Conformal Superspace σ-Models

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Based on arXiv:0809.1046 (with V. Mitev and V. Schomerus)

and arXiv:09080878 (with C. Candu, V. Mitev, H. Saleur and V. Schomerus)

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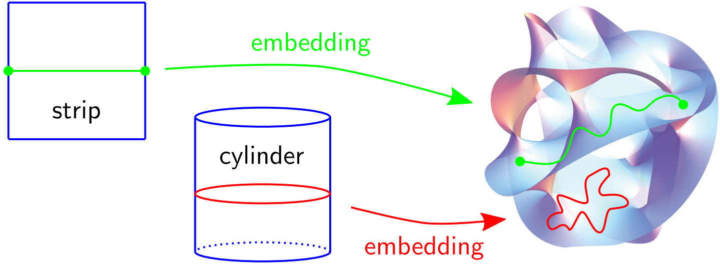
World-sheet

2D surface
(w/wo boundaries or handles)

Target space

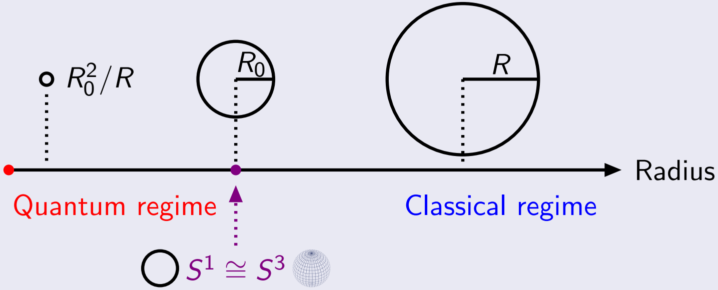
(Pseudo-)Riemannian manifold
(extra structure: gauge fields, ...)



σ-models = (quantum) field theories

A simple example: The circle

The moduli space of circle theories

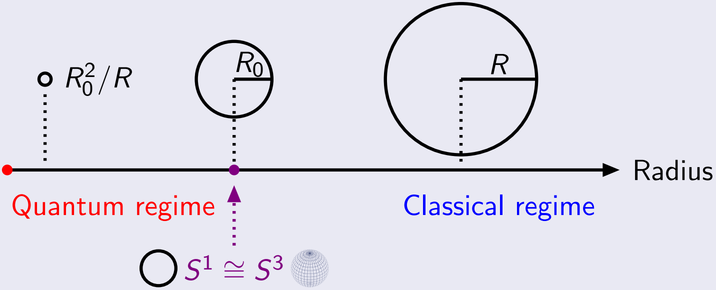


Two lessons

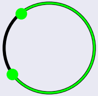
- There is an equivalence: $R \leftrightarrow R_0^2/R$ ("T-duality")
- In the quantum regime geometry starts to loose its meaning

A simple example: The circle

The moduli space of circle theories



An open string partition function



$$Z(q,z|R) = \mathrm{tr}\left[z^{\mathrm{P}}\, q^{\mathrm{Energy}(R)}\right] = \frac{1}{\eta(q)} \sum_{w\in\mathbb{Z}} z^{w}\, q^{\frac{w^2}{2R^2}}$$

Adding curvature and supersymmetry...

Appearances of superspace σ-models

- String theory
  - Quantization of strings in flux backgrounds [Berkovits]
  - String theory / gauge theory correspondence [Maldacena]
  - Moduli stabilization in string phenomenology [KKLT] [...]
- Disordered systems
  - Quantum Hall systems
  - Self avoiding random walks, polymer physics, ...
  - Efetov's supersymmetry trick

Conformal invariance

- String theory: Diffeomorphism + Weyl invariance
- Statistical physics: Critical points / 2^{nd} order phase transitions

String quantization: The pure spinor formalism

Ingredients

- Superspace σ-model encoding geometry and fluxes
- Pure spinors: Curved ghost system
- BRST procedure

[Berkovits at al] [Grassi et al] [...]

Features

- Manifest target space supersymmetry
- Manifest world-sheet conformal symmetry
- Action quantizable, but quantization hard in practice

Strings on $AdS_5 \times S^5$ and $AdS_4 \times \mathbb{CP}^3$

Spectrum accessible because of integrability

- Factorizable S-matrix
- Structure fixed (up to a phase) by $SU(2|2) \ltimes \mathbb{R}^3$-symmetry
- Bethe ansatz, Y-systems, ...

Open issues

- String scattering amplitudes?
- 2D Lorentz invariant formulation?
- Other backgrounds? $\rightarrow$ Conifold, nil-manifolds, ...

The standard perspective on AdS/CFT

Overview

Gauge theory		String theory
$\mathcal{N} = 4$ Super Yang-Mills		$\mathrm{AdS}_5 \times \mathrm{S}^5$
$\mathcal{N} = 6$ Chern-Simons		$\mathrm{AdS}_4 \times \mathbb{CP}^3$
S-matrix, spectrum,...	$\Longleftrightarrow$	S-matrix, spectrum,...
t'Hooft coupling λ, ...		Radius R, ...

Problem

From this perspective, both sides need to be solved separately.

An alternative perspective on AdS/CFT

Proposal: Two step procedure...

Weakly coupled 4D gauge theory

$\Updownarrow$ Feynman diagram expansion

Weakly coupled 2D theory (Topological σ-model)

$\Updownarrow$ "Well-established machinery"

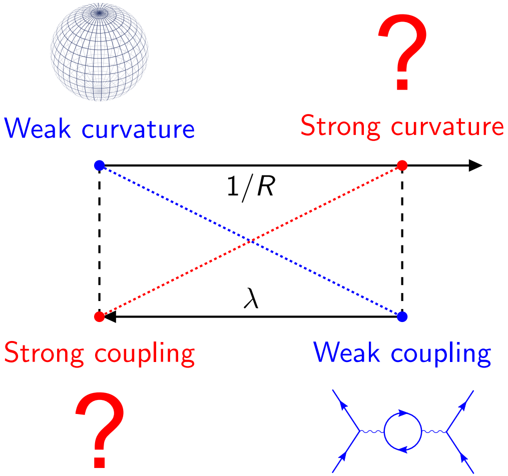
Strongly curved 2D σ-model

[Berkovits] [Berkovits,Vafa] [Berkovits]

Summary: String theory/gauge theory dualities

String theory in 10D
(σ-model with constraints)

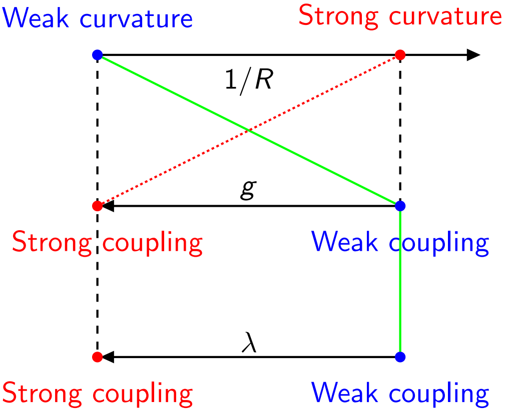
Gauge theory



String theory in 10D
(σ-model with constraints)

“Some dual 2D theory”

Gauge theory



The structure of this talk

Outline

1. Supercoset σ-models
   - Occurrence in string theory and condensed matter theory
   - Ricci flatness and conformal invariance
2. Particular examples
   - Superspheres
   - Projective superspaces
3. Quasi-abelian perturbation theory
   - Exact open string spectra
   - World-sheet duality for supersphere σ-models

Appearance of supercosets

String backgrounds as supercosets...

Minkowski	$\mathrm{AdS}_5 \times S^5$	$\mathrm{AdS}_4 \times \mathbb{CP}^3$	$\mathrm{AdS}_2 \times S^2$
$\frac{\text{super-Poincaré}}{\text{Lorentz}}$	$\frac{\mathrm{PSU}(2,2\|4)}{\mathrm{SO}(1,4)\times\mathrm{SO}(5)}$	$\frac{\mathrm{OSP}(6\|2,2)}{\mathrm{U}(3)\times\mathrm{SO}(1,3)}$	$\frac{\mathrm{PSU}(1,1\|2)}{\mathrm{U}(1)\times\mathrm{U}(1)}$

[Metsaev,Tseytlin] [Berkovits,Bershadsky,Hauer,Zhukov,Zwiebach] [Arutyunov,Frolov]

Supercosets in statistical physics...

IQHE	Dense polymers	Dense polymers
(non-conformal)	$S^{2S+1\|2S}$	$\mathbb{CP}^{S-1\|S}$
$\frac{\mathrm{U}(1,1\|2)}{\mathrm{U}(1\|1)\times\mathrm{U}(1\|1)}$	$\frac{\mathrm{OSP}(2S+2\|2S)}{\mathrm{OSP}(2S+1\|2S)}$	$\frac{\mathrm{U}(S\|S)}{\mathrm{U}(1)\times\mathrm{U}(S-1\|S)}$

[Weidenmüller] [Zirnbauer]

[Read,Saleur] [Candu,Jacobsen,Read,Saleur]

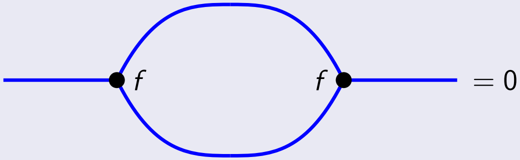
A unifying construction

Definition of the cosets

$$G/H : \quad gh \sim g$$

Some additional requirements for conformal invariance

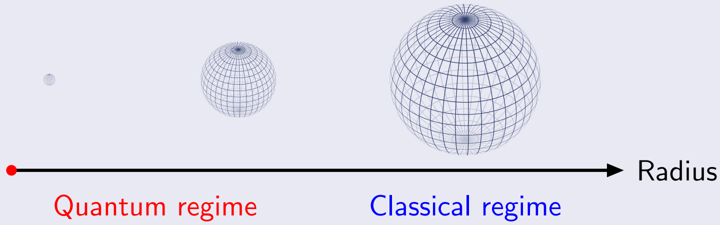
- $H \subset G$ is invariant subgroup under an automorphism
- Ricci flatness ("super Calabi-Yau") $\Leftrightarrow$ vanishing Killing form



Examples: Cosets of $\mathrm{PSU}(N|N)$, $\mathrm{OSP}(2S+2|2S)$, $\mathrm{D}(2,1;\alpha)$.

Properties of conformal supercoset models

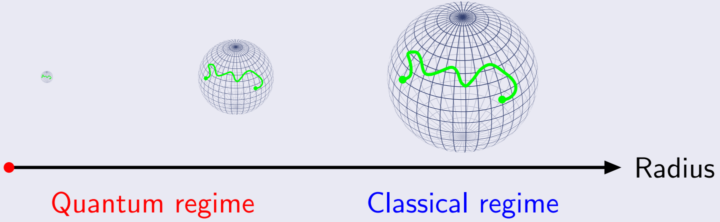
The moduli space of generic supercoset theories



Properties at a glance

- Supersymmetry G: $g \mapsto kg$ (realized geometrically)
- Conformal invariance [Kagan,Young] [Babichenko]
- Integrability [Pohlmeyer] [Lüscher] ... [Bena,Polchinski,Roiban] [Young]

The moduli space of generic supercoset theories

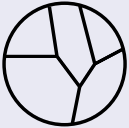


The general open string partition function

$$Z(q,z|R) \;=\; \mathrm{tr}\Big[z^{\text{Cartan}}\, q^{\text{Energy}(R)}\Big] \;=\; \sum_{\Lambda} \underbrace{\psi_\Lambda(q,R)}_{\text{Dynamics}}\; \underbrace{\chi^G_\Lambda(z)}_{\text{Symmetry}}$$

The β-function vanishes identically...

$$\beta = \sum_{\substack{\text{certain} \\ G\text{-invariants}}}$$



$$= 0$$

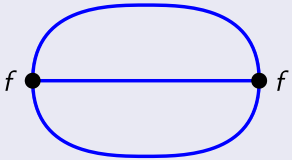
Ingredients:

Invariant form: $\kappa^{\mu\nu}$

Structure constants: $f^{\mu\nu\lambda}$

Sketch of conformal invariance

The β-function vanishes identically...



$= 0$

Ingredients:

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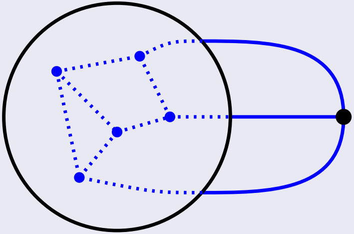
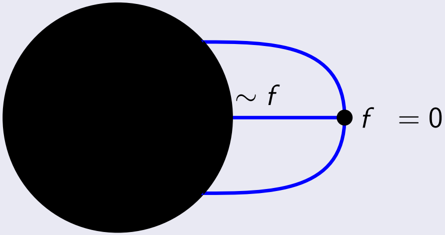


There is a unique invariant rank 3 tensor!

[Bershadsky,Zhukov,Vaintrob'99] [Babichenko'06]

Sketch of conformal invariance

The β-function vanishes identically...



There is a unique invariant rank 3 tensor!

[Bershadsky,Zhukov,Vaintrob'99] [Babichenko'06]

Supersphere σ-models

The supersphere $S^{3|2}$

Realization of $S^{3|2}$ as a submanifold of flat superspace $\mathbb{R}^{4|2}$

$$\vec{X} = \begin{pmatrix} \vec{x} \\ \eta_1 \\ \eta_2 \end{pmatrix} \qquad \text{with} \qquad \vec{X}^2 = \vec{x}^2 + 2\eta_1\eta_2 = R^2$$

Symmetry

$$\mathrm{O}(4) \times \mathrm{SP}(2) \xrightarrow[\text{symmetrization}]{\text{super-}} \mathrm{OSP}(4|2)$$

Realization as a supercoset

$$S^{3|2} = \frac{\mathrm{OSP}(4|2)}{\mathrm{OSP}(3|2)}$$

The supersphere σ-model

Action functional

$$\mathcal{S}_\sigma = \int \partial_\mu \vec{X} \cdot \partial^\mu \vec{X} \qquad \text{with} \qquad \vec{X}^2 = R^2$$

The space of states for freely moving open strings

$$\prod X^{a_i} \prod \partial_t X^{b_j} \prod \partial_t^2 X^{c_k} \cdots \qquad \text{and} \qquad \vec{X}^2 = R^2$$

$\Rightarrow$ Products of coordinate fields and their derivatives

Large volume partition function

- "Single particle energies" add up $\rightarrow$ # derivatives
- Partition function is pure combinatorics [Candu,Saleur] [Mitev,TQ,Schomerus]

The large volume limit

Keeping track of quantum numbers...

- Symmetry

$$\mathrm{OSP}(4|2) \rightarrow \mathrm{SP}(2) \times \mathrm{SO}(4) \cong \mathrm{SU}(2) \times \mathrm{SU}(2) \times \mathrm{SU}(2)$$

- Classify states according to the bosonic symmetry:

$$\vec{X} = (\vec{x}, \eta_1, \eta_2): \qquad V = \underbrace{\left(0, \tfrac{1}{2}, \tfrac{1}{2}\right)}_{\text{bosons}} \oplus \underbrace{\left(\tfrac{1}{2}, 0, 0\right)}_{\text{fermions}}$$

- Other quantum numbers:
  - Energy q^E
  - Polynomial grade t^n (broken by $\vec{X}^2 = R^2$)
- Use this to characterize all monomials

$$\prod X^{a_i} \prod \partial_t X^{b_j} \prod \partial_t^2 X^{c_k} \cdots \quad \text{with} \quad \vec{X}^2 = R^2$$

Constituents of the partition function

A useful dictionary

Field theoretic quantity	**Contribution**	Representation
2 Fermionic coordinates	$t\,z_1^{\pm 1}$	$\frac{1}{2}$
4 Bosonic coordinates	$t\,z_2^{\pm 1} z_3^{\pm 1},\ t\,z_2^{\pm 1} z_3^{\mp 1}$	$\left(\frac{1}{2}, \frac{1}{2}\right)$
Derivative ∂	q	
Constraint $\vec{X}^2 = R^2$	$1 - t^2$	
Constraint $\partial^n \vec{X}^2 = 0$	$1 - t^2 q^n$	

$t \leftrightarrow$ polynomial grade

$z_1, z_2, z_3 \leftrightarrow$ SU(2) quantum numbers

The full σ-model partition function

Summing up all contributions...

$$Z_\sigma(R_\infty) = \lim_{t\to 1}\Bigg[q^{-\frac{1}{24}}\prod_{n=0}^{\infty}(1-t^2q^n)\times$$
$$\times\prod_{n=0}^{\infty}\frac{(1+z_1tq^n)(1+z_1^{-1}tq^n)}{(1-z_2z_3tq^n)(1-z_2z_3^{-1}tq^n)(1-z_2^{-1}z_3tq^n)(1-z_2^{-1}z_3^{-1}tq^n)}\Bigg]$$

The problem...

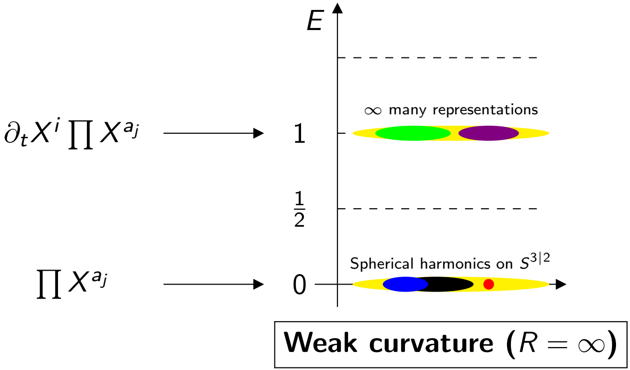
Organize this into representations of OSP(4|2)!

Since the model is symmetric under OSP(4|2) the partition function may be decomposed into characters of OSP(4|2):

$$Z_\sigma(R_\infty) \;=\; \sum_{[j_1,j_2,j_3]} \psi^\sigma_{[j_1,j_2,j_3]}(q)\,\chi_{[j_1,j_2,j_3]}(z)$$

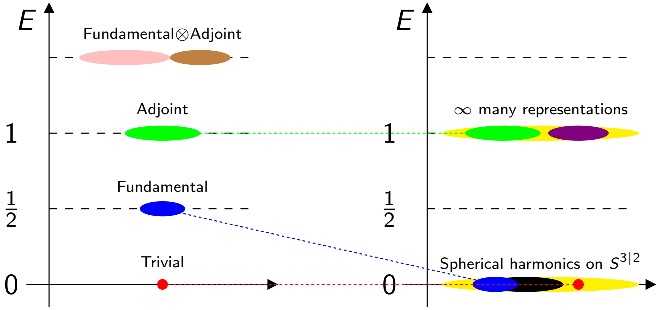
All the non-trivial information is encoded in

$$\psi^\sigma_{[j_1,j_2,j_3]}(q) \;=\; \frac{q^{-C_{[j_1,j_2,j_3]}/2}}{\eta(q)^4}\sum_{n,m=0}^{\infty}(-1)^{m+n}q^{\frac{m}{2}(m+4j_1+2n+1)+\frac{n}{2}+j_1-\frac{1}{8}}$$
$$\times\left(q^{(j_2-\frac{n}{2})^2}-q^{(j_2+\frac{n}{2}+1)^2}\right)\left(q^{(j_3-\frac{n}{2})^2}-q^{(j_3+\frac{n}{2}+1)^2}\right)$$



E
$\partial_t X^i \prod X^{a_j}$
1
∞ many representations
$\frac{1}{2}$
$\prod X^{a_j}$
0
Spherical harmonics on $S^{3|2}$
Weak curvature ($R = \infty$)

Sketch of the large volume partition function

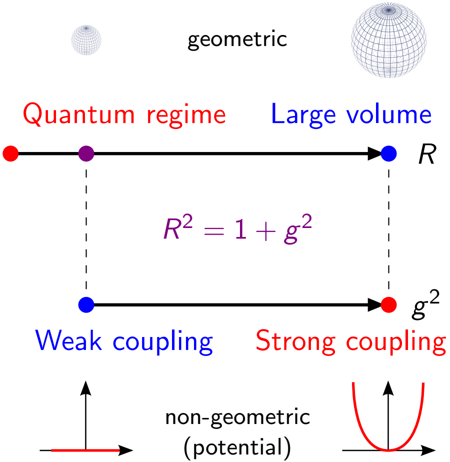


Quantum regime ($R = 1$)

Weak curvature ($R = \infty$)

A world-sheet duality for supersphere σ-models

A world-sheet duality for superspheres?



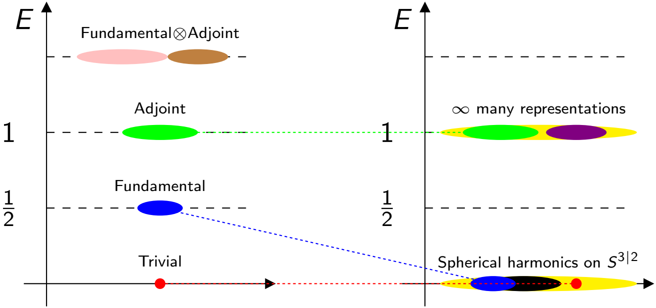
Supersphere σ-model

$$Z_\sigma(q, z|R)$$
$$\|$$
$$Z_{\mathrm{GN}}(q, z|g^2)$$

OSP(4|2) Gross-Neveu model

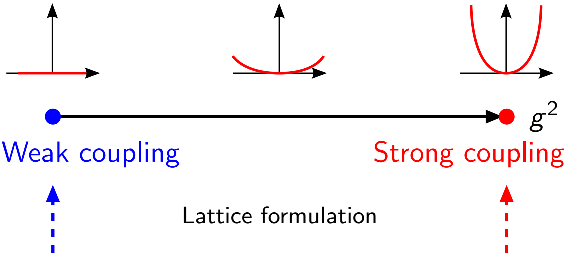
[Candu,Saleur]^2 [Mitev,TQ,Schomerus]

In the two extreme limits the spectrum has the form...



Quantum regime ($R = 1$) | **Weak curvature ($R = \infty$)**

Evidence for the duality

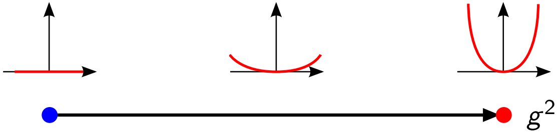


Free ghosts / WZW model
Affine symmetry

Free theory
Combinatorics

[Candu,Saleur]^2 [Mitev,TQ,Schomerus]

Evidence for the duality



Weak coupling — Strong coupling

Lattice formulation

Free ghosts / WZW model
Affine symmetry

Free theory
Combinatorics

Goal: $Z_{\text{GN}}(q,z|g^2) \;=\; \sum_\Lambda \psi^\sigma_\Lambda(q,g^2)\,\chi_\Lambda(z)$

[Candu,Saleur] [Mitev,TQ,Schomerus]

$OSP(4|2)$ Gross-Neveu model

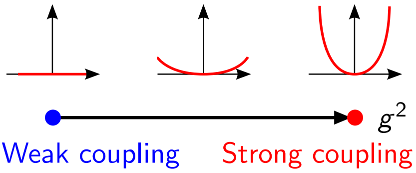
The OSP(4|2) Gross-Neveu model

Field content

- Fundamental OSP(4|2)-multiplet $(\psi_1, \psi_2, \psi_3, \psi_4, \beta, \gamma)$
- All these fields have scaling dimension 1/2

Formulation as a Gross-Neveu model

$$\mathcal{S}_{\text{GN}} = \mathcal{S}_{\text{free}} + g^2\,\mathcal{S}_{\text{int}} \qquad \begin{cases} \mathcal{S}_{\text{free}} = \int \left[\psi\bar{\partial}\psi + 2\beta\bar{\partial}\gamma + h.c.\right] \\ \mathcal{S}_{\text{int}} = \int \left[\psi\bar{\psi} + \beta\bar{\gamma} - \gamma\bar{\beta}\right]^2 \end{cases}$$



Goal: $Z_{\text{GN}}(q, z|g^2)$

An open string spectrum

Formulation as a deformed OSP(4|2) WZW model

$$\mathcal{S}_{\text{GN}} = \mathcal{S}_{\text{WZW}} + g^2\,\mathcal{S}_{\text{def}} \qquad \text{with} \qquad \mathcal{S}_{\text{def}} = \int \text{str}(J\bar{J})$$

Solution at $g = 0$

- At $g = 0$ there is an OSP(4|2) Kac-Moody algebra symmetry
- Partition functions can be constructed using combinatorics

An open string partition function for $g = 0$

$$Z_{\text{GN}}(g^2 = 0) = \sum_{\Lambda} \underbrace{\psi_{\Lambda}^{\text{WZW}}(q)}_{\text{energy levels}} \quad \underbrace{\chi_{\Lambda}(z)}_{\text{OSP}(4|2)\ \text{content}}$$

An open string spectrum

Formulation as a deformed OSP(4|2) WZW model

$$\mathcal{S}_{\text{GN}} = \mathcal{S}_{\text{WZW}} + g^2\,\mathcal{S}_{\text{def}} \qquad \text{with} \qquad \mathcal{S}_{\text{def}} = \int \text{str}(J\bar{J})$$

Solution at $g = 0$

- At $g = 0$ there is an OSP(4|2) Kac-Moody algebra symmetry
- Partition functions can be constructed using combinatorics

An open string partition function for all g

$$Z_{\text{GN}}(g^2) = \sum_{\Lambda} \underbrace{q^{-\frac{1}{2}\frac{g^2}{1+g^2}C_\Lambda}}_{\text{anomalous dimension}} \underbrace{\psi_\Lambda^{\text{WZW}}(q)}_{\text{energy levels}} \underbrace{\chi_\Lambda(z)}_{\text{OSP(4|2) content}}$$

A D-brane spectrum

A specific D-brane in the OSP(4|2) WZW model...

The spectrum of a "twisted D-brane" is

$$Z_{\text{GN}}(g^2=0) \;=\; \underbrace{\chi_{\{0\}}(q,z)}_{\text{vacuum}} + \underbrace{\chi_{\{1/2\}}(q,z)}_{\text{fundamental}}$$

The problem (yet again...)

Organize this into representations of OSP(4|2)!

Decomposition into representations of OSP(4|2)

Plugging in concrete expressions, one obtains

$$\begin{aligned} Z_{\mathrm{GN}}(g^2=0) &= \frac{\eta(q)}{\theta_4(z_1)}\left[\frac{\theta_2(q^2,z_2^2)\theta_2(q^2,z_3^2)}{\eta(q)^2}+\frac{\theta_3(q^2,z_2^2)\theta_3(q^2,z_3^2)}{\eta(q)^2}\right] \\ &= \sum \psi^{\mathrm{WZW}}_{[j_1,j_2,j_3]}(q)\,\chi_{[j_1,j_2,j_3]}(z) \end{aligned}$$

Plugging in concrete expressions, one obtains

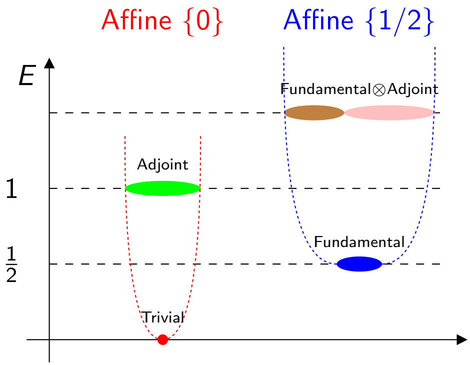
$$Z_{\mathrm{GN}}(g^2=0) = \frac{\eta(q)}{\theta_4(z_1)}\left[\frac{\theta_2(q^2,z_2^2)\theta_2(q^2,z_3^2)}{\eta(q)^2}+\frac{\theta_3(q^2,z_2^2)\theta_3(q^2,z_3^2)}{\eta(q)^2}\right]$$

$$= \sum \psi^{\mathrm{WZW}}_{[j_1,j_2,j_3]}(q)\,\chi_{[j_1,j_2,j_3]}(z)$$

$$\psi^{\mathrm{WZW}}_{[j_1,j_2,j_3]}(q) = \frac{1}{\eta(q)^4}\sum_{n,m=0}^{\infty}(-1)^{n+m}q^{\frac{m}{2}(m+4j_1+2n+1)+j_1+\frac{n}{2}-\frac{1}{8}}$$

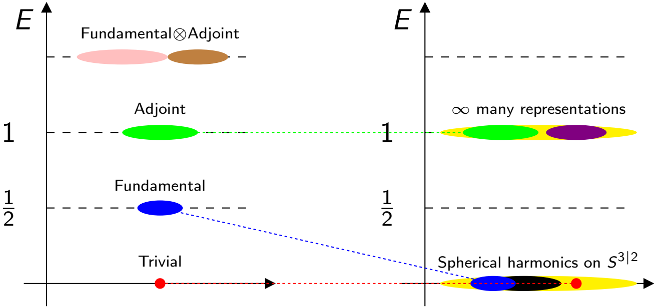
$$\times\left(q^{(j_2-\frac{n}{2})^2}-q^{(j_2+\frac{n}{2}+1)^2}\right)\left(q^{(j_3-\frac{n}{2})^2}-q^{(j_3+\frac{n}{2}+1)^2}\right)$$

What did we achieve now?



WZW model ($g = 0$)

$$Z_{\text{GN}}(g^2) = \sum_{\Lambda} \underbrace{q^{-\frac{1}{2}\frac{g^2}{1+g^2}C_\Lambda}}_{\text{anomalous dimension}} \underbrace{\psi_\Lambda^{\text{WZW}}(q)}_{\text{energy levels}} \underbrace{\chi_\Lambda(z)}_{\text{OSP}(4|2)\text{ content}}$$



WZW model ($g = 0$)

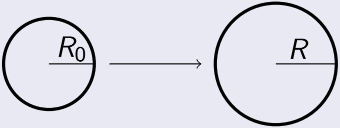
Strong deformation ($g = \infty$)

Supersphere σ-model at $R \to \infty$

Quasi-abelian deformations

Radius deformation of the free boson

Consider a deformation...



$$R = R_0\sqrt{1+\gamma}$$

Freely moving open strings on a circle of radius R...

$$Z(q,z|R) \;=\; \frac{1}{\eta(q)}\sum_{w\in\mathbb{Z}} z^w\, q^{\frac{w^2}{2R^2}} \;=\; \frac{1}{\eta(q)}\sum_{w\in\mathbb{Z}} q^{\frac{w^2}{2R_0^2(1+\gamma)}}\,\chi_w(z)$$

Anomalous dimensions

$$\delta_\gamma E_w \;=\; \frac{w^2}{2R_0^2}\left[\frac{1}{1+\gamma}-1\right] \;=\; -\frac{\gamma}{1+\gamma}\,\frac{w^2}{2R_0^2} \;=\; -\frac{\gamma}{1+\gamma}\,C_2(w)$$

Quasi-abelianness of supergroup WZW theories

The effective deformation for conformal dimensions

- The combinatorics of the perturbation series is determined by the current algebra

$$J^\mu(z)\,J^\nu(w) = \frac{k\,\delta^{\mu\nu}}{(z-w)^2} + \frac{if^{\mu\nu}{}_\lambda J^\lambda(w)}{z-w} \sim \frac{k\,\delta^{\mu\nu}}{(z-w)^2}$$

- Vanishing Killing form ⇒ the perturbation is quasi-abelian (for the purposes of calculating anomalous dimensions)

[Bershadsky,Zhukov,Vaintrob] [TQ,Schomerus,Creutzig]

- In the OSP(4|2) WZW model a representation Λ shifts by

$$\delta E_\Lambda(g^2) \;=\; -\frac{1}{2}\frac{g^2 C_\Lambda}{1+g^2} \;=\; -\frac{1}{2}\left(1-\frac{1}{R^2}\right)C_\Lambda$$

Projective Superspaces

New features

- Family contains supertwistor space $\mathbb{CP}^{3|4}$ → [Witten]
- Non-trivial topology

$\Rightarrow$ Monopoles and θ-term

- Symplectic fermions as a subsector [Candu,Creutzig,Mitev,Schomerus]

θ-term $\Rightarrow$ twists

- σ-model brane spectrum can be argued to be

$$Z_{R,\theta}(q,z) \;=\; \underbrace{q^{-\frac{1}{2}\lambda(R,\theta)\left[1-\lambda(R,\theta)\right]}}_{\text{twist}} \sum_{\Lambda} \underbrace{q^{f(R,\theta)C_\Lambda}}_{\text{Casimir}} \underbrace{\psi_\Lambda^\infty(q)\,\chi_\Lambda(z)}_{\text{result for } R\to\infty}$$

[Candu,Mitev,TQ,Saleur,Schomerus]

- Currently no free field theory point is known...

Conclusions

Conclusions and Outlook

Conclusions

- Using supersymmetry we determined the full spectrum of anomalous dimensions for certain open string spectra in various models as a function of the moduli
- Our results provided strong evidence for a duality between supersphere σ-models and Gross-Neveu models

Outlook

- Conformal invariance $\leftrightarrow$ Integrability
- Closed string spectra?
- Application to more stringy backgrounds...