i, for each $i \in \Lambda$.*

Proposition 1.6 ([14]). *Let R be any ring and let an R -module $M = K \oplus K^{'}$ be the direct sum of submodules K, $K^{'}$ Then K is a fully invariant submodule of M if and only if Hom $(K, K^{'}) = 0$.*

Proposition 1.7 ([6]). *Let M be an R-module and $K \leq L \leq M$ if L/K is a fully invariant submodule of M/K and K is a fully invariant submodule of M, then L is a fully invariant in M.*

2. *FI*-Semisimple Modules

Definition 2.1. An R-module M is called FI-semisimple if for each fully invariant submodule N of M, there exists direct summand K such that $K \leq_{ess} N$.

The following result is a characterization of FI-semisimple modules.

Proposition 2.2. *An R-module M is FI-semisimple if and only if every fully invariant submodule of M is a direct summand.*

Proof. $\Rightarrow$ Let N be a fully invariant submodule of M, so there exists $K \leq^{\square} M$ such that $K \leq_{ess} N$. But $K \leq^{\square} M$ implies K is closed in M, so it has no proper essential extension in M. Thus $K = N$ and so $N \leq^{\square} M$.

$\Leftarrow$ Let N be a fully invariant submodule of M. By hypothesis $N \leq^{\square} M$. But $N \leq_{ess} N$ and $N \leq^{\square} M$. Thus M is FI-semisimple.

Remarks and Examples 2.3 (1) It is clear that every semisimple module is FI-semisimple, but the converse is not true in general, for example: The Z-module Q has only two fully invariant submodules which are (0), Q. Hance Q is FI-semisimple, but it is not semisimple.

(2) t-semisimple module does not implies FI-semisimple in general for example Z_{12} as$Z$-module t-semisimple but it is not FI-semisimple. Also FI-semisimple module does not implies t-semisimple, for example Q as Z-module is FI-semisimple and it is not t-semisimple.

(3) If M is a duo module (hence if M is a multiplication module), then M is a semisimple module if and only if M is FI-semisimple. In particular the Z-modules Z, Z_4, Z_{12} are not FI-semisimple. Also, for every commutative ring R, R is semisimple if and only if R is FI-semisimple.

(4) A fully invariant submodule of FI-semisimple is FI-semisimple.

Proof. Let N be a fully invariant submodule of M and M is a FI-semisimple. Let U be a fully invariant submodule of N, hence U is a fully invariant in M by proposition 1. 3. It follows that $U \leq^{\square} M$. Thus $U \oplus U' = M$ for some $U' \leq M$ and so $N = (U \oplus U') \bigcap N = U \oplus (U' \bigcap N)$ by modular law. Then $U \leq^{\square} N$. Thus N is FI-semisimple by Proposition (2. 2). Z_4 as Z_4 -module is not singular, but it is Z_2-torsion, so it is strongly t-semisimple.

(5) Every FI-semisimple module M is FI-extending. Where M is called FI-extending if every fully invariant submodule is essential in a direct summand.

Proof. Let N be a fully invariant submodule of M. As M is FI-semisimple, $N \leq^{\oplus} M$. But $N \leq_{ess} N$. So that M is FI-extending.

(6) If M and N are isomorphic R-modules, then M is FI-semisimple if and only if N is FI-semisimple.

(7) If $f : M \mapsto M'$ be an epimorophism and M' is FI-semisimple, then it is not necessary that M is FI-semisimple. For example $\Pi : Z \mapsto Z/(6) \cong Z_6, Z_6$ is FI-semisimple, but Z is not.

Proposition 2.4. *Let M be a FI-semisimple R-module and N is a fully invariant submodule in M then M/N is a FI-semisimple module.*

Proof. Let W/N be a fully invariant submodule of M/N. Since N is a fully invariant submodule of M. Then W is fully invariant submodule of M by Proposition (1. 7). But M is FI-semisimple, so $W \leq^{\oplus} M$. Then $W \oplus K = M$ for some $K \leq M$. This implies $W/N \oplus K+N/N = M/N$. Thus $W/N \leq^{\oplus} M/N$ and M/N is FI-semisimple.

Corollary 2.5. *Let $f : M \mapsto M'$ be an R- epimorphism and $Kerf$ is a fully invariant submodule of M. If M is a FI-semisimple R-module, then M' is a FI-semisimple.*

Proof. Since f:$M \mapsto M'$ epimorphism, $M/Kerf \cong M'$. But $M/Kerf$ is a FI-semisimple module by proposition (2. 4), hence M' is FI-semisimple by Remarks and Examples 2. 3(5).

Corollary 2.6. *Let Mbe a FI-semisimple R-module. Then$M/(Z_2(M))$ is FI-semisimple and $M = Z_2(M) \oplus M'$ where M' is nonsingular FI-semisimple.*

Proof. As $Z_2(M)$ is a fully invariant submodule of M, $M/(Z_2(M))$ is FI-semisimple module by Proposition (2. 4). Also, $Z_2(M)$ is a fully invariant submodule in $M implies Z_2(M) \leq^{\oplus} M$, by Proposition (2. 2). Thus $M = Z_2(M) \oplus M'$ for some $M' \leq M$. But $M' \cong M/(Z_2(M))$, so M' is nonsingular FI-semisimple.

Proposition 2.7. *Let $M = M_1 \oplus M_2$, where $M_1, M_2 \leq M$ If $M_1$and M_2 are FI-semisimple, then M is FI-semisimple and converse hold if M_1 and M_2 are FI-submodules of M.*

Proof. $\Rightarrow$ Let N be a fully invariant submodule of M. Then

$$N = N \bigcap M = N \bigcap M_1 \oplus N \bigcap M_2$$

and $N \bigcap M_1, N \bigcap M_2$ are fully invariant submodules of M_1 and M_2 respectively by Lemma 1.3. Put $N_1 = N \bigcap M_1$, $N =_2= N \bigcap M_2$. Hence $N_1 \leq M_1$,

$N_2 \leq M_2$, since M_1 and M_2 are FI-semisimple modules. It follows that $N = N_1 \oplus N_2 \leq^{\oplus} M$ and so M is FI-semisimple.

$\Leftarrow$ Since M_1 is a fully invariant submodule of M, and

$$M/M_1 = (M_1 \oplus M_2)/M_1 \cong M_2.$$

Hence M_2 is FI-semisimple, by Proposition (2. 4). Similarly, M_1 is FI-semisimple.

3. $FI - t$-Semisimple Modules

Definition 3.1. An R-module M is called FI-t-semisimple if for each fully invariant submodule N of M, there exists $K \leq^{\oplus} M$ such that $K \leq_{tes} N$.

Remarks and Examples 3.2. (1) It is clear that every t-semisimple module is $FI - t$-semisimple, but the converse is not true , for Q as Z-module is not t-semisimple and it is clear that it is $FI - t$-semisimple.

(2) It is clear that every FI-semisimple module is $FI - t$-semisimple, hence each of the Z-module Q, $Q \oplus Z_2$, $Z_2 \oplus Z_6$ is $FI - t$-semisimple, since each of them is FI-semisimple module.

(3) The converse of part (2) is not true in general, for example, the Z_{12} is a $FI - t$-semisimple (since it is t-semisimple) but it is not FI-semisimple, and the Z-module Z is not $FI - t$-semisimple.

(4) Let M be a nonsingular R-module. Then M is FI-semisimple if and only if M is $FI - t$-semisimple. In particular, Z as Z-module is not $FI - t$-semisimple, and if $R = Z[x]$, then R_R is not $FI - t$-semisimple.

Proof. $\Rightarrow$It is clear by part (2).

$\Leftarrow$ Let M be a $FI - t$-semisimple module and N be a fully invariant submodule of M, there exists $K \leq^{\oplus} M$ and $K \leq_{tes} N$. But M is nonsingular implies N is nonsingular and hence $K \leq_{ess} N$. But $K \leq^{\oplus} M$ implies K is a closed submodule of M and so that $K = N$. It follows that M is FI-semisimple by Proposition 2.2.

Proposition 3.3. *Every fully invariant submodule of a $FI - t$-semisimple module is $FI - t$-semisimple.*

Proof. Let N be a fully invariant submodule of a $FI - t$-semisimple R-module M. To prove N is $FI - t$-semisimple, let W be fully invariant submodule of N. Hence W is a fully invariant submodule of M. It follows that there exists

$K \leq^{\oplus} M$ and $K \leq_{tes} W$, since M is $FI-t$-semisimple. Hence $M = K \oplus C$ for some $C \leq M$ and so that $N = K \oplus (C \bigcap N)$, thus $K \leq^{\oplus} N$ and so that $N$is $FI-t$-semisimple.

Proposition 3.4. *Let $M = M_1 \oplus M_2$. If M_1 and M_2 are $FI-t$-semisimple, then M is a $FI-t$-semisimple. The converse holds, if M_1 and M_2 are fully invariant submodules.*

Proof. $\Rightarrow$ Let N be a fully submodule of M. Then $N = N_1 \oplus N_2$, where N_1 is fully invariant in M_1 and N_2 is fully invariant in M_2 by Lemma(1. 5). Hence, there exist $K_1 \leq^{\oplus} M_1$ and $K_2 \leq^{\oplus} M_2$ such that $K_1 \leq_{tes} N_1$, $K_2 \leq_{tes} N_2$. Hence $K = K_1 \oplus K_2 \leq^{\oplus} M$ and $K = K_1 \oplus K_2 \leq_{tes} N_1 \oplus N_2 = N$. $\Leftarrow$ It is clear by Proposition (3. 3).

To prove our next result, we need the following lemma.

Lemma 3.5. *Let $K \leq N \leq M$ such that $N \leq^{\oplus} M$. If K is a fully invariant submodule in M, then K is a fully invariant in N.*

Proof. Since $N \leq^{\oplus} M$, $N \oplus L = M$ for some $L \leq M$. Let $\vartheta : N \mapsto N$ be any R-homomorphism. ϑ can be extended to homomorphism $h : M \mapsto M$ where $h(x) = (\vartheta(x) if x \in N 0 otherwise)$. Then $h(K) \leq K$. But $K \leq N$, so $h(K) = \vartheta(K)$ and hence $\vartheta(K) \leq K$; that K is a fully invariant submodule of N.

Let $(*)$ means the following: For an R-module M, the complement of $Z_2(M)$ is stable in M. An R-module M is called self-projective if M is M-projective; equivalently for every submodule N of M and for every homomorphism $\theta : M \mapsto M/N$, θ can be lifted by a homomorphism $\psi : M \mapsto N$such that $\phi \circ \psi = \theta$, where ϕ is the natural projection from $N into M/N$[13].

Theorem 3.6. *Consider the following statements for an R-module M:*

(1) M is an $FI-t$-semisimple module;

(2) $M/Z_2(M)$ is a FI-semisimple module;

(3) $M = Z_2(M) \oplus M'$, where $M'$is nonsingular, FI-semisimple and M' is stable in M;

(4) Every nonsingular FI-submodule of M is a direct summand;

(5) Every FI-submodule of M which contains $Z_2(M)$ is direct summand of M.

Then (3)$\Rightarrow$ (5) $\Rightarrow$ (2) and (3) $\Rightarrow$ (1) $\Rightarrow$ (4).(4) $\Rightarrow$ (3) if condition $(*)$ hold. (2)$\Rightarrow$ (1) if M is self- projective. Thus all statements from (1) to (5) are equivalent if M satisfies $(*)$ and M is self-projective.

Proof. (3)$\Rightarrow$ (5) Let N be a fully invariant submodule of M, $N \supseteq Z_2(M)$. Since $M = Z_2(M) \oplus M'$ where M' is FI-semisimple nonsingular and stable in

M.

Then $N = Z_2(M) \oplus (M)^{'} \bigcap N$ by modular law. As N and $M^{'}$ are fully invariant in M, so $N \bigcap M^{'}$ is fully invariant in M. Hence $(N \bigcap M^{'})$ is fully invariant in $M^{'}$ by Lemma (3. 5). As $M^{'}$ is FI-semisimple, $(N \bigcap M^{'}) \leq^{\oplus} M^{'}$.

It follows that $M^{'} = (N \bigcap M^{'}) \oplus W$, for some $W \leq M^{'}$ and so that $M = Z_2(M) \oplus [(N \bigcap M^{'}) \oplus W] = [Z_2(M) \oplus (N \bigcap M^{'})] \oplus W = N \oplus W$. Therefore $N \leq^{\oplus} M$.

(5) $\Rightarrow$(2) Let $N/(Z_2(M))$ be a fully invariant submodule of $M/(Z_2(M))$. Since $Z_2(M)$ is fully invariant in M, then N is fully invariant in M by Proposition (1. 7). Also $N \oplus Z_2(M)$, so by condition (5), $N \leq^{\oplus} M$. Thus $N \oplus K = M$ for some $K \leq M$. it follows that $M/(Z_2(M))N/(Z_2(M)) \oplus (K + Z_2(M))/(Z_2(M))$ So that $N/(Z_2(M)) \leq^{\oplus} M/(Z_2(M))$ and so $M/(Z_2(M))$ is a FI-semisimple.

(3) $\Rightarrow$ (1) By hypothesis, $M = Z_2(M) \oplus M^{'}$, where $M^{'}$ is nonsingular FI-semisimple and $M^{'}$ is stable in M. Let N be a fully invariant submodule of M. It follows that $(N \bigcap M^{'}) \leq^{\oplus} M$. On the other hand, $N/((N \bigcap M^{'})) \cong ((N + M^{'}))/M^{'} \leq M/M^{'}$ which is Z_2-torsion, hence, $N/((N \bigcap M^{'}))$ is Z_2-torsion and so that $(N \bigcap M^{'}) \leq_{tes} N$ by Proposition (1. 1). Thus $(N \bigcap M^{'}) \leq^{\oplus} M$ and $(N \bigcap M^{'}) \leq_{tes} N$ which implies that M is $FI - t$-semisimple.

(1) $\Rightarrow$(4) Let N be a nonsingular fully invariant submodule of M. By condition (1) there exists $K \leq^{\oplus} M$ such that $K \leq_{tes} N$. As N is nonsingular, $K \leq_{ess} N$. But$K \leq^{\oplus} M$, implies K is closed in M, hence $K = N$. Thus $N \leq^{\oplus} M$.

(4) $\Rightarrow$ (3) Let $M^{'}$ be a complement of $Z_2(M)$. Hence $Z_2(M) \oplus M^{'} \leq_{ess} M$, implies $M^{'} \leq M$ by proposition (1. 1). Hence $M/M^{'}$ is Z_2-torsion. We claim that $M^{'}$is nonsingular. To explain our assertion, suppose $x \in Z(M^{'}), sox \in M^{'} \oplus M$ and $ann(x) \leq_{ess} R$. Hence $ann(x) \leq_{tes} R$ and this implies $x \in Z_2(M)$. Thus $x \in Z_2(M) \bigcap = M^{'} = (0)$, thus $x = 0$ and $M^{'}$ is a nonsingular. By condition $(*)$, $M^{'}$ is stable, hence $M^{'} \leq^{\oplus} M$ by condition (4). Thus$M = M^{'} \oplus L$, for some $L \leq M$ and so $Z_2(M) = Z_2(M^{'}) \oplus Z_2(L)$. But $Z_2(M^{'}) = 0$ and $L \cong M/M^{'}$ is Z_2-torsion, so $Z_2(L) = L$. Hence $Z_2(M) = L$. Thus $M = M^{'} \oplus Z_2(M)$ such that $M^{'}$ is nonsingular and stable. To prove $M^{'}$ is FI-semisimple, let N be a fully invariant submodule of $M^{'}$. As $M^{'}$ is fully unvariant in M, so N is fully invariant in M , and since $M^{'}$ is nonsingular, implies N is nonsingular. Thus N is nonsingular fully invariant in M. Hence by condition (4), $N \leq^{\oplus} M$, and so $N \oplus W = M$, for some $W \leq M$. Then $M^{'} = (N \oplus W) \bigcap M^{'} = N \oplus (W \bigcap M^{'})$ by modular law. Thus $N \leq^{\oplus} M^{'}$. Thus $M^{'}$ is a FI-semisimple module.

(2)$\Rightarrow$ (1) Let N be a fully invariant submodule of M. Then $N + Z_2(M)$ is fully invariant submodule of M. Since M is self-projective, $N{+}Z_2(M))/(Z_2(M))$

is a fully invariant submodule of $(M)/(Z_2(M))$. Hence , $(N+Z_2(M))/(Z_2(M)) \leq^{\oplus} (M)/(Z_2(M))$ because $(M)/(Z_2(M))$ is FI-semisimple. Hence $(M)/(Z_2(M)) = (N+Z_2(M))/(Z_2(M)) \oplus (W)/(Z_2(M))$ for some $(W)/(Z_2(M)) \leq (M)/(Z_2(M))$, and this implies $(N+Z_2(M)) \oplus W = M$. But $Z_2(M) \leq W$ so that $N \oplus W = M$. Thus $N \leq^{\oplus} M$ and M is FI-semisimple and hence M is FIt-semisimple.

Recall that "an R-module M is called $FI-t$-extending if every fully invariant t-closed submodule of M is a direct summand", see [6].

Proposition 3.7. *Let M be an R-module such that condition $(*)$ hold. If M is a $FI-t$-semisimple, then $M$is $FI-t$-extending.*

Proof. By Theorem (3. 6) ($1 \Rightarrow 4 \Rightarrow 3 \Rightarrow 5$) for each fully invariant submodule N such that $N \supseteq Z_2(M), N \leq^{\oplus} M$ and hence for each fully invariant submodule N with that $N \supseteq Z_2(M)$, imply $N \leq_{ess} N \leq^{\oplus} M$. Thus M is $FI-t$-extending by [6, Theorem 2. 2 (6)$\Rightarrow$(1)].

Theorem 3.8. *Let M be an R-module such that complement of a fully invariant submodule is stable. Then M is an $FI-t$-semisimple if and only if M/C is an FI-semisimple, for each t-closed fully invariant submodule of M, and the converse hold if M is self- projective.*

Proof. $\Rightarrow$ By Proposition (3. 7), M is $FI-t$-extending. Hence, any fully invariant t-closed submodule, $C \leq^{\oplus} M$ by Definition. Thus $C \oplus C' = M$ for some $C' \leq M$. By hypothesis C' is a FI-submodule of M. Hence C' is a $FI-t$-semisimple by Proposition (2. 10). But $C' \propto M/C$ is a FI-semisimple.
$\Leftarrow$ $Z_2(M)$ is $FI-t$-closed submodule. Hence by hypothesis, $M/(Z_2(M))$ is FI-semisimple. Thus M is $FI-t$-semisimple by Theorem (2. 6) $2 \Rightarrow 1$.

Proposition 3.9. *Let M be an R-module such that complement of any fully invariant submodule is fully invariant. Then M is a $FI-t$-semisimple if and only if $N+Z_2(M)$ is closed, for each fully invariant submodule N of M.*

Proof. $\Rightarrow$ By Theorem (3. 6) $1 \Rightarrow 5$, for each fully invariant submodule N of M such that $N \supseteq Z_2(M)$, $N$is a direct summand. But $N+Z_2(M) \supset Z_2(M)$ and it is fully invariant submodule of M, so that $N+Z_2(M)$ is a direct summand and hence $N+Z_2(M)$ is a closed submodule of M.

$\Leftarrow$ To prove M is $FI-t$-semisimple. Let K be a nonsingular fully invariant submodule of M. Assume L is a complement of K, then by hypothesis, L is fully invariant submodule of M. Also $K \oplus L \leq_{ess} M$, and $K \oplus L$ is a fully invariant submodule of M. It follows that $K \oplus L) + Z_2(M) \leq_{ess} M$. But $(K \oplus L) + Z_2(M)$ is fully invariant submodule containing $Z_2(M)$, so that $(K \oplus L) + Z_2(M)$ is closed by hypothesis. Thus $(K \oplus L) + Z_2 M = M$ and so $K + (L + Z_2(M)) = M$ is closed by hypothesis. Thus $(K \oplus L) + Z_2 M = M$ and so $K + (L + Z_2(M)) = M$. But we can show that. $K \bigcap (L + Z_2(M)) = 0$ as follows if $0 \neq x \in K \bigcap (L + Z_2(M))$, then $x = l + y, l \in L, y \in Z_2(M)$. Since K is nonsingular, $ann(x) \leq\neq_{ess} R$. But $x - l = y, so ann(x - l) = ann(y) \leq_{ess} R$. It follows that $ann(x) \bigcap ann(l) \leq_{ess} R$, which implies $ann(x) \leq_{ess} R$, that is a contradiction Thus $K \bigcap (L + Z)_2(M)) = 0$, and $K \oplus (L + Z_2(M)) = M$. that is $K \leq^{\square} M$ and hence M is $FI-t$-semisimple by Theorem (3.6)4 $\Rightarrow$ (3) $\Rightarrow$ (1).

Proposition 3.10. *Let M be an R-module such that condition $(*)$ hold. Then M is $FI - tsemisimple$ if and only if M has no proper nonzero fully invariant submodule N containing $Z_2(M)$ with $N \leq_{ess} M$.*

Proof. $\Rightarrow$ By Theorem (3.6)1 $\Rightarrow$ 5, since M is $FI - t$-semisimple, implies for is fully invariant submodule N of M containing $Z_2(M)$, $N \leq^{\square} M$. Hence $N \leq\neq_{ess} M$ for each ($N \supseteq Z_2(M)$, N is fully invariant).

$\Leftarrow$ $Let M'$ be a complement of $Z_2(M)$, so that $M' \oplus Z_2(M) \leq_{ess} M$. But by hypothesis, $M'$is a fully invariant submodule of M and also $M' \oplus Z_2(M)$ is a fully invariant submodule of M. Thus $M' \oplus Z_2(M) = M$. Hence, $M' \cong M/(Z_2(M))$ is nonsingular and stable. Let N be a fully invariant submodule of M'. Since M' is a fully invariant in M, then N is a fully invariant submodule in M. Hence $N + Z_2(M)$ is fully invariant in M. Let K be a complement of $N + Z_2(M)$. So that $(N + Z_2(M)) \oplus K \leq_{ess} M$. But by hypothesis $(N + Z_2(M)) \oplus K = M$, then $N + (Z_2(M) + K) = M$. We can show that $N \bigcap (Z_2(M) + K) = (0)$, as follows. Let $x \in N \bigcap (Z_2(M) + K)$. Then $x = a + b$ for some $a \in Z_2(M), b \in K$. Then $x - a = b \in (N + Z_2(M)) \bigcap K = 0$, hence $x - a = b = 0$, and so that $x = a \in (N \bigcap Z_2(M) = Z_2(N) = 0$. Thus $x = 0$ and $N \bigcap (Z_2(M) + K) = 0$, hence $N \oplus (Z_2(M) + K) = M$, that is $N \leq^{\square} M$. Now $M' = [N \oplus (Z_2(M) + K)] \bigcap M' = N \oplus [(Z_2(M) + K) \bigcap M']$. Thus $N \leq^{\square} M'$. Hence M' is FI-semisimple which implies that $M/(Z_2(M))$is FI-semisimple. Thus by Theorem 3. 7 ((3) $\Rightarrow$ (1))M is $FI - t$-semisimple.

Recall that if N, K are submodules of $M.K$ is called a supplement of N if K is minimal with respect to the property $M = K + N$. Equivalently K is a supplement of Nif $M = K + N$ and $K \bigcap N \ll K$

(the notion $\ll$ denotes a small submodule)[8]. K is called a weak supplement of N if $M = K + N$ and $K \bigcap N \ll M$, [8].

Proposition 3.11. **(3. 11):** *Let M be an R-module such that condition $(*)$ hold. A module M is $FI - t$-semisimple if $Rad(M)$ is Z_2-torsion and every nonsingular fully invariant submodule of M has a weak supplement.*

Proof. Let N be nonsingular fully invariant submodule of M. By hypothesis there exists a submodule K of M such that $M = K + N$ and $K \bigcap N \ll M$. Clearly $M = (K + Rad(M)) + N$. Now we show that $(K + Rad(M)) \bigcap N = 0$.

Assume that $x \in (K + Rad(M)) \bigcap N$. Then $x = y + z$ where $y \in K$ and $z \in Rad(M)$. Since $Rad(M)$ is Z_2-torsion there exists a t-essential right ideal I of R such that $(x-y)I = 0$. Thus $xI = yI \leq K \bigcap N \leq Rad(M) \leq Z_2(M)$. So $(x + Z_2(M))I = Z_2(M)$ and $x + Z_2(M) \in Z_2(M/(Z_2(M))) = 0$. Hence $x \in Z_2(M)$. Thus $x \in Z_2(M) \bigcap N = Z_2(N) = 0$ and this implies that N is direct summand. Hence by Theorem 3. 6 $(4 \Rightarrow 3 \Rightarrow 1)M$ is $FI-t$-semisimple.

Proposition 3.12. *The following assertions are equivalent for an module M which satisfies, that for any $B \leq M$, a complement of a fully invariant submodule A of B is a fully invariant in B.*

(1) M is $FI-t$-semisimple

(2) For each fully invariant submodule N of M, there exists a decomposition $M = K \oplus L$ such that $K \leq L$ and L is stable in M and $N \bigcap L \leq Z_2(L)$.

(3) For each fully invariant submodule N of $M, N = K \oplus K^{'}$ such that K is a direct summand stable in M and $K^{'}$ is Z_2-torsion.

Proof. (1) $\Rightarrow$ (2) Let N be a fully invariant submodule of M. Let K be a complement of $Z_2(N)$ in N. Then K is a fully invariant in N and $K \oplus Z_2(N) \leq_{ess} N$. By proposition (3. 3) and proposition (3. 10), $K \oplus Z_2(N) = N$. Let C be a complement of $K \oplus Z_2(M)$, so C is a fully invariant submodule of M and $(K \oplus Z_2(M)) \oplus C \leq_{ess} M$. But M is $FI-t$-semisimple, hence by proposition (3. 10), $K \oplus Z_2(M)) \oplus C = M$. Put $Z_2(M)C = L$, hence is a fully invariant in M. Moreover, $N = (K \oplus L) \bigcap N = K \oplus (N \bigcap L)$. But $K \oplus Z_2(N) = N$ implies $N/K \cong Z_2(N)$ which is Z_2-torsion. On other hand, $N/K \cong N \bigcap L$, so that $N \bigcap L$ is Z_2-torsion. Then $N \bigcap L = Z_2(N \bigcap L) \leq Z_2(L)$. Thus $M = K \oplus L$ is the desired decomposition.

(2) $\Rightarrow$ (3) Let N be a fully invariant submodule of M. By condition (2), $M = K \oplus L$ where $K \leq N$ and $L$is stable in M and $N \bigcap L \subseteq Z_2(L)$. Hence $N = (K \oplus L) \bigcap N = K \oplus (L \bigcap N)$. Put $K^{'} = N \bigcap L$, so that $N = K \oplus K^{'}$, and $N/K \cong K^{'} = N \bigcap L$ which is Z_2-torsion. Also K stable in M, since K is a complement of L in M.

(3) $\Rightarrow$ (1)$LetN$ be a fully invariant submodule of M. By condition (3), $N = K \oplus K^{'}$, where $K \leq^{\square} M$ and stable in M and $K^{'}$is Z_2-torsion. Now $K \leq N$ and $N/K \cong K^{'}$ which is Z_2-torsion. Hence $K \leq_{tes} N$ and so that M is $FI-t$-semisimple.

An R-module M is said to be t-Baer, if $t_M(I) = m \in M | Im \leq Z_2(M)$ is a direct summand of M for each left ideal I of $End(M)$. An R- module M is $FI-t$-Baer if $t_M(I)$ is a direct summand of M for any two-sided ideal I of $End(M)$. $t_S(N) = \varphi \in S : \varphi N \leq Z_2(M)[6]$.

Proposition 3.13. *Let M be an R-module such that complement of*

$Z_2(M)$ is stable. Then the following statements are equivalent:

(1) M is $FI-t$-semisimple.

(2) M is $FI-t$-extending and $N = t_M t_S(N)$ for every fully invariant submodule N of M contain $Z_2(M)$.

(3) M is $FI-t$-Baer and $N = t_M t_S(N)$ for every fully invariant submodule N of M contain $Z_2(M)$.

Proof. (1) $\Rightarrow$ (2)M is $FI-t$-semisimple implies M is $FI-t$-extending by Proposition (3. 7). Now, let N be a fully invariant submodule of M and $N \supseteq Z_2(M)$. Hence $N \leq^{\oplus} M$ by Theorem 3. 6 (4 $\Rightarrow$ 3 $\Rightarrow$ 5). Hence, $M = N \oplus N'$ for some $N' \leq M$. It is obvious, that $N \leq t_M t_S(N)$. Let Π' be the canonical projection on N', that is $\pi' : N \oplus N' \mapsto N' \leq N \oplus N'$, so $\pi' \in S$, $\pi'(N) = 0 \leq Z_2(M)$, so $\pi' \int_S(N)$, $m \in t_M t_S(N), \pi'(m) \in Z_2(M) \leq N$. Hence $\pi'(m) = 0$, and then $m \in N$.

(2) $\Rightarrow$ (3) It is obvious, since every $FI-t$-extending is $FI-t-$ Baer [6, Thorem 3. 9].

(3) $\Rightarrow$ (1) Since M is $FI-t$-Baer, $Z_2(M) = t_M(S)$ is a direct summand and $M = Z_2(M) \oplus M'$, where M' is nonsingular. Hence M' is a complement of $Z_2(M)$, so it is stable.

Now, let N' be a fully invariant submodule of M', so that N' is a fully invariant submodule of M. Put $N = Z_2(M) \oplus N'$. Then N is a fully invariant submodule of M containing $Z_2(M)$. On the other hand, M is $FI-t$-Baer and $t_{(N)}$ is a two sided ideal of S, hence $t_M t_S(N) \leq^{\oplus} M$. Thus $N \leq^{\oplus} M$. It follows that $M = N \oplus W$ for some $W \leq M$, hence $M = Z_2(M) \oplus N' \oplus W$. But by hypothesis complement of $Z_2(M)$ is stable so by [1], $N' \oplus W = M'$ and hence $N' \leq^{\oplus} M'$, and this implies M' is FI-semisimple. Therefore M is $FI-t-$ semisimple by Theorem 3. 6 (3 $\Rightarrow$ 1).

4. Strongly $FI-t$-Semisimple

Definition 4.1. An R-module M is called strongly $FI-t$-semisimple if for each fully invariant submodule N of M, there exists a fully invariant direct summand K such that $K \leq_{tes} N$.

Remarks and Examples (4. 2). (1) Every strongly $FI-t$-semisimple is $FI-t$-semisimple and every strongly t-semisimple is strongly $FI-t$-semisimple.

(2) Consider Q as Z-module is strongly $FI-t$-semisimple, since Q has only two fully invariant submodules (0), Q. But Q is not strongly t-semisimple.

(3) Every FI-semisimple module M is strongly $FI-t$-semisimple.

Proof. Let N be a FI-submodule of M. Then $N \leq^{\square} M$, since M is a FI-semisimple. But $N \leq_{tes} N$, hence M is strongly $FI-t$-semisimple.

Proposition 4.3. *Let M be an R-module with property, complement of any submodule of M is stable. The following statements are equivalent:*
(1) M is strongly $FI-t$-semisimple;
(2) M is $FI-t$-semisimple;

Proof. (1) $\Rightarrow$ (2)It is clear.
(2) $\Rightarrow$ (1) Let N be a fully invariant submodule of M. Since M is $FI-t$-semisimple, there exists $K \leq^{\square} M$ and $K \leq_{tes} N$. Hence $M = K \oplus W$ for some $W \leq M$. Hence K is a complement of W. But by hypothesis K is stable. Thus M is strongly $FI-t$-semisimple.

Proposition 4.4. *A fully invariant submodule N of a strongly $FI-t$-semisimple module M is strongly $FI-t$-semisimple.*

Proof. Let W be a fully invariant submodule of N. Then Wis a fully invariant submodule of M by Proposition (1. 3). Since M is strongly $FI-t$-semisimple, there exists $K \leq^{\square} M$, K is a fully invariant submodule of M and $K \leq_{tes} W$. But $K \leq^{\square} M$ implies $M = K \oplus A$ for some $A \leq M$ and this implies $N = K \oplus (A \bigcap N)$; that is $K \leq^{\square} N$. Beside this by Lemma (3. 5), K is a fully invariant submodule of N. Thus N is strongly $FI-t$-semisimple.

Remark 4.5. The condition a fully invariant submodule of M cannot be dropped from Proposition 4. 4 as the following example shows. Q as Z-module is strongly $FI-t$-semisimple, and $Z < Q$. But Z is not strongly $FI-t$-semisimple and, Z is not fully invariant submodule of Q.

We can set the following corollaries.

Corollary 4.6. *For any strongly $FI-t$-semisimple module M, $Z_2(M)$ is strongly $FI-t$-semisimple.*

Proof. It follows directly by Proposition 4.4.

Proposition 4.7. *Let M be an R-module and satisfies $(*)$. If M is strongly $FI-t$-semisimple, then $M/(Z_2(M))$ is FI-semisimple, and hence it is strongly $FI-t$-semisimple. The converse is hold if M is self-projective.*

Proof. $\Rightarrow$ As M is strongly $FI-t$-semisimple, M is $FI-t$-semisimple and hence by Theorem 3. 6 (1 $\Rightarrow$ 2), $M/(Z_2(M))$ is FI-semisimple. $\Leftarrow$If $M/(Z_2(M))$ is a FI-semisimple, then by the proof of Theorem 3. 6 (2 $\Rightarrow$ 1)M is a FI-semisimple module and hence M is strongly $FI-t$-semisimple.

Corollary 4.8. *Let M be a self-projective and satisfies $(*)$. Then the statements are equivalent:*

(1) M is strongly $FI-t$-semisimple;

(2) $M/(Z_2(M))$ is FI-semisimple;

(3) M is $FI-t$-semisimple.

Proof. (1) $\Leftrightarrow$ (2) *It follows by proposition (4. 7).*

(2) $\Leftrightarrow$ (3) *It follows by Theorem 3. 6 (2 $\Leftrightarrow$ 1).*

The following result follows by combining Proposition (4. 3) and Proposition (3. 10).

Proposition 4.9. *Let M be an R-module such that complement of any submodule of M is stable. Then the following conditions are equivalent:*

(1) M is strongly $FI-t$-semisimple;

(2) M is $FI-t$-semisimple;

(3) M has no proper nonzero fully invariant submodule N containg $Z_2(M)$ and $N \leq_{ess} M$.

Lemma 4.10. *Let $M = M_1 \oplus M_2$ where M_1 and M_2 be R-modules, such that M_1 and M_2 are fully invariant in M. Then M is strongly $FI-t$-semisimple if and only if M_1 and M_2 are strongly $FI-t$-semisimple.*

Proof. $\Rightarrow$It follows by Proposition 4. 4.

$\Leftarrow$ Let N be a fully invariant submodule of M. Then by Proposition 1. 5, $N = (N \bigcap M_1) \oplus (N \bigcap M_2)$ and $N \bigcap M_1$, $N \bigcap M_2$ are fully invariant submodules of M_1 and M_2 respectively. Put $N_1 = N \bigcap M_1$, $N_2 = N \bigcap M_2$. As M_1 and M_2 are strongly FI-t-semisimple, there exists $K_1 \leq^{\oplus} M_1$, K_1 is a fully invariant in M_1 with $K_1 \leq_{tes} N_1$ and there exists $K_2 \leq^{\oplus} M_2$, K_2 is a fully invariant in M_2 with $K_2 \leq_{tes} N_2$. But $K_1 \leq^{\oplus} M_1$, $K_2 \leq^{\oplus} M_2$ implies $K = K_1 \oplus K_2 \leq^{\oplus} M$. By Proposition 1.6, $(M_1, M_2) = 0$, $Hom(M_2, M_1) = 0$,

$$End(M_1, M_2) \cong \begin{pmatrix} EndM_1 & Hom(M_2, M_1) \\ Hom(M_1, M_2) & EndM_2 \end{pmatrix} = \begin{pmatrix} EndM_1 & \alpha_1 \\ \alpha_2 & EndM_2 \end{pmatrix}.$$

Therefore

$$\theta = \begin{pmatrix} \alpha_1 & 0 \\ 0 & \alpha_2 \end{pmatrix},$$

$\alpha_1 \in EndM_1$, $\alpha_2 \in EndM_2$. It follows that $\theta(K_1 \oplus K_2) = \alpha_1(K_1) \oplus \alpha_2(K_2) \leq K_1 \oplus K_2 = K$. Thus K is fully invariant in M. Also $K_1 \leq_{tes} N_1$ and $K_2 \leq_{tes} N_2$ imply $K \leq_{tes} N$ by Proposition 1. 2(2). Thus M is strongly $FI-t$-semisimple.

Lemma 4.11. *Let $M = M_1 \oplus M_2$ such that $annM_1 + annM_2 = R$. Then:*
(1) $Hom(M_1, M_2) = 0$ and $Hom(M_2, M_1) = 0$.
(2) M_1 and M_2 are fully invariant in M.

Proof. (1) Since $R = annM_1 + annM_2$, then $M_1 = M_1(annM_1) + M_1(annM_2)$, $M_2 = M_2(annM_1) + M_1(annM_2$. Put $annM_1 = A_1$, $annM_2 = A_2$, therefore $M_1 = M_1A_2$, and $M_2 = M_2A_1$. Then for each $\varphi \in Hom(M_1, M_2)$, $\varphi(M_1) = \varphi(M_1, A_2) = \varphi(M_1)A_2 \leq M_2A_2 = 0$, hence $\varphi = 0$. Thus $Hom(M_1, M_2) = 0$. Similarly, $Hom(M_2, M_1) = 0$.
(2) It follows directly by Proposition (1. 6).

Proposition 4.12. *Let $M = M_1 \oplus M_2$ where M_1 and M_2 be R-modules with $M_1 + annM_2 = R$. Then M is strongly $FI - t$-semisimple if and only if M_1 and M_2 are strongly $FI - t$-semisimple.*

Proof. $\Rightarrow$ It follows by Proposition (4. 4).

Proposition 4.13. *If M is an R-module and $M = M_1 \oplus M_2$, where M_1 and M_2 are fully invariant submodules of M. Then M is strongly $FI - t$-semisimple if and only if M_1 and M_2 are strongly $FI - t$-semisimple.*

Proof. $\Rightarrow$ By Lemma 4.11 (2) $M_1$and M_2 are fully invariant submodule of M and so the result follows by Proposition 4.4.
$\Leftarrow$ It follows by Lemma 4.11 (1).

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