

Article

Generalized Maxwell Relations in Thermodynamics with Metric Derivatives

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Abstract: In this contribution, we develop the Maxwell generalized thermodynamical relations via the metric derivative model upon the mapping to a continuous fractal space. This study also introduces the total q -derivative expressions depending on two variables, to describe nonextensive statistical mechanics and also the α -total differentiation with conformable derivatives. Some results in the literature are re-obtained, such as the physical temperature defined by Sumiyoshi Abe.

Keywords: deformed maxwell relations; metric derivatives; fractal continuum; generalized statistical mechanics; Legendre structure

1. Introduction

Maxwell relations are thermodynamic equations which establish the relations between various thermodynamic quantities (e.g., pressure, P , volume, V , Entropy, S , and temperature, T) in equilibrium thermodynamics via other fundamental quantities known as thermodynamical potentials—the most important being internal energy, U , Helmholtz free energy, F , enthalpy, H , and Gibbs free energy, G .

A Legendre transform converts from a function of one set of variables to another function of a conjugate set of variables. With the use of Legendre transforms, we can study different equations of state. For example, with the adequate change in variables, $U(S, V)$ transforms to $F = F(V, T)$.

In this study, we pursue the development of Maxwell's generalized thermodynamic relations, considering complex systems in which the thermodynamic quantities are expressed by partial metric derivatives (MDs) [1,2]. In order to fulfill this task, we start with the redefinition of the mathematical concept of the derivatives of functions under the physical basis of mappings to a continuous fractal space as proposed in References [3–5]. According to Reference [6], the generalized equations of state (called “fractal” equations of state) are derived on the basis of the experimentally measured thermodynamic variables P , V , and T , which achieve good agreement with experimental data for argon gas.

To pursue a mathematical basis for physics on fractals, differential vector calculus in a three-dimensional continuum with fractal metric were studied in Reference [5], where the mapping of physical problems on nowhere differentiable fractals into the corresponding problems for a continuum, with a proper fractal metric, is discussed. The authors also showed that there exists an isomorphism of algebras and present a mathematical formalism to differential vector calculus in a continuum with the induced fractal metric. In Reference [7], while studying anomalous diffusion of fluid momentum and Darcy-like law for laminar flow in media with fractal porosity, the authors showed that in the context studied, there exists a homomorphism in certain cases of dimensional parameters. The physical problems on a fractal ($F \subset E^3$) can be mapped into problems in the ν -dimensional space $F^\nu \subseteq E^3$

equipped with the metric induced by the fractal topology, while ν is the number of effective degrees of freedom of a random walker on the fractal. The mapping implies a change from the Cartesian coordinates in the embedding Euclidean space to the fractal coordinates in the fractional dimensional space allied to the fractal. Furthermore, the mapping $F \subset E^3 \rightarrow F^\nu \subseteq E^3$ implies the use of metric partial derivatives instead of conventional partial derivatives. So, we can claim that the mapping to a continuous fractal space [3–5] leads naturally to the necessity of modifications in the definition of derivatives. Consequently, the use of the MD [1,2] is justified. The modification of the derivatives in terms of the metric implies a change in the algebra involved, which in turn gives rise to a generalized statistical mechanics with certain generalized definition of entropy, such as in nonextensive statistical mechanics. The use of these deformed-operators can be justified on our proposition that there exists an intimate relation between dissipation, coarse-grained media, and the limit scale of energy among the interactions. It seems that a suitable way to describe the dynamics of complex systems requires the use of the MD and the associated operators instead of the popular fractional derivatives (FD). The q -deformed derivative and the correspondent q -deformed algebras consequently appear in the context of Tsallis' statistical mechanics and other deformed operators like the Hausdorff and the conformable derivatives.

In considering the appearance of the concept of general multiscale metric derivatives, we can cast the interesting work in Reference [8] (reviewed in Reference [9]) with the effort to build up geometry and field theory in multi-fractional space-time, but there with the name of q -derivative. In [9], the author shows, among a vast number of other things, that the multifractal paradigm not only makes the "fractal space-time" idea systematic, but it also provides an explicit form for the dynamics (questions 31 and 40 in the reference). Moreover, the author works with a mapping between the fractional and the integer picture that includes the deformed derivative involved there.

All of the above-referred approaches indicate the systematic attempts to build up mappings that lead to the redefinition of derivatives as structural deformed operators that can be used to describe the dynamics of complex and open systems in the most diverse research lines.

Following this path, in a recent work it was demonstrated that there is a possible relation between q -deformed algebras in the two different contexts of statistical mechanics (namely, the Tsallis' framework and the Kaniadakis' scenario), with a local form of a deformed-derivative operator for fractal media—the so-called Hausdorff derivatives—mapped into a continuous medium with a fractal measure, considering the relevant space-time/phase space as fractal or multifractal [1]. One of the authors shows that this approach could be capable of describing dissipative systems, where the traditional Lagrangian formalism fails [2]. Instead, a method of dealing with a mass-dependent Lagrangian was presented.

Following this line of thought, in a recent article [10] it was claimed that the use of deformed operators could be justified based on our proposition that there exists an intimate relation between dissipation, coarse-grained media, and a limit energy scale for the interactions. Concepts and connections like open systems, quasi-particles, energy scale, and the change in the geometry of space-time at its topological level, nonconservative systems, and noninteger dimensions of space-time connected to a coarse-grained medium have been discussed. With this perspective, we argued that deformed (or metric or structural derivatives)—similar to the fractional calculus (FC)—could allow us to describe and emulate certain dynamics without explicit many-body, dissipation, or geometrical terms in the dynamical governing equations. Additionally, we emphasized that the paradigm we adopted was different from the standard approach in the generalized statistical mechanics context [11–13], where the modification in the definition of entropy leads to the modification of the algebra—and consequently, the concept of a derivative [1,2]. This was set up by a mapping into a continuous fractal space [3–5], which naturally yields the need for modifications in the derivatives which we have named deformed, or better, metric derivatives [1,2]. In connection with the metric, the modifications of the derivatives yields a change in the algebra involved, which in turn may lead to a generalized statistical mechanics with some adequate definition of entropy.

According to Reference [14], there is a relation between entropy and the fractal dimension; the latter can be considered a statistical index, measuring the complexity of a given pattern embedded in the topological space and characterizing a measure of the space-filling capacity. It is possible to define the statistical complexity by using different distance measures [15]. Thus, we claim that with the physical definition of the MD, the complexity can be inherently considered by the generalized thermodynamics characterized in entropic expressions.

To establish a solid basis for the MD and its applications, some experimental tests can be indicated. For instance, the so called fractal differential equations [16]—particular forms of the metric differential equations—were applied to predict the surface temperature of air-cooled cylindrical Li-ion cells. The authors [16] also argue that it can be used to model a variety of non-integer power law scaling phenomena such as turbulence, fractional quantum mechanics, anomalous diffusion, and they elaborate an interesting argument that fractal time models can describe systems with temporal behavior that present strong intermittence and, self-similarity, without a characteristic time scale. Under fractal time average, the time of an event must be infinite instead of the usual finite for systems that possess time scale. Such an example relates the time in which a fluid particle spends in a given vortex. The lack of an average time induces fractional exponents. For details, see references therein.

In numerical contexts, it has been shown that the MD models have distinct advantages over the existing models from the perspective of computational accuracy and efficiency [17].

Here, it seems pertinent to stress the fact that the mathematical tool of MD has advantages when compared to the formalism with the FC (in addition to the numerical ones already observed above). This also includes attempts to formulate non-equilibrium thermodynamics. The Leibniz differentiation rules and the chain rule are preserved for MD, which facilitates the design, development, and research in linear and nonlinear systems, including quantum systems and hydrodynamic models. However, the equivalence between these formalisms needs to be verified in each case. For example, there seems to be some equivalence in the case of water transport in an unsaturated medium [18]. Recall that the formalism of FC is based on non-local operators in its formulation, while that of MD is based on definitions of local operators. However, one claim that non-locality can also be considered with the argument is as follows: we are dealing with open systems and the hereditary effects or the nonlocal effects (memory and spatial effects) are inherent to the fractality, connected to the geometry or the topology of the medium. In an attempt to outline possible connections between formalisms of FC and the MD (the latter being presented in this work), we can indicate Reference [18], where nonlocality is considered with the use of FC. The authors in Reference [18] propose a fractal Richards equation, in which the classical integer-order time derivative of the water content is replaced by a fractal derivative underlying a power law rule in time, whose solution exhibits anomalous non-Boltzmann scaling, attributed to the fractal nature of heterogeneous media. The model was verified by accurately fitting the eight different benchmark experiment data for the water content curves reported in the literature. The wide applicability of this Richards model demonstrates a clear edge of the MD approach over the other existing models.

Additionally, in terms of applications, the authors in Reference [19] show the applicability of what they call fractal models with fractal derivatives to characterize the creep of viscoelastic materials. The applicability of a MD formalism is reinforced by the recent publication in Reference [20], where aspects of Tsallis' entropy in water engineering are studied. Additionally, an effort to unified formalisms can be found in Reference [21].

In this contribution, our focus is on the development of generalized Maxwell relations based on a mapping to a fractal continuum [4,5,22] with the use of MDs. We work with two kinds of MD: the first in the context of generalized nonadditive statistical mechanics called q -derivative, and the second called conformable derivatives [23], in the context of fractals and multifractals. All these operators are local metric derivatives.

In the following section, the reader will see that the concept of physical temperature defined by Abe in Reference [24] emerges naturally in our approach, just like the emergence of the Rényi entropy from the Tsallis entropy and the relation between Sharma–Mittal and Tsallis entropic forms.

The remainder of this paper is outlined as follows: In Section 2, the metric derivatives and mathematical aspects are presented. In Section 3, the deformed Maxwell relations are developed, followed by the Conclusions and Outlook in Section 4.

2. Metric Derivatives: Mathematical Aspects

In this Section, we provide some brief information to review the main forms of the MD. For more details, the readers may see References [1–3,19,25].

2.1. Conformable Derivative

This kind of MD was proposed in Reference [23], preserves classical operational properties, and is given by

$$T_\alpha f(t) = \lim_{\epsilon \rightarrow 0} \frac{f(t + \epsilon t^{1-\alpha}) - f(t)}{\epsilon}. \quad (1)$$

If the function f is differentiable in a classical sense, the definition above yields

$$T_\alpha f(t) = t^{1-\alpha} \frac{df(t)}{dt}. \quad (2)$$

Changing the variable $t \rightarrow 1 + \frac{x}{t_0}$, one of the authors shows [1] that Equation (2) is nothing but the Hausdorff derivative [1,3–5,19,25] up to a constant, and is valid for differentiable functions.

We can see that the above form of the MD has the same mathematical structure of generic MD for differentiable functions.

Interconnections and equivalence of the metric derivatives was presented in Reference [1]—particularly between definitions of fractal derivatives, called Hausdorff derivative in Reference [25], q -derivatives in Reference [26], and conformable derivatives in Reference [23]. Therein, a simple scale change in the variable was used to show that for differentiable functions, the conformable derivatives are equivalent to Hausdorff derivatives.

We provide some comments here: The first is that the Maxwell relations that will be stated in Section 3 may involve multiscales, since we are working with different state variables that can be described by different parameters (e.g., different entropic indices). Moreover, we reinforce that all the metric derivatives referred to here are local operators, and for this reason cannot be simply proportional to the Riemann–Liouville derivative in the context of FC.

2.2. The q -Derivative in a Nonextensive Context

With the generalized nonadditive q -entropy as the main motivation, the q -derivative sets up a deformed algebra and considers that the q -exponential is an eigenfunction of $D_{(q)}$ [26]. Borges [26] proposed the operator for q -derivative as given below:

$$D_{(q)}f(x) \equiv \lim_{x' \rightarrow x} \frac{f(x') - f(x)}{x' \ominus_q x} = [1 + (1 - q)x] \frac{df(x)}{dx}. \quad (3)$$

Here, \ominus_q is the deformed difference operator,

$$x' \ominus_q x \equiv \frac{x' - x}{1 + (1 - q)x} \quad (x \neq 1/(q - 1)). \quad (4)$$

If we write $x' = x \oplus_q \Delta x = x + \Delta x + (1 - q)x\Delta x$, we will have $x' \ominus_q x = \Delta x$, where \oplus_q is the deformed sum operator.

Additionally,

$$d_q(x) \equiv \lim_{x' \rightarrow x} x' \ominus_q x = \frac{dx}{1 + (1 - q)x} \quad (x \neq 1/(q - 1)). \tag{5}$$

In this way, the expression (3) can be also rewritten as in Reference [27],

$$D_{(q)}f(x) \equiv \lim_{\Delta x \rightarrow 0} \frac{f(x \oplus_q \Delta x) - f(x)}{\Delta x}. \tag{6}$$

This is the key for the multivariable q -Taylor expansion.

Recently, one of the authors has shown [1] that the main parameters of MD, Hausdorff derivative, and the q -derivative can be connected by

$$1 - q = \frac{(1 - \zeta)}{l_0}, \tag{7}$$

where the characteristic scale l_0 appears in the context of the mappings in References [3–5].

We concluded that the deformed or metric q -derivative is the first-order expansion of the Hausdorff derivative, and that there is a strong connection between these formalisms by means of a fractal metric.

An important point to emphasize here concerns the appearance of multifractals, since multifractal measures are formally used to explain systems with scaling properties [28]. In Reference [28], an approach based on generalized Tsallis dimensions is used for the formulation of mutual information related dependence coefficients in the multifractal domain. This is only one example indicating that the concepts of fractals, scales, and multifractals are present in our approach, since q -derivatives are one of several possible forms of metric derivatives. In Reference [2], the authors pointed out that the deformation parameter or entropic index, q —occupying an important place in the description of those complex systems—describes deviations from standard Lie symmetries, and the formalism of metric derivatives aimed to accommodate scale invariance in a system with multifractal properties to the thermodynamic formalism, and considered the relevant space-time/phase space as fractal or multifractal.

2.3. The q -Total Differentiation in Nonextensive Context

For one variable, let us expand $f(x') = f(x \oplus_q \Delta x)$ around x [27]:

$$f(x') = f(x) + (x' - x) \frac{df(x')}{dx'} \Big|_{x'=x} + O((\Delta x)^2), \tag{8}$$

which can be written as

$$f(x \oplus_q \Delta x) = f(x) + \Delta x [1 + (1 - q)x] \frac{df(x')}{dx'} \Big|_{x'=x} + O((\Delta x)^2), \tag{9}$$

where $\Delta x \rightarrow 0$.

Now, we can extend to more than one variable with the expansion around x, y . Assuming a possible different q -parameter for x, y variables, a total differential is:

$$f(x \oplus_q \Delta x, y \oplus_{q'} \Delta y) = f(x, y) + \Delta x [1 + (1 - q)x] \frac{\partial f(x')}{\partial x'} \Big|_{x'=x, y=cte} + \Delta y [1 + (1 - q')y] \frac{\partial f(y')}{\partial y'} \Big|_{y'=y, x=cte} + O((\Delta x)^2, (\Delta y)^2). \tag{10}$$

In the limit $\Delta x \rightarrow 0, \Delta y \rightarrow 0$, the expression above can be written as

$$f(x \oplus_q dx, y \oplus_q dy) = f(x, y) + \frac{\partial_q f(x, y)}{\partial x} \Big|_y dx + \frac{\partial_{q'} f(x, y)}{\partial y} \Big|_x dy. \tag{11}$$

2.4. The Leibiniz and the Chain Rule Hold for Metric Derivative

As the reader can readily check, for the MD the Leibniz rule holds $D^\alpha(fg) = gD^\alpha f + fD^\alpha g$; and similarly, for q -derivative: $D_q(fg) = gD_q f + fD_q g$. For composed functions, the chain rule holds true as well:

$$D^\alpha[f \circ g](x) = \frac{df(g(x))}{dg} D^\alpha g(x), \tag{12}$$

$$D_q[f \circ g](x) = \frac{df(g(x))}{dg} D_q g(x). \tag{13}$$

3. Deformed Maxwell Relations

In this section, we pursue the investigation of a version of the MD to study a possible version of generalized thermodynamics. The intention is to build a tool to study the dynamics of complex systems such as porous media or some viscous systems.

Before proceeding further, in order to clarify some points here, we now make a few brief remarks about entropy. There are some controversies regarding the connections of the concept of entropy in the sense of Clausius and in the context of information theory—that is, with entropy and disorder [29]. For example, in Reference [30], the author uses liquid crystals to show that an increase in the order of the refereed system follows the increase in entropy, which could be considered impossible in the context of the connection between entropy and disorder. Additionally, against the idea that entropy and disorder are always the same, see Reference [31]. In the sense of information theory, entropy is related to aspects and problems of information and communication [32]. In addition, in the context of information theory, the important concept of informational energy is that related to classical mechanics, and can be considered as a measure of the uncertainty or randomness of a probability system [32]. Here, one adopts the conciliatory sense—namely, that entropy describes the amount of disorder or randomness in a system bearing energy or information [32], and that can be also used to evaluate aspects of uncertainty. In this broader sense, there are some generalized entropic forms which are particularly inspired in the context of information theory; for instance, Havdra–Charvat–Tsallis (HCT), Sharma–Mittal (SM), Sharma–Taneja (ST), and attempts to construct a generalized statistics with these forms. Examples of these attempts are based on the entropy concept like Rényi, HCT, Kaniadakis, and incomplete information-based theory [33,34].

Now, consider a mapping to a continuous fractal space [3–5] which leads naturally to the necessity of modifications in the derivatives, called the MD [1,2]. In the new continuous space, we call S_q, S_α metric entropies, depending on the metric considered.

3.1. q -Deformed Maxwell Relations

Consider U_q as some internal energy that is a function of the entropy S_q and volume $V_{q'}$: $U_q = U_q(S_q, V_{q'})$, where the entropic indices q and q' are not necessarily the same.

The total q -derivative can be written as

$$\begin{aligned} &U_q(S_q \oplus_q \Delta S_q, V_{q'} \oplus_{q'} \Delta V_{q'}) - U_q(S_q, V_{q'}) \\ = &U_q(S_q \oplus_q \Delta S_q, V \oplus_{q'} \Delta V_{q'}) - U_q(S_q, V \oplus_{q'} \Delta V_{q'}) \\ &+ U_q(S_q, V \oplus_{q'} \Delta V_{q'}) - U_q(S_q, V_{q'}). \end{aligned} \tag{14}$$

In the limit of $\Delta V_{q'}, \nabla S_q \rightarrow 0$, we find for the total q -derivative of U_q

$$\begin{aligned} D_q U_q &= [1 + (1 - q)S_q] \frac{\partial U_q}{\partial S_q} |_{V_{q'}} dS_q + [1 + (1 - q')V_{q'}] \frac{\partial U_q}{\partial V_{q'}} |_{S_q} dV_{q'} \\ &= \frac{\partial_q U_q}{\partial S_q} |_V dS_q + \frac{\partial_{q'} U_{q'}}{\partial V_{q'}} |_S dV_{q'}. \end{aligned} \tag{15}$$

Now, in the same way, consider $U_q = U_q(T_q, S_q)$, where the variable T_q is introduced and will be made clear in what follows.

The total q -derivative can be written as:

$$D_q U_q(T_q, S_q) = \frac{\partial_q U_q}{\partial T_q} |_{S_q} dT_q + \frac{\partial_q U_q}{\partial S_q} |_{T_q} dS_q. \tag{16}$$

Here, $T_q = \frac{\partial U_q}{\partial S_q} |_{V_{q'}}$ is not a temperature. It is similar to $k^T \beta^{-1}$ in the nonextensive formalisms, and is sometimes related to the Lagrange multiplier in the maximum entropy variational approach.

3.1.1. Legendre Transform and Helmholtz Free Energy, Enthalpy, and Gibbs Free Energy

Suppose a transform with change of variables in such a way that $U_q(S_q, V_q)$ transfers to $F_q = F_q(V_q, T_q)$, with $F_q = U_q - T_q S_q$ being the Helmholtz free energy.

Adding and subtracting Equation (16) to Equation (15) and considering that F_q only depends on (V_q, T_q) , yields

$$D_q(U_q - T_q S_q) = D_q F_q = \frac{\partial_q U_q}{\partial V_q} |_S dV_q - \frac{\partial_q(T_q S_q)}{\partial T_q} |_S dT_q, \tag{17}$$

$$\frac{\partial_q U_q}{\partial S_q} |_{V_q} = T_q [1 + (1 - q)S_q]. \tag{18}$$

The expression in Equation (18) is nothing else than the physical temperature defined by Abe in Reference [24].

Now consider the partial q -derivatives of $T_q S_q$ and of $P_q V_q$

$$\frac{\partial_q(P_q V_{q'})}{\partial P_q} |_V = V_{q'} [1 + (1 - q')P_{q'}], \tag{19}$$

$$\frac{\partial_q(P_q V_q)}{\partial V_q} |_P = P_{q'} [1 + (1 - q')V_{q'}]. \tag{20}$$

Analogously to U_q , consider the enthalpy function given by $H_q = U_q + P_q V = H_q(S_q, P_{q'})$. The total q -derivative of H_q can be written as

$$D_q(U_q + P_q V) = D_q H_q = \frac{\partial_q U_q}{\partial S_q} |_{V_q} dS + \frac{\partial_q(P_q V_q)}{\partial P_q} |_{V_q} dP_q; \tag{21}$$

$$\frac{\partial_q U_q}{\partial V} |_{S_q} = - P_q [1 + (1 - q)V]. \tag{22}$$

Finally, the q -deformed differential forms of U_q, F_q, H_q are given by

$$D_q U_q = T_q [1 + (1 - q)S_q] dS_q - P_q [1 + (1 - q)V_q] dV_q, \tag{23}$$

$$D_q F_q = -P_q [1 + (1 - q)V_q] - S_q [1 + (1 - q)T_q] dT_q, \tag{24}$$

$$D_q H_q = T_q [1 + (1 - q)S_q] dS_q + V_q [1 + (1 - q)P_q] dP_q. \tag{25}$$

In a similar way, for the Gibbs free energy G_q , we have

$$D_q G = -S_q[1 + (1 - q)T_q]dT_q + V_q[1 + (1 - q)P_q]dP_q. \tag{26}$$

With the help of the above equations, we can rewrite Equation (15) as

$$D_q U_q = T_{Phys}dS_q - P_{Phys}dV_q. \tag{27}$$

3.1.2. The q -Deformed Maxwell Relations

Below we collect all the Maxwell relations obtained by the present approach with q -derivatives. They are collected in Table 1.

Table 1. q -Deformed Maxwell relations.

$\frac{\partial_q U_q}{\partial V_q} _{S_q} = -P_q[1 + (1 - q)V_q],$	$\frac{\partial_q U_q}{\partial S_q} _{V_q} = T_q[1 + (1 - q)S_q],$
$\frac{\partial_q F_q}{\partial V_q} _{T_q} = -P_q[1 + (1 - q)V_q],$	$\frac{\partial_q F_q}{\partial T_q} _{V_q} = -S_q[1 + (1 - q)T_q],$
$\frac{\partial_q H_q}{\partial P_q} _{S_q} = V_q[1 + (1 - q)P_q],$	$\frac{\partial_q H_q}{\partial S_q} _{P_q} = T_q[1 + (1 - q)S_q],$
$\frac{\partial_q G_q}{\partial P_q} _{T_q} = -V[1 + (1 - q)P_q],$	$\frac{\partial_q G_q}{\partial T_q} _{P_q} = -S_q[1 + (1 - q)T_q],$

One important point that can be emphasized is the difference between the entropic indices in the Maxwell relations. Those differences could indicate that there are additional ways to interpret the possible existence of the so-called q -triplet [35]. A large updated bibliography can be found in Reference [36] pursuing the task of contextualizing the entropic index as something beyond a simple parameter of data adjustment, and that has been researched and justified. Even though it is not one of the objectives of this article, we suggest that the entropic index is related to the metric, which in turn is related to the existing interactions. So, we call the reader’s attention to the possible relation between the entropic index and the metric, justifying the use of the appropriate metric derivative to each physical phenomenon.

3.1.3. Emergence of Rényi Entropy from S_q as Tsallis Entropy

Consider that from Equation (27) $dQ_q = T_{Phys}dS_q$ as the inexact deformed heat transfer, we are led to

$$dS_q = \frac{dQ_q}{T_{Phys}}, \tag{28}$$

or

$$(dS)_q = \frac{dQ_q}{T_q} \times \frac{1}{[1 + (1 - q)S_q]} = \frac{ds_q}{[1 + (1 - q)S_q]}. \tag{29}$$

Note that the above expression is consistent with the differentiation in the fractal continuous space given by [26]

$$(d_q x) = \frac{dx_q}{[1 + (1 - q)x_q]}. \tag{30}$$

A direct integration leads to

$$(\Delta S)_q^R = \frac{1}{1 - q} \ln[1 + (1 - q)S_q^T]. \tag{31}$$

That is the relation between Rényi–Tsallis entropies which is also given in Reference [24] and can indicate an emergence of micro to macro properties.

Rényi introduced a family of entropy functionals called the alpha entropies by using the Kolmogorov–Nagumo averages [37]. According to Reference [37], the role of scale translation would

be studied in a systematic manner, since the procedure of estimating parameters is sensitive to the scale on which the experimental data are presented.

The Rényi entropy is additive, while the Tsallis entropy is not. This means that the thermodynamic macroscopic additivity for the entropy emerges even from a starting point for entropy based on nonadditivity. There is a nonlinear scale for the connection of the Rényi–Tsallis entropies, and this could indicate the possible emergence of additivity. This can be a consequence of complexity, in the sense that there are repeating patterns to form new complex patterns.

Here, the entropic form S_q can be seen as metric entropy from a microscopic view. On the other hand, considering the standard second law in the Clausius form, $(\Delta S)_q^R$ leads to the validity of the second law for the metric entropy S_q :

$$\frac{dQ_q}{T_q} \geq 0, \tag{32}$$

That is, $dS_q \geq 0$.

3.2. Conformable Deformed Maxwell Relations

In order to build up the Maxwell relations with the conformable derivatives, we expand a given function f with an equivalent form of Taylor expansion using integer polynomials to the first order [38], instead of noninteger [39] order polynomial. This merely reflects the fact that the space of concerned functions is considered differentiable.

The α -total differentiation of a differentiable function can be developed by the infinitesimal difference:

$$d_\alpha f \equiv f(x + \varepsilon_x x^{1-\alpha}, y + \varepsilon_y y^{1-\alpha}) - f(x, y), \tag{33}$$

which can be written in terms of conformable partial derivatives as

$$\begin{aligned} d_\alpha f &= f(x + \varepsilon_x x^{1-\alpha}, y + \varepsilon_y y^{1-\alpha}) - f(x, y) \\ &= f(x + \varepsilon_x x^{1-\alpha}, y) - f(x, y) + f(x + \varepsilon_x x^{1-\alpha}, y + \varepsilon_y y^{1-\alpha}) - f(x + \varepsilon_x x^{1-\alpha}, y) \\ &= \frac{\partial_\alpha f}{\partial x} \varepsilon_x + \frac{\partial_\alpha f}{\partial y} \varepsilon_y = \frac{\partial_\alpha f}{\partial x} (dx) + \frac{\partial_\alpha f}{\partial y} (dy). \end{aligned} \tag{34}$$

Next, following the analogous approach of q -derivative to obtain Legendre Transform and Helmholtz free energy, enthalpy, and Gibbs free energy, we can obtain the α -deformed Maxwell relations that we collect in Table 2.

Table 2. α -Deformed Maxwell relations.

$\frac{\partial_\alpha U_\alpha}{\partial V} _{S_\alpha} = -P_\alpha V^{1-\alpha},$	$\frac{\partial_\alpha U_\alpha}{\partial S_\alpha} _{V} = T_\alpha S_\alpha^{1-\alpha},$
$\frac{\partial_\alpha F_\alpha}{\partial V} _{T_\alpha} = -P_\alpha V^{1-\alpha},$	$\frac{\partial_\alpha F_\alpha}{\partial T_\alpha} _{V} = -S_\alpha T_\alpha^{1-\alpha},$
$\frac{\partial_\alpha H_\alpha}{\partial P_\alpha} _{S_\alpha} = V P_\alpha^{1-\alpha},$	$\frac{\partial_\alpha H_\alpha}{\partial S_\alpha} _{P_\alpha} = T_\alpha S_\alpha^{1-\alpha},$
$\frac{\partial_\alpha G_\alpha}{\partial P_\alpha} _{T_\alpha} = -V P_\alpha^{1-\alpha},$	$\frac{\partial_\alpha G_\alpha}{\partial T_\alpha} _{P_\alpha} = -S_\alpha T_\alpha^{1-\alpha},$

This is almost identical to the results in Reference [6], except for the gamma function pre factor. Here α is different from that work as well. All procedures to obtain a fractal equation of state are simpler than in Reference [6], and can be done to compare with experimental data. A further investigation along this line will be presented in a future work.

In each of the above Maxwell relations, the α -parameter does not necessarily need to be the same for all Maxwell relations. In this study, we maintain the same in order to compare to results in Reference [6].

Note that in Reference [6], the parameter α is very close to 1. This is important to for the approximation there to be valid; that is, only lower order for i parameter ($i = 0$) in the expansion

A_3, A_4 are kept. This in fact makes the procedure used there local. Apparently, the use of complicated fractional calculus seems unnecessary compared to the conformable or the q -deformed MD procedures.

Considering $dU = T_{phys}dS_\alpha$, we can write

$$dS_\alpha = \frac{dQ}{T_{phys}} = \frac{dQ}{T} \times \frac{1}{S^{1-\alpha}} = \frac{dS}{S^{1-\alpha}}, \quad (35)$$

which is consistent with the Improper Riemann integral [23]:

$$I_a^\alpha(f)(t) = \int_a^t f(x) d^\alpha x = \int_a^t f(x) x^{\alpha-1} dx, \quad (36)$$

where $d^\alpha x = \frac{1}{x^{1-\alpha}} dx$. Integrating Equation (35) will give

$$(\Delta S)_\alpha = \frac{S^\alpha}{\alpha}. \quad (37)$$

Changing the variable $S \rightarrow 1 + \frac{S}{l_0}$ in Equation (35), we obtain, after integration, the result $l_0 \left(1 + \frac{S}{l_0}\right)^\alpha$, which is related to the Hausdorff derivative up to a constant and is valid for differentiable functions [2]. Consequently, we have

$$(\Delta S)_\alpha = \frac{l_0 \left(1 + \frac{S}{l_0}\right)^\alpha}{\alpha}. \quad (38)$$

If we consider $\alpha = \frac{1-r}{1-q}$ and $l_0 = \frac{l'_0}{1-q}$, the result is

$$(\Delta S)_\alpha = \frac{l'_0 \left(1 + (1-q) \frac{S}{l'_0}\right)^{\frac{1-r}{1-q}}}{1-r}. \quad (39)$$

For $l'_0 = 1$, this is related to the Sharma–Mittal (SM) and Tsallis entropic forms

$$(\Delta S)_\alpha = S_{q,r}^{SM} + \frac{1}{1-r} = \frac{[1 + (1-q)S_q^T]^{\frac{1-r}{1-q}}}{1-r}. \quad (40)$$

It is well-known that for $q = r$, the Tsallis entropy is the result. For the limit $r \rightarrow 1, q \neq 1$, the Rényi entropy emerges. For $r \rightarrow 1, q \rightarrow 1$, the Boltzmann–Gibbs is the limiting entropic form. We conclude that the SM entropy is a kind of re-parametrized geometric average of the Tsallis entropy, which is on its own an exponentiation of Rényi entropic form. The Rényi entropy is a smoothing of both the Tsallis and the Sharma–Mittal entropic forms.

4. Conclusions and Outlook

All foregoing presented results give strong evidence to support the paradigm of the MD in the attempt to generalize the thermodynamics. In short, we summarize our findings by the following remarks:

Considering a mapping to the continuous fractal space, we have obtained the generalized Maxwell relations with the use of the MD such as the q -derivative in the nonextensive context and the conformable derivative.

To obtain the q -deformed and α -deformed Maxwell relations, the q -total derivative and the α -total derivative were developed. The physical temperature, T_{phys} , in Reference [24] appears naturally in our formalism.

The emergence of the Rényi entropy in the nonextensive context and the Sharma–Mittal-like entropy is obvious.

For the cases studied, the Maxwell relations of classical thermodynamics are recovered for $\alpha \rightarrow 1$, $q \rightarrow 1$.

In future works, we intend to study the applications of the MD toward the development of a generalized thermodynamics on a solid physical basis within fractal, multifractal under complex systems context.

Additionally, as an outlook for possible research, Reference [40] and some approximate approaches in References [41–43], there are indications that the use of MD and variational principles, that perspectives of applications seem to open for modeling quantum systems, including the representation of quantum mechanics by hydrodynamic models, including those with stochastic processes [44].

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