Analysis of the Simulated Aggregate Interference in Random Ad-hoc networks

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Abstract—In this paper, we propose a new method for bias correction in the simulation of random wireless ad-hoc networks (WANETs), when the distribution of the node locations is modeled as a Poisson-Point-Process (PPP). The aggregate interference is the main limiting factor in WANETs, and dominates the achievable rate and thus also the network capacity. In the proposed method, a bias correction constant is added to the aggregate interference that is measured in each simulation iteration. The value of the constant is derived through stochastic geometry analysis. We prove that the proposed method can reduce the computational complexity by several orders of magnitude, while producing more accurate simulation results. This improved accuracy is also demonstrated by simulations. As an example, we prove that a bias corrected simulation with only 100 transmitters is sufficient to estimate the aggregate interference with an accuracy of 1%.

I. INTRODUCTION

The analysis of random wireless ad hoc networks (WANETs) has attracted much attention in recent years. In particular, the use of Poisson Point Processes (PPP) to model the node locations in WANETs has gained much popularity (e.g., [1], [2]).

Performance analysis of random WANETs mainly requires a good characterization of the aggregate interference received by each node (which results from the accumulation of many simultaneous transmissions). This aggregate interference has been studied extensively both in theory and by simulations.

Aggregate interference modeling using PPP was first introduced to analyze the transmission capacity (TC) of WANETs [3]. The TC quantifies the spatial spectral efficiency of WANETs, defined as the density of successful communication links subject to an outage constraint. Thus, the evaluation of the TC requires an evaluation of the probability density function (PDF) of the aggregate interference.

An alternative throughput measure, commonly termed ergodic rate density (ERD), measures the achievable rate density when the nodes use coding techniques that take advantage of the varying nature of the interference power [4]. The ERD is higher than the transport capacity, at the price of higher complexity (and in some cases also larger delays). The evaluation of the ERD requires the evaluation of the expectation of a function of the aggregate interference.

These works motivated the further study of various interference models [5], [6]. The interference studies showed that the choice of an appropriate interference model is important for the analysis and design of WANETs. For example, a proper interference model needs to include a good representation of the physical channel parameters such as the attenuation of the electromagnetic signal due to the spatial location of nodes, and the channel variations due to shadowing, fading and multipath. A proper interference model should also take into account functions related to the WANET design such as transmission powers, routing decisions and possible use of multiple antennas.

However, closed form throughput expressions are currently known only for a limited number of WANETs models, while all other models were studied mostly through simulations. As an example, one can consider the case of CSMA (e.g., [7],[8]), where no exact analytic expression is known so far, and all throughput evaluations rely on simulations.

In this paper we consider the simulation of the aggregate interference in a homogenous PPP model. We show that the evaluation of the aggregate interference by finite simulation is not trivial, and accurate evaluations require a very large number of transmitters per iteration. We also propose a simple method to significantly improve the simulation efficiency. Using this method, which is based on the addition of a bias correction term to the simulated interference, a simulation can achieve the same accuracy with a much smaller number of transmitters. The proposed method can significantly boost the simulations of random ad-hoc networks, and hence contribute to improve the research on the subject.

The rest of this paper is organized as follows. Section II describes the system model. Section III presents the main results and the suggested simulation method. Sections IV and V present the proofs of Lemma 1 and Lemma 2, respectively. Numerical results that demonstrate the effectiveness of our correction are presented in Section VI, followed by conclusions in Section VII.

II. SYSTEM MODEL

We assume a decentralized WANET over an infinite area. The locations of the nodes are modeled by a homogeneous Poisson Point Process (PPP) [9] with density $\lambda_T$ (i.e., the number of transmitters in any area of size $A$ has a Poisson distribution with a mean of $\lambda_T A$). Using some medium
access (MAC) protocol (e.g., ALOHA [10]), some nodes are actively transmitting while the others are idle. We assume herein that the transmission decisions are performed in an independent manner. Hence, by the thinning property of PPP, the distribution of the active transmitters is also a PPP. In the following we will denote the density of the active transmitters by $\lambda$.

We use the shift invariant property of the system [9], to analyze the distribution of the interference using a probe receiver. The aggregate interference, measured at the probe receiver, subject to path loss and fading effects can be written as:

$$I = \sum_i \rho_i r_i^{-\alpha} V_i$$

where $\alpha > 2$ is the channel exponential decay factor, $r_i$ is the distance between the $i$-th transmitter and the probe receiver, and $V_i$ is the fading between the $i$-th transmitter and the probe receiver (which can include any random change in the received power, e.g., channel fading, transmitter preprocessing, receiver postprocessing, etc.). The fading variables $V_i$ are assumed to be statistically independent and identically distributed (i.i.d.) random variables.

The model also allows the transmission power of each transmitter to be adapted in any manner that is independent of all other transmitters (e.g., according to the gain of the desired channel). The (random) transmission power of the $i$-th transmitter is denoted by $\rho_i$.

As any simulation can only consider a finite number of nodes, we wish to evaluate the effect of the unsimulated nodes on the simulation accuracy. For simplicity, we consider a simulation of a circular area of size $A$, centered at the probe receiver. Thus, the number of active transmitters in the simulation has a Poisson distribution with an average of: $\bar{N} = A \cdot \lambda$ transmitters. The theoretical aggregate interference, (1), can be written as

$$I \triangleq I_{\text{sim}}(A) + I_{\text{unsim}}(A)$$

where $I_{\text{sim}}(A)$ and $I_{\text{unsim}}(A)$ denote the aggregate interference of the transmitters within and outside the circular simulated area, respectively. Mathematically,

$$I_{\text{sim}}(A) \triangleq \sum_{i; r_i \leq \sqrt{A/\pi}} \rho_i r_i^{-\alpha} V_i$$

$$I_{\text{unsim}}(A) \triangleq \sum_{i; r_i > \sqrt{A/\pi}} \rho_i r_i^{-\alpha} V_i$$

Obviously, the aggregate interference in the simulation over the finite area, $I_{\text{sim}}(A)$, is smaller than the theoretical aggregate interference: $I_{\text{sim}}(A) < I$. Thus, the simulation will always give an underestimate of the aggregate interference. The effect of the transmitters outside of the simulated area (with $r_i > \sqrt{A/\pi}$) is captured by $I_{\text{unsim}}(A)$ in Equation (4). This term represents the simulation inaccuracy, and hence will take most of our attention in this paper.

The following section presents our suggested simulation approach and our main results on the simulation accuracy.

### III. MAIN RESULTS

#### A. Simulation correction

To partly compensate for the effect of the unsimulated area, we suggest to add the expectation of $I_{\text{unsim}}(A)$ to the simulated aggregate interference, $I_{\text{sim}}(A)$. Thus, the corrected aggregate interference is given by:

$$I_c \triangleq I_{\text{sim}}(A) + \mathbb{E}\{I_{\text{unsim}}(A)\}$$

$$= I_{\text{sim}}(A) + \frac{2\pi^{\alpha/2} \lambda}{\alpha - 2} A^{1-\alpha/2} \mathbb{E}\{V\} \mathbb{E}\{\rho\}$$

where the expectation of the unsimulated interference, $\mathbb{E}\{I_{\text{unsim}}(A)\}$, is evaluated in Appendix A. Note that this correction results in an unbiased interference estimate, i.e., after the proposed correction in (5), we have $\mathbb{E}\{I_c\} = \mathbb{E}\{I\}$.

#### B. Simulation accuracy with correction

The following lemma shows that the accuracy of the simulation after the suggested correction is inversely proportional to the average number of transmitters in the simulation, $N$.

**Lemma 1.** After the correction of Subsection III-A, The simulation accuracy is bounded by

$$\xi_c \triangleq \mathbb{E}\left\{ \left( \frac{I - I_c}{I_c} \right)^2 \right\} \leq \frac{(\alpha - 2)^2}{4(\alpha - 1)} \mathbb{E}\{V\}^2 \mathbb{E}\{\rho^2\} \frac{1}{N}$$

**Proof:** See proof in Section IV.

Interestingly, (6) does not depend directly on the transmitters density $\lambda$, or the simulation area, $A$, but only depends on the average number of simulated transmitters, $\bar{N}$, and the channel exponential decay factor $\alpha$.

Although the bound in (6) is typically not tight, it gives a practical bound on the required simulation size that can guarantee any desired accuracy. The accuracy bound of Lemma 1 is demonstrated by the solid lines in Fig. 1. The solid lines depict the minimal average number of transmitters in the simulation (according to (6)) that is required to guarantee different accuracy levels, as a function of the path loss exponent, $\alpha$. For example, a simulation accuracy of at most 1% can be guaranteed for any $\alpha$ smaller than $4$, by using simulations with 70 transmitters (and the proposed correction).

#### C. Simulation accuracy without correction

We next wish to show the importance of the suggested correction by upper bounding the accuracy without the correction. This task is not trivial, because the expectation of the aggregate interference received at the probe receiver is not bounded due to the probability to have interferers that are very close to the probe receiver. Thus, it is somewhat difficult to quantify the effect of the unsimulated area. Motivated by George et al. [4], we consider the truncated aggregate interference,

$$I_\eta \triangleq \sum_{i; r_i > \eta} \rho_i r_i^{-\alpha} V_i$$

See proof in Section IV. 

- **Lemma 1.** After the correction of Subsection III-A, The simulation accuracy is bounded by

$$\xi_c \triangleq \mathbb{E}\left\{ \left( \frac{I - I_c}{I_c} \right)^2 \right\} \leq \frac{(\alpha - 2)^2}{4(\alpha - 1)} \mathbb{E}\{V\}^2 \mathbb{E}\{\rho^2\} \frac{1}{N}$$

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Although the bound in (6) is typically not tight, it gives a practical bound on the required simulation size that can guarantee any desired accuracy. The accuracy bound of Lemma 1 is demonstrated by the solid lines in Fig. 1. The solid lines depict the minimal average number of transmitters in the simulation (according to (6)) that is required to guarantee different accuracy levels, as a function of the path loss exponent, $\alpha$. For example, a simulation accuracy of at most 1% can be guaranteed for any $\alpha$ smaller than $4$, by using simulations with 70 transmitters (and the proposed correction).
George et al. [4] have shown that the expectation of $I_{\eta}$ can be used to produce a useful lower bound on the ERD. In this work we focus on the case that $\eta = \sqrt{\frac{\alpha^{-2}}{\alpha - 2}}$ as detailed in appendix B. The simulated version of $I_{\eta}$ is defined by:

$$I_{\eta,\text{sim}}(A) \triangleq \sum_{i: \eta < r_i < \sqrt{A/\pi}} \rho_i r_i^{-\alpha} V_i.$$  

(8)

Lemma 2 shows that the accuracy of the evaluation of $I_{\eta}$ from a simulation with no correction, improves at most as $N^{2 - \alpha}$. Thus, at least if the path loss exponent is small (recall that $\alpha > 2$) the evaluation of the ERD from simulations requires a very large number of transmitters.

**Lemma 2. Without the correction of Subsection III-A, The simulation accuracy is bounded by**

$$\xi_u \triangleq \frac{\mathbb{E} \left\{ |I_{\eta} - I_{\eta,\text{sim}}(A)|^2 \right\}}{\mathbb{E}^2 \{ I_{\eta} \}} > \left( \frac{\alpha}{\alpha - 2} \right)^{1 - \alpha} N^{2 - \alpha}.$$  

(9)

**Proof:** See proof in Section V.

The bound of Lemma 2 is demonstrated by the dashed curves in Fig. 1, which depict the minimal average number of simulated transmitters, $N$, which is required to achieve a desired accuracy. The figure shows that, for $\alpha < 3$, the required number of transmitters without the correction is much higher than the required number with bias correction. More specifically, a bias corrected simulation can guarantee a given accuracy that is not feasible with an uncorrected simulation in which the number of transmitters is several orders of magnitude higher.

**IV. PROOF OF LEMMA 1**

The error of the proposed correction can be upper-bounded by the normalized difference between the theoretical term $I$ and the corrected measurement $I_c$:

$$\frac{I - I_c}{I_c} = \frac{I_{\bar{A}}(A) - \mathbb{E} \{ I_{\text{sim}}(A) \} - \mathbb{E} \{ I_{\text{sim}}(A) \}}{I_{\text{sim}}(A) + \mathbb{E} \{ I_{\text{sim}}(A) \}} \leq \frac{\mathbb{E} \{ I_{\text{sim}}(A) \} - \mathbb{E} \{ I_{\text{sim}}(A) \}}{\mathbb{E} \{ I_{\text{sim}}(A) \}}.$$  

Thus, the mean-square-error (MSE) of the normalized simulation error is bounded by

$$\xi_c = \mathbb{E} \left\{ \left( \frac{I - I_c}{I_c} \right)^2 \right\} \leq \frac{\mathbb{E} \left\{ [I_{\text{sim}}(A) - \mathbb{E} \{ I_{\text{sim}}(A) \}]^2 \right\}}{\mathbb{E}^2 \{ I_{\text{sim}}(A) \}} = \frac{\text{var}(I_{\text{sim}}(A))}{\mathbb{E}^2 \{ I_{\text{sim}}(A) \}} = \frac{(\alpha - 2)^2 \mathbb{E} \{ V^2 \} \mathbb{E} \{ \rho^2 \}}{4(\alpha - 1) \mathbb{E}^2 \{ V \} \mathbb{E}^2 \{ \rho \}} \frac{1}{N}$$  

(10)

where $\text{var}(I_{\text{sim}}(A))$ and $\mathbb{E} \{ I_{\text{sim}}(A) \}$ are evaluated in Appendix A, where we used $\gamma = \sqrt{A/\pi}$ and $N = \lambda A$.

**V. PROOF OF LEMMA 2**

Comparing (7) and (8), and assuming a reasonable simulation area so that $\sqrt{A/\pi} > \eta$, the gap between the theoretical truncated aggregate interference, $I_{\eta}$, and the simulated truncated aggregate interference, $I_{\eta,\text{sim}}(A)$, depends on the transmitters outside of the area $A$. Thus we have $I_{\eta} - I_{\eta,\text{sim}}(A) = I_{\text{sim}}(A)$, and the simulation accuracy can be written as:

$$\xi_u = \frac{\mathbb{E} \left\{ |I_{\eta} - I_{\eta,\text{sim}}(A)|^2 \right\}}{\mathbb{E}^2 \{ I_{\eta} \}} = \frac{\mathbb{E} \{ I_{\text{sim}}^2(A) \}}{\mathbb{E}^2 \{ I_{\text{sim}}(A) \}} = \frac{\mathbb{E}^2 \{ I_{\text{sim}}(A) \} + \text{var}(I_{\text{sim}}(A))}{\mathbb{E}^2 \{ I_{\text{sim}}(A) \}} > \frac{\mathbb{E}^2 \{ I_{\text{sim}}(A) \}}{\mathbb{E}^2 \{ I_{\eta} \}}$$  

where $\mathbb{E}^2 \{ I_{\text{sim}}(A) \} > \text{var}(I_{\text{sim}}(A))$. Using the results of appendix A and using $\eta = \sqrt{\frac{\alpha^{-2}}{\alpha - 2}}$, we get

$$\xi_u > \left( \frac{\alpha}{\alpha - 2} \right)^{1 - \alpha} N^{2 - \alpha}.$$  

(11)

**VI. NUMERICAL RESULTS**

In this section, we present monte-carlo simulation results with 20,000 network realizations. The simulation results demonstrate the efficiency and usefulness of the proposed correction, (5). To simulate a PPP within a circle, we set the number of active transmitters to have a Poisson distribution with an average of $N$. The location of each transmitter is uniformly distributed inside a circle with area $A = \bar{A}/\lambda$ centered at the origin, independently of any other transmitters. Each transmitter is equipped with a single antenna, and the probe receiver is also equipped with a single antenna. We consider a constant transmission power scheme ($\rho_i = 1, \forall i$)
ERD vs the active transmitters density, \( \lambda \)

The TC of a network with an active transmitters density of \( \lambda \) is given by:

\[
\text{TC}(\lambda) = \lambda \cdot \frac{(1 - P_{\text{out}}(\zeta)) \cdot \log_2 (1 + \zeta)}{1 + \zeta}.
\]

The performance is measured by the average achievable rate over a single link is \( R \triangleq \log_2 (1 + \text{SIR}) \).

The outage probability is defined as \( P_{\text{out}}(\zeta) \triangleq P(\text{SIR} < \zeta) \).

The ERD of a network with an active transmitter density of \( \lambda \) is given by:

\[
\text{ERD}(\lambda) = \lambda \cdot \mathbb{E}\left\{ \log_2 \left( 1 + \frac{S}{T} \right) \right\}.
\]

Without loss of generality, we consider normalized distance. i.e., the distance between the desired transmitter and the probe receiver is 1.

The upper sub-figure in Fig. 2 depicts the ‘normalized’ ERD vs the active transmitters density, \( \lambda \). The ‘normalized’ ERD is defined as the ratio between the simulated ERD and the upper bound on the ERD from [11]. Fig. 2 depicts the ERD for the case of \( \alpha = 2.5 \). In this case, the aggregate interference from far transmitters (outside the simulated area) is not negligible, and the proposed correction, (5), has a significant contribution to the accuracy of the simulated ERD. Taking the \( \tilde{N} = 10^7 \) curve as the most accurate estimation of the ERD, we can see that bias corrected simulation with \( \tilde{N} = 100 \) is at least as accurate as an uncorrected simulation with \( \tilde{N} = 10^6 \) (which requires much higher computational complexity). Furthermore, even the \( \tilde{N} = 10^7 \) curve exceeds the upper bound for high densities. Thus, it is reasonable to assume that bias corrected simulation with \( \tilde{N} = 100 \) is even more accurate than an uncorrected simulation with \( \tilde{N} = 10^7 \). Due to lack of space, we do not present herein simulation results for higher values of \( \alpha \). In such case, the need for very high number of simulated transmitters is somewhat relaxed even in a non-corrected simulation. Yet, the bias corrected simulation always remains more accurate.

The lower sub-figure in Fig. 2 depicts the ‘normalized’ TC vs the active transmitters density \( \lambda \), for an outage threshold of \( \zeta = 0.02 \). In this case, we do not have a convenient upper bound, and hence, the ‘normalized’ TC is defined as the ratio between the simulated TC and the TC curve of \( \tilde{N} = 10^7 \) (without bias correction). Note that in the TC evaluation, the proposed correction allows a more efficient calculation of the outage probability. Again, a bias corrected simulation with only \( \tilde{N} = 10^2 \) transmitters is shown to be at least as good as a simulation without the correction using of \( \tilde{N} = 10^6 \).

\[7\] Interference

In this appendix, we evaluate the moments of the aggregate interference.

Define \( I_{\gamma} \triangleq \sum_{i \in \mathcal{F}(\gamma)} \rho_i r_i^{-\alpha} V_i \). In the following, we determine \( \mathbb{E}\{I_{\gamma}\} \) and \( \text{var}(I_{\gamma}) \), using stochastic geometry tools [9]. The characteristic function of the aggregate interference, measured at the middle of the circular guard zone, is given by [12]:

\[
\Phi_{I_{\gamma}}(s) = \exp \left( -2\pi \rho \mathbb{E} \left\{ \int_{\gamma} \left( 1 - e^{-s \rho \sqrt{\gamma} r^{-\alpha}} \right) rdr \right\} \right).
\]
The expectation of $I_r$ can be found [12] by
\[
\mathbb{E}\{I_r\} = -\frac{\partial \ln \Phi_{r_1}(s)}{\partial s}\bigg|_{s=0} = 2\pi \lambda \mathbb{E}\left\{ \int_{r}^{\infty} \rho V r^{1-\alpha} dr \right\} = \frac{2\pi \lambda}{\alpha - 2} \gamma^{2-\alpha} \mathbb{E}\{V\} \mathbb{E}\{\rho\}.
\]
(15)

The variance of $I_r$ is given by
\[
\text{var}(I_r) = \frac{\partial^2 \ln \Phi_{r_1}(s)}{\partial s^2}\bigg|_{s=0} = 2\pi \lambda \mathbb{E}\left\{ \int_{r}^{\infty} \rho^2 V^2 r^{1-2\alpha} dr \right\} = \frac{\pi \lambda}{\alpha - 1} \gamma^{2-2\alpha} \mathbb{E}\{V^2\} \mathbb{E}\{\rho^2\}.
\]
(16)

**APPENDIX B**

**ERD LOWER-BOUND**

In this appendix we present the modified ERD lower bound (which is a slight variation of the ERD lower-bound of George et al. [4]).

**Lemma 3.** The ERD lower-bound is given by
\[
R_{LB}(\lambda) = \lambda e^{-\lambda \pi \eta^2} \mathbb{E}\left\{ \log_2 \left( 1 + \frac{\rho S}{2\pi \lambda} \right) \right\}
\]
(17)

and good selection of $\eta$ is $\sqrt{\frac{\alpha - 2}{\alpha \pi \lambda}}$.

**Proof:** Denote the distance between the probe and the closest transmitter as $r_{\min} = \min_i r_i$. The ERD, (13), can be written as
\[
R_{ERD}(\lambda) = \lambda \mathbb{P}(r_{\min} \leq \eta) \mathbb{E}\left\{ \log_2 \left( 1 + \frac{\rho S}{T} r_{\min} \leq \eta \right) \right\} + \lambda \mathbb{P}(r_{\min} > \eta) \mathbb{E}\left\{ \log_2 \left( 1 + \frac{\rho S}{T} r_{\min} > \eta \right) \right\}
\]
(18)

where we use the fact that the first line of (18) is non-negative, the Jensen inequality, and the statistical independence of different areas in a PPP such that: $\mathbb{E}\left\{ \log_2 \left( 1 + \frac{\rho S}{T} r_{\min} \geq \eta \right) \right\} = \mathbb{E}\left\{ \log_2 \left( 1 + \frac{\rho S}{T} \right) \right\}$. Using the properties of PPP, the probability that no transmitter is located within distance of $\eta$ from the origin is $\mathbb{P}(r_{\min} > \eta) = e^{-\lambda \pi \eta^2}$. Combining (15) into (18), the ERD is lower bounded by
\[
R_{ERD}(\lambda) \geq R_{LB}(\eta, \lambda)
\]
\[
= \lambda e^{-\lambda \pi \eta^2} \mathbb{E}\left\{ \log_2 \left( 1 + \frac{\rho S}{2\pi \lambda} \eta^{2-\alpha} \mathbb{E}\{V\} \mathbb{E}\{\rho\} \right) \right\}
\]
(19)

The best bound can be obtained by maximizing (19) with respect to $\eta$. However, this approach turns out to be too complicated. Instead, recalling that any value of $\eta$ will result in a legitimate lower bound, we produce a simpler lower bound that is motivated by the optimization of the bound with respect to both $\eta$ and $\lambda$.

This maximization can be written as
\[
\max_{\eta, \lambda} \lambda e^{-\lambda \pi \eta^2} \mathbb{E}\left\{ \log_2 \left( 1 + \frac{\rho S}{2\pi \lambda} \eta^{2-\alpha} \mathbb{E}\{V\} \mathbb{E}\{\rho\} \right) \right\}
\]
(20)

Using the same approach as [4], the optimal $\eta$ must satisfy
\[
\eta_{opt} = \sqrt{\frac{\alpha - 2}{\alpha \pi \lambda_{opt}}}
\]
(21)

As in [4], we adopt the relation in (21) for any $\lambda$.

**REFERENCES**


