Differential calculus on Hopf Group Coalgebra

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Abstract

In this paper we construct the Differential calculus on the Hopf Group Coalgebra introduced by Turaev [10]. We proved that the concepts introduced by S.L.Woronowicz in constructing Differential calculus on Hopf Compact Matrix Pseudogroups (Quantum Groups)[7] can be adapted to serve again in our construction.

Introduction

Quantum groups, from a mathematical point of view, may be introduced by making emphasis on their $q-$deformed enveloping algebra aspects [1,2], which leads to the quantized enveloping algebras, or by making emphasis in the $R-$matrix formalism that describes the deformed group algebra. Also, they are mathematically well defined in the framework of Hopf algebra [3]. Quantum groups provide an interesting example of non-commutative geometry[4]. Non-commutative differential calculus on quantum groups is a fundamental tool needed for many applications [5,6].

S.L.Woronowicz [7] gave the general framework for bicovariant differential calculus on quantum groups following general ideas of A.Connes. Also, He showed that all important notions and formulae of classical Lie group theory admit a generalization to the quantum group case and he has restricted himself to compact matrix pseudogroups as introduced in [8].In contrast to the classical differential geometry on Lie groups, there is no functorial method to obtain a unique bicovariant differential calculus on a given quantum group [9].

Recently, Quasitriangular Hopf $\pi-$coalgebras are introduced by Turaev [10]. He has showed that they give rise to crossed $\pi-$categories. Virelizier [11] studied the algebraic properties of the Hopf $\pi-$coalgebras, also he has showed that the existence of integrals and trace for such coalgebras and has generalized the main properties of the quasitriangular Hopf algebras to the setting of Hopf $\pi-$coalgebra.

In this paper we will use the concepts introduced by S.L.Woronowicz [7] to construct the Differential calculus on the Hopf group coalgebra(introduced by Turaev [10]). We briefly describe the content of the paper.In section one we give the definition of Hopf group coalgebras [11]. In section two, we give the main definitions and theorems concerning first order differential calculus. Section three contains the construction of the $\pi-$graded Bicovariant bimodules. Finally, in section four we construct the first order differential calculus on the Hopf group coalgebra.
Now let us give some basic definitions about Hopf $\pi-$coalgebra

1 Hopf Group Coalgebra

Definition 1.1. A $\pi-$coalgebra is a family $C = \{C_\alpha\}_{\alpha\in\pi}$ of $k-$linear spaces endowed with a family $\Delta = \{\Delta_{\alpha,\beta} : C_{\alpha\beta} \to C_\alpha \otimes C_\beta\}_{\alpha,\beta\in\pi}$ of $k-$linear maps (the comultiplication) and a $k-$linear map $\varepsilon : C_1 \to k$ such that

- Δ is coassociative in the sense that for any $\alpha, \beta, \gamma \in \pi$,

$$(\Delta_{\alpha,\beta} \otimes id)\Delta_{\alpha\beta,\gamma} = (id \otimes \Delta_{\beta,\gamma})\Delta_{\alpha,\beta\gamma} \ ,$$

- for all $\alpha \in \pi$,

$$(id \otimes \varepsilon)\Delta_{\alpha,1} = (\varepsilon \otimes id)\Delta_{1,\alpha}.$$

Sweedler's notation In the case of Hopf group coalgebra Sweedler's notations have been extended by Turaev and Virelizier in the following way: for any $\alpha, \beta \in \pi$ and $c \in C_{\alpha\beta}$, they defined

$$\Delta_{\alpha,\beta}(c) = \sum_{(c)} c_{(1,\alpha)} \otimes c_{(2,\beta)} \in C_\alpha \otimes C_\beta.$$

or shortly, if we have the summation implicit

$$\Delta_{\alpha,\beta}(c) = c_{(1,\alpha)} \otimes c_{(2,\beta)}.$$

The coassociativity axiom gives that , for any $\alpha, \beta, \gamma \in \pi$ and $c \in C_{\alpha\beta,\gamma}$

$$c_{(1,\alpha\beta)(1,\alpha)} \otimes c_{(1,\alpha\beta)(2,\beta)} \otimes c_{(2,\gamma)} = c_{(1,\alpha)} \otimes c_{(2,\beta\gamma)(1,\beta)} \otimes c_{(2,\beta\gamma)(2,\gamma)}.$$

Let $C = (\{C_\alpha\}_{\alpha\in\pi}, \Delta, \varepsilon)$ be a $\pi-$coalgebra and A be an algebra with multiplication m and unit element 1_A. The family Δ and the map m induce a map

$$* : conv(C, A) \otimes conv(C, A) \to conv(C, A)$$

defined by the composition

$$Hom(C_\alpha, A) \otimes Hom(C_\beta, A) \xrightarrow{\rho} Hom(C_\alpha \otimes C_\beta, A \otimes A) \xrightarrow{Hom(\Delta_{\alpha,\beta}, m)} Hom(C_{\alpha\beta}, A)$$

where ρ is the natural injection of $Hom(C_\alpha, A) \otimes Hom(C_\beta, A)$ into $Hom(C_\alpha \otimes C_\beta, A \otimes A)$
The map $*$ is called convolution product of f, g
Also, the maps

$$\varepsilon : C_1 \longrightarrow k \quad \text{and} \quad \eta : k \longrightarrow A$$

induce a map

$$\eta_{Conv(C,A)} : k \longrightarrow Conv(C, A)$$

defined by

$$(\eta_{Conv(C,A)}(\lambda))(c) = \varepsilon(c)\eta(\lambda)$$

for all $c \in C_1$.

Lemma 1.1. *The k-space*

$$Conv(C, A) = \bigoplus_{\alpha\in\pi} Hom(C_\alpha, A)$$

endowed with the convolution product $*$ and the unit element $\varepsilon 1_A$ is a $\pi-$graded algebra called the convolution algebra.

Remark 1.1. If we put $A = k$ in the above lemma the $\pi-$graded algebra $Conv(C, k) = \bigoplus_{\alpha\in\pi} C_\alpha^*$ is called dual to C and denoted by C^*.

Definition 1.2. A Hopf $\pi-$ coalgebra is a $\pi-$coalgebra $H = (\{H_\alpha\}_{\alpha\in\pi}, \Delta, \varepsilon)$ endowed with a family

$$S = \{S_\alpha : H_\alpha \to H_{\alpha^{-1}}\}_{\alpha\in\pi}$$

of $k-$linear maps called the antipode such that

(1) Each H_α is an algebra with multiplication m_α and unit element $1_\alpha \in H_\alpha$,

(2) The linear maps

$$\begin{aligned} \Delta_{\alpha,\beta} &: H_{\alpha\beta} \to H_\alpha \otimes H_\beta, \\ \varepsilon &: H_1 \to k. \end{aligned}$$

are algebra maps for all $\alpha, \beta \in A$,

(3) For any $\alpha \in \pi$

$$m_\alpha(S_{\alpha^{-1}} \otimes id)\Delta_{\alpha^{-1},\alpha} = m_\alpha(id \otimes S_{\alpha^{-1}})\Delta_{\alpha,\alpha^{-1}}.$$

Remark 1.2. If $H = (\{H_\alpha\}_{\alpha\in\pi}, \Delta, \varepsilon, S)$ is a Hopf $\pi-$coalgebra then axiom (3) says that S_α is the inverse of $I_{H_{\alpha^{-1}}}$ in the convolution algebra $Conv(H, H_{\alpha^{-1}})$.

Remark 1.3. $(H_1, \Delta_{1,1}, \varepsilon, S_1)$ is a classical Hopf algebra

Lemma 1.2. *Let $H = (\{H_\alpha\}_{\alpha\in\pi}, \Delta, \varepsilon, S)$ be a Hopf $\pi-$coalgebra. then*

1. $\Delta_{\beta^{-1},\alpha^{-1}} S_{\alpha\beta} = \sigma_{H^{\alpha^{-1}},H^{\beta^{-1}}}(S_\alpha \otimes S_\beta)\Delta_{\alpha,\beta}$ *for any* $\alpha, \beta \in \pi$,

2. $\varepsilon(S_1) = \varepsilon$,

3. $S_\alpha(ab) = S_\alpha(b)S_\alpha(a)$ *for any* $\alpha \in \pi$ *and* $a, b \in A$,

4. $S_{1_\alpha} = 1_{\alpha^{-1}}$ *for any* $\alpha \in \pi$.

Definition 1.3. Let $C = (\{C_\alpha\}_{\alpha\in\pi}, \Delta, \varepsilon)$ be a $\pi-$coalgebra. A right $\pi-$comodule over C is a family $M = \{M_\alpha\}_{\alpha\in\pi}$ of $k-$linear spaces endowed with a family $\rho = \{\rho_{\alpha,\beta} : M_{\alpha\beta \longrightarrow M_\alpha \otimes C_\beta}\}$ of $k-$linear maps (the structure maps) such that

- For any $\alpha, \beta, \gamma \in \pi$

$$(\rho_{\alpha,\beta} \otimes id)\rho_{\alpha\beta,\gamma} = (id \otimes \Delta_{\beta,\gamma})\rho_{\alpha,\beta\gamma} \qquad *$$

- For any $\alpha \in \pi$

$$(id \otimes \varepsilon)\rho_{\alpha,1} = id \qquad **$$

Definition 1.4. A $\pi-$subcomodule of M is a family $N = \{N_\alpha\}_{\alpha\in\pi}$ where N_α is a $k-$linear subspace of M_α such that for all $\alpha, \beta \in \pi$

$$\rho_{\alpha,\beta}(N_{\alpha\beta}) \subset N_\alpha \otimes C_\beta$$

Definition 1.5. A $\pi-$comodule morphism between to right $\pi-$comodules M and M' over a $\pi-$coalgebra C (with structure maps ρ and ρ', respectively) is a family $f = \{f_\alpha : M_\alpha \longrightarrow M'_\alpha\}$ of $k-$linear maps such that for all $\alpha, \beta \in \pi$

$$\rho'_{\alpha,\beta}(f_{\alpha\beta}) = (f_\alpha \otimes id)\rho_{\alpha,\beta}$$

Sweedler's notation

For any $\alpha, \beta \in \pi$ and $m \in M_{\alpha,\beta}$ we write

$$\rho_{\alpha,\beta}(m) = m_{(0,\alpha)} \otimes m_{(1,\beta)} \in M_\alpha \otimes C_\beta$$

also the axiom

$$(\rho_{\alpha,\beta} \otimes id)\rho_{\alpha\beta,\gamma} = (id \otimes \Delta_{\beta,\gamma})\rho_{\alpha,\beta\gamma}$$

can be written as

$$m_{(0,\alpha\beta)(0,\alpha)} \otimes m_{(0,\alpha\beta)(1,\beta)} \otimes m_{(1,\gamma)} = m_{(0,\alpha)} \otimes m_{(1,\beta\gamma)(1,beta)} \otimes m_{(1,\beta\gamma)(2,\gamma)}$$

This elements of $M_\alpha \otimes C_\beta \otimes C_\gamma$ is written as $m_{(0,\alpha)} \otimes m_{(1,\beta)} \otimes m_{(2,\gamma)}$

2 Basic Definitions of differential calculus

Definition 2.1. Let $A = \{A_\alpha\}_{\alpha\in\pi}$ be a Hopf group coalgebra ,$\Gamma = \{\Gamma_\alpha\}_{\alpha\in\pi}$ be a $\pi-$ graded bimodule over A , and

$$d = \{d_\alpha : A_\alpha \longrightarrow \Gamma_\alpha\} \qquad 2.1$$

be a family of linear maps. We say that (Γ, d) is a $\pi-$graded first order differential calculus over A if for any $\alpha \in \pi$

1. For any $a, b \in A_\alpha$

$$d_\alpha(ab) = d_\alpha(a)b + ad_\alpha(b) \qquad 2.2$$

2. Any element $\rho \in \Gamma_\alpha$ is of the form

$$\rho = \sum_{k=1}^{n} a_k d_\alpha b_k \quad , \quad a_k, b_k \in A_\alpha$$

Definition 2.2. Two $\pi-$graded first order differential calculi are said to be isomorphic if there exists a family of bimodule isomorphisms $i = \left\{i_\alpha : \Gamma_\alpha \longrightarrow \Gamma'_\alpha\right\}$ such that

$$i_\alpha \left(d_\alpha a\right) = d'_\alpha a, \text{ for all } a \in A_\alpha, \alpha \in \pi.$$

Let $A = \{A_\alpha\}_{\alpha\in\pi}$ be a Hopf group coalgebra , $m_\alpha : A_\alpha \otimes A_\alpha \longrightarrow A_\alpha$ be the multiplication defined on A_α for each α. Define $A^2 = \{A^2_\alpha\}_{\alpha\in\pi}$ such that

$$A^2_\alpha = \{q \in A_\alpha \otimes A_\alpha, m_\alpha (q) = 0\} \qquad 2.3$$

By definition A^2_α is a linear subspace of $A_\alpha \otimes A_\alpha$ for each $\alpha \in \pi$ .On A^2 define an $A-$bimodule structure as

For any $\alpha \in \pi, c \in A_\alpha, \sum_k a_k \otimes b_k \in A^2_\alpha$

$$c\left(\sum_k a_k \otimes b_k\right) = \sum_k ca_k \otimes b_k \qquad 2.4$$

$$\left(\sum_k a_k \otimes b_k\right) c = \sum_k a_k \otimes b_k c \qquad 2.5$$

Define $D = \{D_\alpha\}_{\alpha\in\pi}$ by

$$D_\alpha (b) = 1_\alpha \otimes b - b \otimes 1_\alpha,$$

for all $b \in A_\alpha, \alpha \in \pi$

It is clear that $m_\alpha (D_\alpha (b)) = 0$,i.e.$D_\alpha (b) \in A^2_\alpha$. Moreover

$$\begin{aligned} D_\alpha (ab) &= 1_\alpha \otimes ab - ab \otimes 1_\alpha \\ &= 1_\alpha \otimes ab - a \otimes b + a \otimes b - ab \otimes 1_\alpha \\ &= (1_\alpha \otimes a - a \otimes 1_\alpha) b + a (1_\alpha \otimes b - b \otimes 1_\alpha) \\ &= D_\alpha (a) b + aD_\alpha (b) \end{aligned}$$

Proposition 2.1. *Let $N = \{N_\alpha\}_{\alpha\in\pi}$ be a $\pi-$graded sub-bimodule of $A^2, \Gamma = A^2/N$, $\pi = \{\pi_\alpha : A^2_\alpha \longrightarrow \Gamma_\alpha\}$ be the family of canonical epimorphisms , and $d = \{d_\alpha = \pi_\alpha \circ D_\alpha\}_{\alpha\in\pi}$. Then $\Gamma = \left(\{\Gamma_\alpha\}_{\alpha\in\pi}, d\right)$ is a first order differential calculus over A.Any other $\pi-$graded first order differential calculus over A can be obtained in this way.*

Proof. By definition of $\Gamma = \{\Gamma_\alpha\}_{\alpha\in\pi}, \Gamma$ is a $\pi-$graded bimodule over A.Moreover ,by definition of $d = \{d_\alpha = \pi_\alpha \circ D_\alpha\}_{\alpha\in\pi}$ we find that $\Gamma = \left(\{\Gamma_\alpha\}_{\alpha\in\pi}, d\right)$ is a $\pi-$graded first order differential calculus over A. It remains to show that any $\pi-$graded first order differential calculus over A can be obtained in this way.

Let $\Gamma = \left(\{\Gamma_\alpha\}_{\alpha\in\pi}, d\right)$ be any other $\pi-$graded first order differential calculus over A .We have for each $\alpha \in \pi$, $\sum_k a_k \otimes b_k \in A^2_\alpha, c \in A_\alpha$

$$\sum_k ca_k d_\alpha b_k = c\left(\sum_k a_k d_\alpha b_k\right)$$

and

$$\begin{aligned}\sum_k a_k d_\alpha (b_k c) &= \left(\sum_k a_k d_\alpha b_k\right) c + \left(\sum_k a_k b_k\right) d_\alpha c \\ &= \left(\sum_k a_k d_\alpha b_k\right) c\end{aligned}$$

i.e. the family $\pi = \{\pi_\alpha : A^2_\alpha \longrightarrow \Gamma_\alpha\}$defined by the formula

$$\pi_\alpha\left(\sum_k a_k \otimes b_k\right) = \sum_k a_k d_\alpha b_k \qquad 2.6$$

is a bimodule morphism.We will show that π_α is surjective for each $\alpha \in \pi$.

Let $\rho \in \Gamma_\alpha$ such that

$$\rho = \sum_k a_k d_\alpha b_k, \qquad a_k, b_k \in A_\alpha$$

Define an element $q \in A_\alpha \otimes A_\alpha$ by

$$q = \sum_k a_k \otimes b_k - a_k b_k \otimes 1_\alpha$$

It is clear that $m_\alpha q = 0$, i.e. $q \in A^2_\alpha$ .Moreover ,

$$\begin{aligned}\pi_\alpha(q) &= \sum_k a_k d_\alpha b_k - a_k b_k d_\alpha 1_\alpha \\ &= \sum_k a_k d_\alpha b_k \\ &= \rho\end{aligned}$$

therefore π_α is surjective for each $\alpha \in \pi$.

$$\begin{aligned}\ker\pi &= \{\ker\pi_\alpha\}_{\alpha\in\pi} \\ &= \left\{\sum_k a_k \otimes b_k \in A^2_\alpha, \sum_k a_k d_\alpha b_k = 0\right\}_{\alpha\in\pi}\end{aligned}$$

Taking

$$N = \{N_\alpha = \ker \pi_\alpha = \{\sum_k a_k \otimes b_k \in A_\alpha^2 \sum_k a_k d_\alpha b_k = 0\}\}_{\alpha \in \pi} \qquad 2.7$$

then Γ can be identified by A^2/N and for any $b \in A_\alpha$

$$\begin{aligned} \pi_\alpha D_\alpha (b) &= \pi_\alpha (1_\alpha \otimes b - b \otimes 1_\alpha) \\ &= d_\alpha b - b d_\alpha 1_\alpha \\ &= d_\alpha b. \end{aligned}$$

□

Definition 2.3. Let $\Gamma = \left(\{\Gamma_\alpha\}_{\alpha \in \pi}, d\right)$ be a $\pi-$graded first order differential calculus over A.We say that $\Gamma = \left(\{\Gamma_\alpha\}_{\alpha \in \pi}, d\right)$ is left covariant if for any $\alpha, \beta \in \pi$

$$\sum_k a_k d_{\alpha\beta} b_k \Longrightarrow \sum_k \Delta_{\alpha,\beta}(a_k)(id \otimes d_\beta)\Delta_{\alpha,\beta}(b_k) = 0 \qquad 2.8$$

for any $a_k, b_k \in A_{\alpha\beta}, k = 1, 2, ..., n$.

Proposition 2.2. *Let* $\Gamma = \left(\{\Gamma_\alpha\}_{\alpha \in \pi}, d\right)$ *be a left covariant* $\pi-$*graded first order differential calculus over* A*.Then there exists a family of linear mappings*

$$\Delta^l = \left\{\Delta^l_{\alpha,\beta} : \Gamma_{\alpha\beta} \longrightarrow A_\alpha \otimes \Gamma_\beta\right\} \qquad 2.9$$

such that

1. *For any* $a \in A_{\alpha\beta}, \rho \in \Gamma_{\alpha\beta}$

$$\Delta^l_{\alpha,\beta}(a\rho) = \Delta_{\alpha,\beta}(a)\,\Delta^l_{\alpha,\beta}(\rho) \qquad 2.10$$

$$\Delta^l_{\alpha,\beta}(\rho a) = \Delta^l_{\alpha,\beta}(\rho)\,\Delta_{\alpha,\beta}(a) \qquad 2.11$$

2. *For any* $\alpha, \beta, \gamma \in \pi$

$$(\Delta_{\alpha,\beta} \otimes id)\,\Delta^l_{\alpha\beta,\gamma} = \left(id \otimes \Delta^l_{\beta,\gamma}\right)\Delta^l_{\alpha,\beta\gamma} \qquad 2.12$$

3. *For any* $\rho \in \Gamma_\alpha$

$$(\varepsilon \otimes id)\,\Delta^l_{1,\alpha}(\rho) = \rho \qquad 2.13$$

4. For any $\alpha, \beta \in \pi$

$$\Delta^l_{\alpha,\beta} d_{\alpha\beta} = (id \otimes d_\beta)\, \Delta_{\alpha,\beta}(a)$$

Proof. Let $\Delta^l = \left\{\Delta^l_{\alpha,\beta}\right\}_{\alpha,\beta\in\pi}$ where $\Delta^l_{\alpha,\beta} : \Gamma_{\alpha\beta} \longrightarrow A_\alpha \otimes \Gamma_\beta$ is defined by

$$\Delta^l_{\alpha,\beta}\left(\sum_{k=1}^{n} a_k d_{\alpha\beta} b_k\right) = \sum_{k=1}^{n} \Delta_{\alpha,\beta}(a_k)\,(id \otimes d_\beta)\,\Delta_{\alpha,\beta}(b_k)$$

where $a_k, b_k \in A_{\alpha\beta}, \alpha, \beta \in \pi$. Then by definition for each $\alpha, \beta \in \pi$ $\Delta^l_{\alpha,\beta}$ is a well defined linear map.

1. Let $a \in A_{\alpha\beta}$, $\rho \in \Gamma_{\alpha\beta}$, $\rho = \sum_{k=1}^{n} a_k d_{\alpha\beta} b_k, a_k, b_k \in A_{\alpha\beta}$

$$\begin{aligned}
\Delta^l_{\alpha,\beta}(\rho a) &= \Delta^l_{\alpha,\beta}\left(\left(\sum_k a_k d_{\alpha\beta} b_k\right) a\right) \\
&= \Delta^l_{\alpha,\beta}\left(\sum_k a_k d_{\alpha\beta}(b_k a) - \sum_k a_k b_k d_{\alpha\beta} a\right) \\
&= \sum_k \left(\Delta_{\alpha,\beta}(a_k)\,(id \otimes d_\beta)\,\Delta_{\alpha,\beta}(b_k a) - \Delta_{\alpha,\beta}(a_k b_k)\,(id \otimes d_\beta)\,\Delta_{\alpha,\beta}(a)\right) \\
&= \sum_k \Delta_{\alpha,\beta}(a_k)\left((id \otimes d_\beta)\,\Delta_{\alpha,\beta}(b_k a) - \Delta_{\alpha,\beta}(b_k)\,(id \otimes d_\beta)\,\Delta_{\alpha,\beta}(a)\right) \\
&= \sum_k \Delta_{\alpha,\beta}(a_k)\left((id \otimes d_\beta)\left(b_{k(1,\alpha)} a_{(1,\alpha)} \otimes b_{k(2,\beta)} a_{(2,\beta)}\right) - b_{k(1,\alpha)} a_{(1,\alpha)} \otimes b_{k(2,\beta)} d_\beta a_{(2,\beta)}\right) \\
&= \sum_k \Delta_{\alpha,\beta}(a_k)\left(b_{k(1,\alpha)} a_{(1,\alpha)} \otimes d_\beta\left(b_{k(2,\beta)} a_{(2,\beta)}\right) - b_{k(1,\alpha)} a_{(1,\alpha)} \otimes b_{k(2,\beta)} d_\beta a_{(2,\beta)}\right) \\
&= \sum_k \Delta_{\alpha,\beta}(a_k)\left[\left(b_{k(1,\alpha)} a_{(1,\alpha)} \otimes d_\beta b_{k(2,\beta)} a_{(2,\beta)} + b_{k(1,\alpha)} a_{(1,\alpha)} \otimes b_{k(2,\beta)} d_\beta a_{(2,\beta)}\right) - b_{k(1,\alpha)} a_{(1,\alpha)}\right. \\
&\quad \left.\otimes b_{k(2,\beta)} d_\beta a_{(2,\beta)}\right] \\
&= \sum_k \Delta_{\alpha,\beta}(a_k)\left(b_{k(1,\alpha)} a_{(1,\alpha)} \otimes d_\beta b_{k(2,\beta)} a_{(2,\beta)}\right) \\
&= \left(\sum_k \Delta_{\alpha,\beta}(a_k)\,(id \otimes d_\beta)\,\Delta_{\alpha,\beta}(b_k)\right)\Delta_{\alpha,\beta}(a) \\
&= \Delta^l_{\alpha,\beta}\left(\sum_k a_k d_{\alpha\beta} b_k\right)\Delta_{\alpha,\beta}(a) \\
&= \Delta^l_{\alpha,\beta}(\rho)\,\Delta_{\alpha,\beta}(a)
\end{aligned}$$

and

$$\begin{aligned}
\Delta^l_{\alpha,\beta}(a\rho) &= \Delta^l_{\alpha,\beta}\left(a\sum_k a_k d_{\alpha\beta} b_k\right)\\
&= \Delta^l_{\alpha,\beta}\left(\sum_k a a_k d_{\alpha\beta} b_k\right)\\
&= \sum_k \Delta_{\alpha,\beta}(a a_k)(id\otimes d_\beta)\Delta_{\alpha,\beta}(b_k)\\
&= \Delta_{\alpha,\beta}(a)\sum_k \Delta_{\alpha,\beta}(a_k)(id\otimes d_\beta)\Delta_{\alpha,\beta}(b_k)\\
&= \Delta_{\alpha,\beta}(a)\,\Delta^l_{\alpha,\beta}\left(\sum_k a_k d_{\alpha\beta} b_k\right)\\
&= \Delta_{\alpha,\beta}(a)\,\Delta^l_{\alpha,\beta}(\rho)
\end{aligned}$$

2. Let $ad_{\alpha\beta\gamma}b \in \Gamma_{\alpha\beta\gamma}$,with $a, b \in A_{\alpha\beta\gamma}$, then we compute

$$\begin{aligned}
(\Delta_{\alpha,\beta}\otimes id)\,\Delta^l_{\alpha\beta,\gamma}(ad_{\alpha\beta\gamma}b) &= (\Delta_{\alpha,\beta}\otimes id)\left(\Delta_{\alpha\beta,\gamma}(a)(id\otimes d_\gamma)\Delta_{\alpha\beta,\gamma}(b)\right)\\
&= (\Delta_{\alpha,\beta}\otimes id)\,\Delta_{\alpha\beta,\gamma}(a)\,(\Delta_{\alpha,\beta}\otimes id)(id\otimes d_\gamma)\,\Delta_{\alpha\beta,\gamma}(b)\\
&= (\Delta_{\alpha,\beta}\otimes id)\,\Delta_{\alpha\beta,\gamma}(a)\,(id\otimes id\otimes d_\gamma)(\Delta_{\alpha,\beta}\otimes id)\,\Delta_{\alpha\beta,\gamma}(b)
\end{aligned}$$

On the other hand

$$\begin{aligned}
\left(id\otimes\Delta^l_{\beta,\gamma}\right)\Delta^l_{\alpha,\beta\gamma}(ad_{\alpha\beta\gamma}b) &= \left(id\otimes\Delta^l_{\beta,\gamma}\right)\left(\Delta_{\alpha,\beta\gamma}(a)(id\otimes d_{\beta\gamma})\Delta_{\alpha,\beta\gamma}(b)\right)\\
&= \left(id\otimes\Delta^l_{\beta,\gamma}\right)\left(a_{(1,\alpha)}b_{(1,\alpha)}\otimes a_{(2,\beta\gamma)}d_{\beta\gamma}b_{(2,\beta\gamma)}\right)\\
&= a_{(1,\alpha)}b_{(1,\alpha)}\otimes a_{(2,\beta)}b_{(2,\beta)}\otimes a_{(3,\gamma)}d_\gamma b_{(3,\gamma)}\\
&= (id\otimes\Delta_{\beta,\gamma})\,\Delta_{\alpha,\beta\gamma}(a)\,(id\otimes id\otimes d_\gamma)(id\otimes\Delta_{\beta,\gamma})\,\Delta_{\alpha,\beta\gamma}(b)
\end{aligned}$$

3. For $\alpha\in\pi$ let $ad_\alpha b\in\Gamma_\alpha$, $a, b\in A_\alpha$

$$\begin{aligned}
(\varepsilon\otimes id)\,\Delta^l_{1,\alpha}(ad_\alpha b) &= (\varepsilon\otimes id)\left(\Delta_{1,\alpha}(a)(id\otimes d_\alpha)\Delta_{1,\alpha}(b)\right)\\
&= \varepsilon\left(a_{(1,1)}b_{(1,1)}\right)a_{(2,\alpha)}d_\alpha b_{(2,\alpha)}\\
&= \varepsilon\left(a_{(1,1)}\right)a_{(2,\alpha)}\varepsilon\left(b_{(1,1)}\right)d_\alpha b_{(2,\alpha)}\\
&= ad_\alpha b.
\end{aligned}$$

4. Let $a \in A_{\alpha\beta}$

$$\begin{aligned}
\Delta^{l}_{\alpha,\beta} d_{\alpha\beta}(a) &= \Delta^{l}_{\alpha,\beta}\left(d_{\alpha\beta}(a)\right) \\
&= \Delta_{\alpha,\beta}(1_{\alpha\beta})(id \otimes d_{\beta})\Delta_{\alpha,\beta}(a) \\
&= (1_{\alpha} \otimes 1_{\beta})(id \otimes d_{\beta})\Delta_{\alpha,\beta}(a) \\
&= (id \otimes d_{\beta})\Delta_{\alpha,\beta}(a)
\end{aligned}$$

□

Definition 2.4. Let $\Gamma = \left(\{\Gamma_{\alpha}\}_{\alpha\in\pi}, d\right)$ be a $\pi-$graded first order differential calculus over A.We say that $\Gamma = \left(\{\Gamma_{\alpha}\}_{\alpha\in\pi}, d\right)$ is right covariant if for any $\alpha, \beta \in \pi$

$$\sum_{k=1}^{n} a_k d_{\alpha\beta} b_k \Longrightarrow \sum_{k=1}^{n} \Delta_{\alpha,\beta}(a_k)(d_{\alpha} \otimes id)\Delta_{\alpha,\beta}(b_k) = 0 \qquad 2.14$$

We say that $\Gamma = \left(\{\Gamma_{\alpha}\}_{\alpha\in\pi}, d\right)$ is bicovariant if it is left and right covariant.

Proposition 2.3. *Let $\Gamma = \left(\{\Gamma_{\alpha}\}_{\alpha\in\pi}, d\right)$ be a right covariant $\pi-$graded first order differential calculus over A. Then there exists a family of linear mappings*

$$\Delta^{r} = \left\{\Delta^{r}_{\alpha,\beta} : \Gamma_{\alpha\beta} \longrightarrow \Gamma_{\alpha} \otimes A_{\beta}\right\} \qquad 2.15$$

such that

1. *For any $a \in A_{\alpha\beta}, \rho \in \Gamma_{\alpha\beta}$*

$$\begin{aligned}
\Delta^{r}_{\alpha,\beta}(a\rho) &= \Delta_{\alpha,\beta}(a)\Delta^{r}_{\alpha,\beta}(\rho) \\
\Delta^{r}_{\alpha,\beta}(\rho a) &= \Delta^{r}_{\alpha,\beta}(\rho)\Delta_{\alpha,\beta}(a)
\end{aligned} \qquad 2.16$$

2. *for any $\alpha, \beta, \gamma \in \pi$*

$$(id \otimes \Delta_{\beta,\gamma})\Delta^{r}_{\alpha,\beta\gamma} = (\Delta^{r}_{\alpha,\beta} \otimes id)\Delta^{r}_{\alpha\beta,\gamma} \qquad 2.17$$

3. *For any $\rho \in \Gamma_{\alpha}$*

$$(id \otimes \varepsilon)\Delta^{r}_{\alpha,1}(\rho) = \rho \qquad 2.18$$

4. *for any $\alpha, \beta, \gamma \in \pi$*

$$\Delta^{r}_{\alpha,\beta} d_{\alpha,\beta} = (d_{\alpha} \otimes id)\Delta_{\alpha,\beta}$$

Proof. Similar to that of proposition 2.2 , where for any $\alpha, \beta, \gamma \in \pi$ $a_k, b_k \in A_{\alpha\beta}$.

$$\Delta^{r}_{\alpha,\beta}\left(\sum_{k} a_k d_{\alpha\beta} b_k\right) = \sum_{k} \Delta_{\alpha,\beta}(a_k)(d_{\alpha} \otimes id)\Delta_{\alpha,\beta}(b_k) \qquad 2.19$$

□

Proposition 2.4. *Let* $\Gamma = \left(\{\Gamma_\alpha\}_{\alpha\in\pi}, d\right)$ *be a bicovariant* $\pi-$*graded first order differential calculus over*A, Δ^l, Δ^r *be the families of linear mappings introduced by proposition 2.2,2.3.Then we have.*

$$\left(id \otimes \Delta^r_{\beta,\gamma}\right) \Delta^l_{\alpha,\beta\gamma} \left(ad_{\alpha\beta\gamma} b\right) = \left(\Delta^l_{\alpha,\beta} \otimes id\right) \Delta^r_{\alpha\beta,\gamma} \left(ad_{\alpha\beta\gamma} b\right) \qquad 2.20$$

Proof. Let $a, b \in A_{\alpha\beta\gamma}$

$$\begin{aligned}\left(id \otimes \Delta^r_{\beta,\gamma}\right) \Delta^l_{\alpha,\beta\gamma} \left(ad_{\alpha\beta\gamma} b\right) &= \left(id \otimes \Delta^r_{\beta,\gamma}\right) \left(\Delta_{\alpha,\beta\gamma}(a) \left(id \otimes d_{\beta\gamma}\right) \Delta_{\alpha,\beta\gamma}(b)\right) \\ &= \left(id \otimes \Delta_{\beta,\gamma}\right) \Delta_{\alpha,\beta\gamma}(a) \left(id \otimes d_\beta \otimes id\right) \left(id \otimes \Delta_{\beta,\gamma}\right) \Delta_{\alpha,\beta\gamma}(b)\end{aligned}$$

On the other hand

$$\begin{aligned}\left(\Delta^l_{\alpha,\beta} \otimes id\right) \Delta^r_{\alpha\beta,\gamma} \left(ad_{\alpha\beta\gamma} b\right) &= \left(\Delta^l_{\alpha,\beta} \otimes id\right) \left(\Delta_{\alpha\beta,\gamma}(a) \left(id \otimes d_\gamma\right) \Delta_{\alpha\beta,\gamma}(b)\right) \\ &= \left(\Delta_{\alpha,\beta} \otimes id\right) \Delta_{\alpha\beta,\gamma}(a) \left(id \otimes d_\beta \otimes id\right) \left(\Delta_{\alpha,\beta} \otimes id\right) \Delta_{\alpha,\beta\gamma}(b)\end{aligned}$$

Using the coassociativity property we find that equation 2.20 holds. □

3 $\pi-$graded Bicovariant bimodules

Throughout this section let $A = \{A_\alpha\}_{\alpha\in\pi}$ be a hopf group coalgebra

Definition 3.1. let $\Gamma = \{\Gamma_\alpha\}_{\alpha\in\pi}$ be a $\pi-$graded bimodule over A , $\Delta^l = \left\{\Delta^l_{\alpha,\beta} : \Gamma_{\alpha\beta} \longrightarrow A_\alpha \otimes \Gamma_\beta\right\}_{\alpha,\beta\in\pi}$ be a family of linear maps .We say that $\Gamma = \left(\{\Gamma_\alpha\}_{\alpha\in\pi}, \Delta^l\right)$ is a left covariant π-graded bimodule over A if

1. For any $a \in A_{\alpha\beta}$, $\rho \in \Gamma_{\alpha\beta}$ $\alpha, \beta \in \pi$

$$\Delta^l_{\alpha,\beta}(a\rho) = \Delta_{\alpha,\beta}(a)\, \Delta^l_{\alpha,\beta}(\rho) \qquad (3.1)$$

$$\Delta^l_{\alpha,\beta}(\rho a) = \Delta^l_{\alpha,\beta}(\rho)\, \Delta_{\alpha,\beta}(a) \qquad (3.2)$$

2. For all $\alpha, \beta, \gamma \in \pi$.

$$(\Delta_{\alpha,\beta} \otimes id)\Delta^l_{\alpha\beta,\gamma} = (id \otimes \Delta^l_{\beta,\gamma})\Delta^l_{\alpha,\beta\gamma} \qquad (3.3)$$

3. For any $\rho \in \Gamma_\alpha$,$\alpha \in \pi$

$$(\varepsilon \otimes id)\, \Delta^l_{1,\alpha}(\rho) = \rho \qquad (3.4)$$

Definition 3.2. Let $\Gamma = \{\Gamma_\alpha\}_{\alpha\in\pi}$be a $\pi-$graded bimodule over A , $\Delta^r = \left\{\Delta^r_{\alpha,\beta} : \Gamma_{\alpha\beta} \longrightarrow \Gamma_\alpha \otimes A_\beta\right\}$ be a family of linear maps .We say that $\Gamma = \left(\{\Gamma_\alpha\}_{\alpha\in\pi}, \Delta^r\right)$is a right covariant $\pi-$graded bimodule over A if

1. For any $a \in A_{\alpha\beta}, \rho \in \Gamma_{\alpha\beta}$

$$\Delta^r_{\alpha,\beta}(a\rho) = \Delta_{\alpha,\beta}(a)\,\Delta^r_{\alpha,\beta}(\rho) \tag{3.5}$$

$$\Delta^r_{\alpha,\beta}(\rho a) = \Delta^r_{\alpha,\beta}(\rho)\,\Delta_{\alpha,\beta}(a) \tag{3.6}$$

2. For $\alpha, \beta, \gamma \in \pi$.

$$(\Delta_{\alpha,\beta} \otimes id)\Delta^r_{\alpha\beta,\gamma} = (id \otimes \Delta^r_{\beta,\gamma})\Delta^r_{\alpha,\beta\gamma} \tag{3.7}$$

3. For any $\rho \in \Gamma_\alpha$,$\alpha \in \pi$

$$(id \otimes \varepsilon)\,\Delta^r_{\alpha,1}(\rho) = \rho \tag{3.8}$$

Definition 3.3. let $\Gamma = \{\Gamma_\alpha\}_{\alpha\in\pi}$ be a $\pi-$graded bimodule over A , $\Delta^l = \left\{\Delta^l_{\alpha,\beta} : \Gamma_{\alpha\beta} \longrightarrow A_\alpha \otimes \Gamma_\beta\right\}_{\alpha,\beta\in\pi}$, and $\Delta^r = \left\{\Delta^r_{\alpha,\beta} : \Gamma_{\alpha\beta} \longrightarrow \Gamma_\alpha \otimes A_\beta\right\}$be two families of linear maps .We say that $\Gamma = \left(\{\Gamma_\alpha\}_{\alpha\in\pi}, \Delta^l, \Delta^r\right)$ is a bicovariant π-graded bimodule over A if

1. $\Gamma = \left(\{\Gamma_\alpha\}_{\alpha\in\pi}, \Delta^l\right)$ is a left covariant π-graded bimodule over A.

2. $\Gamma = \left(\{\Gamma_\alpha\}_{\alpha\in\pi}, \Delta^r\right)$ is a right covariant π-graded bimodule over A.

3. For all $\alpha, \beta, \gamma \in \pi$.

$$(\Delta^l_{\alpha,\beta} \otimes id)\Delta^r_{\alpha\beta,\gamma} = (id \otimes \Delta^r_{\beta,\gamma})\Delta^l_{\alpha,\beta\gamma} \tag{3.9}$$

Definition 3.4. Let $\Gamma = \left(\{\Gamma_\alpha\}_{\alpha\in\pi}, \Delta^l\right)$ be a left covariant π-graded bimodule over A .For any $\alpha \in \pi$ an element $\rho \in \Gamma_\alpha$ is said to be left invariant if

$$\Delta^l_{1,\alpha}(\rho) = 1_1 \otimes \rho \tag{3.10}$$

Denote by ${}_{inv}\Gamma = \{{}_{inv}\Gamma_\alpha\}_{\alpha\in\pi}$ the set of all left invariant elements of Γ. Clearly , ${}_{inv}\Gamma_\alpha$ is a linear subspace of Γ_α for each $\alpha \in \pi$.

Lemma 3.1. *Let* $\Gamma = \left(\{\Gamma_\alpha\}_{\alpha\in\pi}, \Delta^l\right)$ *be a left covariant* π*-graded bimodule over* A*,*${}_{inv}\Gamma = \{{}_{inv}\Gamma_\alpha\}_{\alpha\in\pi}$ *be the linear subspace of all left invariant elements of* Γ*. Then there exists a family*

$$P = \{P_\alpha : \Gamma_1 \longrightarrow \Gamma_\alpha\}_{\alpha\in\pi} \tag{3.11}$$

of mappings such that

$$P_\alpha (b\rho) = \varepsilon (b) P_\alpha (\rho) \tag{3.12}$$

for any $b \in A_1, \rho \in \Gamma_1, \alpha \in \pi$*.*

Moreover , for any $\rho \in \Gamma_\alpha$ *,* $\alpha \in \pi$ *we have*

$$\rho = \sum_k a_k P_\alpha (\rho_k) \tag{3.13}$$

where a_k *,* ρ_k *are elements of* A_α *,* Γ_1 *respectively such that*

$$\Delta^l_{\alpha,1} (\rho) = \sum_k a_k \otimes \rho_k \tag{3.14}$$

And equation 3.13 can be uniquely written in this form.

Proof. For any $\alpha \in \pi$, $\rho \in \Gamma_1$ set

$$P_\alpha (\rho) = \sum_k S_{\alpha^{-1}} (a_k) \rho_k \tag{3.15}$$

where

$$\Delta^l_{\alpha^{-1},\alpha} (\rho) = \sum_{k=1}^{n} a_k \otimes \rho_k$$

Recall that for any $\alpha, \beta \in \pi$, $a \in A_{\beta^{-1}}$ where $\Delta_{\beta^{-1}\alpha^{-1},\alpha} (a) = a_{(1,\beta^{-1}\alpha^{-1})} \otimes a_{(2,\alpha)}$we have

$$\begin{aligned}
\Delta_{\alpha,\beta} \left(S_{\beta^{-1}\alpha^{-1}} \left(a_{(1,\beta^{-1}\alpha^{-1})}\right)\right) \left(a_{(2,\alpha)} \otimes 1_\beta\right) &= \sigma_{A_\beta,A_\alpha} \left(S_{\beta^{-1}} \left(a_{(1,\beta^{-1})}\right) \otimes S_{\alpha^{-1}} \left(a_{(2,\alpha^{-1})}\right)\right) \left(a_{(3,\alpha)} \otimes 1_\beta\right) \\
&= S_{\alpha^{-1}} \left(a_{(2,\alpha^{-1})}\right) a_{(3,\alpha)} \otimes S_{\beta^{-1}} \left(a_{(1,\beta^{-1})}\right) \\
&= \varepsilon \left(a_{(2,1)}\right) 1_\alpha \otimes S_{\beta^{-1}} \left(a_{(1,\beta^{-1})}\right) \\
&= 1_\alpha \otimes \varepsilon \left(a_{(2,1)}\right) S_{\beta^{-1}} \left(a_{(1,\beta^{-1})}\right) \\
&= 1_\alpha \otimes S_{\beta^{-1}} \left(a_{(1,\beta^{-1})} \varepsilon \left(a_{(2,1)}\right)\right) \\
&= 1_\alpha \otimes S_{\beta^{-1}} (a)
\end{aligned}$$

then we have

$$\Delta_{\alpha,\beta}\left(S_{\beta^{-1}\alpha^{-1}}\left(a_{(1,\beta^{-1}\alpha^{-1})}\right)\right)\left(a_{(2,\alpha)}\otimes 1_{\beta}\right)=1_{\alpha}\otimes S_{\beta^{-1}}(a) \tag{3.16}$$

For any $\rho\in\Gamma_1$, $\alpha\in\pi$ set

$$\begin{aligned}\Delta^{l}_{\alpha^{-1},\alpha}(\rho) &= \sum_k a_k\otimes\rho_k\\ \Delta^{l}_{1,\alpha}(\rho_k) &= \sum_l b_{kl}\otimes\rho_{kl}\\ \Delta_{\alpha^{-1},1}(a_k) &= \sum_m c_{km}\otimes d_{km}\end{aligned}$$

Using equation 3.3 we have

$$\sum_{k,l} a_k\otimes b_{kl}\otimes\rho_{kl}=\sum_{k,m} c_{km}\otimes d_{km}\otimes\rho_k \tag{3.17}$$

We compute

$$\begin{aligned}\Delta^{l}_{1,\alpha}(P_{\alpha}(\rho)) &= \Delta^{l}_{1,\alpha}\left(\sum_k S_{\alpha^{-1}}(a_k)\,\rho_k\right)\\ &= \sum_k \Delta_{1,\alpha}(S_{\alpha^{-1}}(a_k))\,\Delta^{l}_{1,\alpha}(\rho_k)\\ &= \sum_{k,l}\Delta_{1,\alpha}(S_{\alpha^{-1}}(a_k))\,(b_{kl}\otimes\rho_{kl})\\ &= \sum_{k,l}\Delta_{1,\alpha}(S_{\alpha^{-1}}(c_{km}))\,(d_{km}\otimes\rho_k)\\ &= \sum_{k,l}\Delta_{1,\alpha}(\otimes S_{\alpha^{-1}}(c_{km}))\,(d_{km}\otimes 1_{\alpha})\,(1_1\otimes\rho_k)\\ &= \sum_k (1_1\otimes S_{\alpha^{-1}}(a_k))\,(1_1\otimes\rho_k)\end{aligned}$$

(using 3.16 for $\alpha=1,\beta=\alpha$)

$$\begin{aligned}&= 1_1\otimes\sum_k S_{\alpha^{-1}}(a_k)\,\rho_k\\ &= 1_1\otimes P_{\alpha}(\rho)\end{aligned}$$

This shows that $P_{\alpha}(\rho)$ is left invariant element in Γ_{α} for each $\alpha\in\pi$.

To prove 3.12 , let $b \in A_1$, $\rho \in \Gamma_1$, set

$$\begin{aligned}
\Delta_{\alpha^{-1},\alpha}(b) &= \sum_k b_k \otimes d_k \\
\Delta^l_{\alpha^{-1},\alpha}(\rho) &= \sum_l c_l \otimes \rho_l \\
\Delta^l_{\alpha^{-1},\alpha}(b\rho) &= \Delta_{\alpha^{-1},\alpha}(b)\,\Delta^l_{\alpha^{-1},\alpha}(\rho) \\
&= \sum_{k,l} b_k c_l \otimes d_k \rho_l
\end{aligned}$$

Then

$$\begin{aligned}
P_\alpha(b\rho) &= \sum_{k,l} S_{\alpha^{-1}}(b_k c_l)\, d_k \rho_l \\
&= \sum_{k,l} S_{\alpha^{-1}}(c_l)\, S_{\alpha^{-1}}(b_k)\, d_k \rho_l \\
&= \sum_l S_{\alpha^{-1}}(c_l)\, \varepsilon(b)\, \rho_l \\
&= \varepsilon(b) \sum_l S_{\alpha^{-1}}(c_l)\, \rho_l \\
&= \varepsilon(b)\, P_\alpha(\rho)
\end{aligned}$$

To prove 3.13. Let $\alpha \in \pi, \rho \in \Gamma_\alpha$.Set

$$\begin{aligned}
\Delta^l_{1,\alpha}(\rho) &= \sum_m d_m \otimes \varrho_m \\
\Delta^l_{\alpha^{-1},\alpha}(\rho_k) &= \sum_n b_{kn} \otimes \rho_{kn} \\
\Delta_{\alpha,\alpha^{-1}}(d_m) &= \sum_l d_{ml} \otimes c_{ml}
\end{aligned}$$

where

$$\Delta^l_{\alpha,1}(\rho) = \sum_k a_k \otimes \rho_k \tag{3.18}$$

using equation 3.3 we have

$$\left(\Delta_{\alpha,\alpha^{-1}} \otimes id\right) \Delta^l_{1,\alpha} = \left(id \otimes \Delta^l_{\alpha^{-1},\alpha}\right) \Delta^l_{\alpha,1}$$

i.e.

$$\sum_{m,l} d_{ml} \otimes c_{ml} \otimes \varrho_m = \sum_{k,n} a_k \otimes b_{kn} \otimes \rho_{kn} \tag{3.19}$$

Then using equation 3.4 we have

$$\begin{aligned}\rho &= (\varepsilon \otimes id)\,\Delta^{l}_{1,\alpha}(\rho)\\ &= \sum_{m} \varepsilon(d_m)\,\varrho_m\\ &= \sum_{m,l} d_{ml} S_{\alpha^{-1}}(c_{ml})\,\varrho_m\\ &= \sum_{k,n} a_k S_{\alpha^{-1}}(b_{kn})\,\rho_{kn}\\ &= \sum_{m} a_k P_{\alpha}(\rho_k)\end{aligned}$$

Finally ,to prove the uniqueness of expression 3.13 let $P^{'} = \left\{P^{'}_{\alpha} : \Gamma_1 \longrightarrow_{inv} \Gamma_\alpha\right\}$ be another family of mappings satisfying that for $\rho \in \Gamma_\alpha$

$$\rho = \sum_{k} a_k P'_{\alpha}(\rho_k) \tag{3.20}$$

where a_k , ρ_k are elements of A_α , Γ_1 respectively such that

$$\Delta^{l}_{\alpha,1}(\rho) = \sum_{k} a_k \otimes \rho_k$$

Let $\rho \in \Gamma_\alpha$ such that $\Delta^{l}_{\alpha,1}(\rho) = \sum_k a_k \otimes \rho_k$ .Then using 3.13

$$\rho = \sum_{k} a_k P_{\alpha}(\rho_k)$$

But using 3.20 we have

$$\rho = \sum_{k} a_k P^{'}_{\alpha}(\rho_k)$$

Subtracting the above two equations we obtain

$$0 = \sum_{k} a_k \left(P_{\alpha}(\rho_k) - P^{'}_{\alpha}(\rho_k)\right)$$

Assuming that all $a_k^{,}s$ all linearly independent we get

$$P_{\alpha}(\rho_k) = P^{'}_{\alpha}(\rho_k) \qquad k = 1, 2, ..., n.$$

which proves the uniqueness of expression 3.13.

Lemma 3.2. *Let $\Gamma = \left(\{\Gamma_\alpha\}_{\alpha\in\pi}, \Delta^{l}\right)$ be a left covariant π-graded bimodule over A . Then , for any $\alpha, \beta \in \pi, \rho \in_{inv} \Gamma_{\alpha\beta}$ we have*

$$\Delta^l_{\alpha,\beta}(\rho) = 1_\alpha \otimes \varrho \tag{3.21}$$

where $\varrho \in_{inv} \Gamma_\beta$.

Proof. Let $\alpha, \beta \in \pi, \rho \in_{inv} \Gamma_{\alpha\beta}$,then using lemma 3.1 , and since the mappings P_α are onto for each $\alpha \in \pi$ then there exists an element $\xi \in \Gamma_1$ such that

$$\rho = P_{\alpha\beta}(\xi) \tag{3.22}$$

Set

$$\begin{aligned}
\Delta^l_{\beta^{-1}\alpha^{-1},\alpha\beta}(\xi) &= \sum_k a_k \otimes \xi_k \\
\Delta^l_{\alpha,\beta}(\xi_k) &= \sum_l c_{kl} \otimes \xi_{kl} \\
\Delta^l_{\beta^{-1},\beta}(\xi) &= \sum_m b_m \otimes \rho_m
\end{aligned}$$

and

$$\Delta^l_{\beta^{-1}\alpha^{-1},\alpha}(b_m) = \sum_n b_{mn} \otimes d_{mn} \tag{3.23}$$

Using equation 3.3

$$\sum_{k,l} a_k \otimes c_{kl} \otimes \xi_{kl} = \sum_{m,n} b_{mn} \otimes d_{mn} \otimes \rho_m \tag{3.24}$$

Applying $\Delta^l_{\alpha,\beta}$ to both sides of 3.22, using 3.24 and 3.16 , we get

$$\begin{aligned}
\Delta^l_{\alpha,\beta}(\rho) &= \Delta^l_{\alpha,\beta}(P_{\alpha\beta}(\xi)) \\
&= \sum_k \Delta^l_{\alpha,\beta}(S_{\beta^{-1}\alpha^{-1}}(a_k)\,\xi_k) \\
&= \sum_k \Delta_{\alpha,\beta}(S_{\beta^{-1}\alpha^{-1}}(a_k))\,\Delta^l_{\alpha,\beta}(\xi_k) \\
&= \sum_{k,l} \Delta_{\alpha,\beta}(S_{\beta^{-1}\alpha^{-1}}(a_k))\,(c_{kl} \otimes \xi_{kl}) \\
&= \sum_{m,n} \Delta_{\alpha,\beta}(S_{\beta^{-1}\alpha^{-1}}(b_{mn}))\,(d_{mn} \otimes \rho_m) \\
&= \sum_{m,n} \Delta_{\alpha,\beta}(S_{\beta^{-1}\alpha^{-1}}(b_{mn}))\,(d_{mn} \otimes 1_\beta)\,(1_\alpha \otimes \rho_m) \\
&= \sum_m (1_\alpha \otimes S_{\beta^{-1}}(b_m))\,(1_\alpha \otimes \rho_m) \\
&= 1_\alpha \otimes \sum_m S_{\beta^{-1}}(b_m)\,\rho_m \\
&= 1_\alpha \otimes P_\beta(\xi)
\end{aligned}$$

But from lemma 3.1 $P_\beta(\xi) \in_{inv} \Gamma_\beta$ and hence the lemma is proved. $\square$

Let $A = (\{A_\alpha\}_{\alpha\in\pi}, \Delta, \varepsilon, S)$ be a hopf $\pi-$coalgebra. Throughtout the next dealing we will consider that A is endowed with a family of linear maps $\Psi = \{\Psi_\alpha : A_\alpha \longrightarrow A_1\}$ of k-linear maps such that for each $\alpha \in \pi$,Ψ_α is an algebra map.For each $\alpha \in \pi$,define the map E_α to be the composition

$$A_\alpha \longrightarrow A_1 \longrightarrow k$$

i.e.

$$E_\alpha = \varepsilon\Psi_\alpha \tag{3.25}$$

Clearly, for each $\alpha \in \pi$ E_α is an algebra map for let $a, b \in A_\alpha$ .Then

$$\begin{aligned}
E_\alpha(ab) &= \varepsilon(\Psi_\alpha(ab)) \\
&= \varepsilon(\Psi_\alpha(a)\ \ \Psi_\alpha(b)) \\
&= \varepsilon(\Psi_\alpha(a)\ )\,\varepsilon(\ \Psi_\alpha(b)) \\
&= E_\alpha(a)\,E_\alpha(b) \\
E_\alpha(1_\alpha) &= \varepsilon(\Psi_\alpha(1_\alpha)) \\
&= \varepsilon(1_1) \\
&= 1_k
\end{aligned}$$

Moreover, E_α is linear being the composition of two linear maps.

Theorem 3.3. *Let $\Gamma = \left(\{\Gamma_\alpha\}_{\alpha\in\pi}, \Delta^l\right)$ be a $\pi-$graded left covariant bimodule over A,$\{\omega_i^\alpha\}_{\alpha\in\pi}$be a basis of ${}_{inv}\Gamma_\alpha$, of all left invariant elements of Γ_α for each $\alpha \in \pi$ .Then*

1. *For any $\alpha \in \pi$,any element $\rho \in \Gamma_\alpha$ is of the form*

$$\rho = \sum_i a_i\omega_i \tag{3.26}$$

where a_i 's $\in A_\alpha$ are uniquely determined , ω_i's $\in_{inv} \Gamma_\alpha$,for any $\alpha \in \pi$.

2. *For any $\alpha \in \pi$,any element $\rho \in \Gamma_\alpha$ is of the form*

$$\rho = \sum_i \omega_i b_i \tag{3.27}$$

where b_i 's $\in A_\alpha$ are uniquely determined , ω_i's $\in_{inv} \Gamma_\alpha$,for any $\alpha \in \pi$.

3. There exists linear functionals $f_{ij} \in A' = \oplus_{\alpha\in\pi} A'_\alpha$ *such that for any* $\alpha \in \pi$

$$\omega_i b = \sum_j (f_{ij} * b)\,\omega_j \tag{3.28}$$

$$a\omega_i = \sum_j \omega_j \left(\left(f_{ij} \circ S_1^{-1}\right) * a\right) \tag{3.29}$$

where $a, b \in A_\alpha, \omega_i^{,}s, \omega_j^{,}s \in_{inv} \Gamma_\alpha$ *.These functionals are uniquely determined by* 3.28. *They satisfy the following relations*

$$f_{ij}(ab) = \sum_k f_{ik}(a) f_{kj}(b) \tag{3.30}$$

for any $i, j \in I$, $a, b \in A_\alpha$ *.Moreover*

$$f_{ij}(1_\alpha) = \delta_{ij} \tag{3.31}$$

Remark 3.1. Any functional $f_{ij} \in A' = \oplus_{\alpha\in\pi} A'_\alpha$ is of the form $f_{ij} = \sum_\alpha f_{ij}^\alpha$ where

$$f_{ij}^\alpha(a) = 0 \quad if\ a \notin A_\alpha$$

Proof. To prove 1: For any $\alpha \in \pi$ let $\rho \in \Gamma_\alpha$. Using 3.13 we have that $\rho = \sum_i a_i\omega_i$, with $\omega_i^{,}s \in_{inv} \Gamma_\alpha$. To prove uniqueness assume that $\rho = \sum_i a_i\omega_i$. Then , using 3.1, 3.21

$$\begin{aligned}
\Delta^l_{\alpha,1}(\rho) &= \Delta^l_{\alpha,1}\left(\sum_i a_i\omega_i\right) \\
&= \sum_i \Delta_{\alpha,1}(a_i)\,\Delta^l_{\alpha,1}(\omega_i) \\
&= \sum_i \left(a_{i(1,\alpha)} \otimes a_{i(2,1)}\right)(1_\alpha \otimes \xi_i), \xi_i \in_{inv} \Gamma_1 \\
&= \sum_i a_{i(1,\alpha)} \otimes a_{i(2,1)}\xi_i
\end{aligned}$$

Applying $(id \otimes P_1)$ to both sides of the above equation ,we get

$$\begin{aligned}
(id \otimes P_1)\,\Delta^l_{\alpha,1}(\rho) &= (id \otimes P_1)\sum_i a_{i(1,\alpha)} \otimes a_{i(2,1)}\xi_i \\
&= \sum_i a_{i(1,\alpha)} \otimes P_1\left(a_{i(2,1)}\xi_i\right) \\
&= \sum_i a_{i(1,\alpha)} \otimes \varepsilon\left(a_{i(2,1)}\right) P_1(\xi_i) \\
&= \sum_i a_{i(1,\alpha)}\varepsilon\left(a_{i(2,1)}\right) \otimes P_1(\xi_i) \\
&= \sum_i a_i \otimes P_1(\xi_i) \\
&= \sum_i a_i \otimes \xi_i
\end{aligned}$$

since $P_1(\xi_i) = \xi_i$ for any $\xi_i \in_{inv} \Gamma_1$ .Since $\omega_i^{,}s$,$i \in I$ are linearly independent , then by linearty of $\Delta^l_{\alpha,1}$, $\xi_i^{,}s$ are also linearly independent and so the coefficients $a_i^{,}s$ are uniquely determined, and this proves the uniqueness of the decomposition 3.26. To prove 3: For any $\alpha \in \pi$, let $b \in A_\alpha, \omega_j \in_{inv} \Gamma_\alpha, j \in I$. Then $\omega_j b$ admits a decomposition in the of the form 3.26 .Let $F^\alpha_{ji}(b)$ be the coefficients preceding ω_i in the decomposition 3.26 i.e.

$$\omega_j b = \sum_i F^\alpha_{ji}(b)\,\omega_i \tag{3.32}$$

Clearly, $F^\alpha_{ji}(b)$ are linear mappings acting on A_α. For any $a, b \in A_\alpha$,and any $j \in I$ we have

$$\begin{aligned}\sum_i F^\alpha_{ji}(ab)\,\omega_i &= \omega_j ab\\ &= \sum_h F^\alpha_{jh}(a)\,\omega_h b\\ &= \sum_{h,i} F^\alpha_{jh}(a)\,F^\alpha_{hi}(b)\,\omega_i\end{aligned}$$

using the uniqueness of the decomposition 3.26 we have

$$F^\alpha_{ji}(ab) = \sum_h F^\alpha_{jh}(a)\,F^\alpha_{hi}(b) \tag{3.33}$$

for all $i, j \in I, \alpha \in \pi, a, b \in A_\alpha$. Let f^α_{ji} be linear functionals defined on A_α introduced by the formula

$$f^\alpha_{ji}\ (a) = E_\alpha\left(F^\alpha_{ji}(a)\right) = \varepsilon\left(\Psi_\alpha\left(F^\alpha_{ji}(a)\right)\right) \tag{3.34}$$

Define $f_{ji} \in A'$ by

$$f_{ji} = \sum_{\alpha\in\pi} f^\alpha_{ji}$$

where for any $\beta \in \pi, a \in A_\beta$

$$f_{ji}(a) = \sum_{\alpha\in\pi} f^\alpha_{ji}\ (a) = f^\beta_{ji}(a) \tag{3.35}$$

Applying E_α to both sides of 3.33 and using 3.34 and 3.35 we have

$$f_{ji}(ab) = \sum_h f_{jh}(a)\,f_{hi}(b)$$

for any $a, b \in A_\alpha$, and hence 3.30 is proven. From 3.30 we get

$$f_{ji} m_\alpha(a \otimes b) = \sum_h (f_{jh} \otimes f_{hi})(a \otimes b)$$

i.e.

$$f_{ji} m_\alpha = \sum_h (f_{jh} \otimes f_{hi}) \tag{3.36}$$

Inserting $b = 1_\alpha$ in 3.32 we get

$$\omega_j = \sum_i F^\alpha_{ji}(1_\alpha)\,\omega_i$$

i.e.

$$F^\alpha_{ji}(1_\alpha) = \delta_{ji} 1_\alpha$$

Applying E_α to both sides of the above equation ,and summing over α we get

$$f_{ji}(1_\alpha) = \delta_{ji}$$

and hence 3.31 is proven. To prove 3.28 Recall that from equation 3.32 for any $\alpha \in \pi, \omega_j \in_{inv} \Gamma_\alpha$, $b \in A_\alpha$

$$\omega_j b = \sum_i F^\alpha_{ji}(b)\,\omega_i$$

Applying $\Delta^l_{\alpha,1}$ to both sides of the above equation we obtain

$$\begin{aligned} \Delta^l_{\alpha,1}(\omega_j b) &= \Delta^l_{\alpha,1}\left(\sum_i F^\alpha_{ji}(b)\,\omega_i\right) \\ (1_\alpha \otimes \xi_j)\,\Delta_{\alpha,1}(b) &= \sum_i \Delta_{\alpha,1}\left(F^\alpha_{ji}(b)\right)(1_\alpha \otimes \xi_i) \end{aligned}$$

where $\xi_j, \xi_i \in_{inv} \Gamma_1$, $i, j \in I$. On the other hand using 3.32

$$(1_\alpha \otimes \xi_j)\,\Delta_{\alpha,1}(b) = \sum_i \left(id \otimes F^1_{ji}\right)\Delta_{\alpha,1}(b)\,(1_\alpha \otimes \xi_i)$$

Comparing the last two equations we get

$$\Delta_{\alpha,1}\left(F^\alpha_{ji}(b)\right) = \left(id \otimes F^1_{ji}\right)\Delta_{\alpha,1}(b)$$

Applying $(id \otimes \varepsilon)$ to both sides of the above equation , using 3.35 we get

$$\begin{aligned} F^\alpha_{ji}(b) &= (id \otimes f_{ji})\,\Delta_{\alpha,1}(b) \\ &= f_{ji} * b \end{aligned}$$

Inserting this result into 3.32 we obtain 3.28. In order to prove 3.29 we have to show that

$$\sum_j \left(f_{ji} * f_{hj} \circ S_1^{-1}\right) = \delta_{ih}\varepsilon \tag{3.37}$$

Let $a \in A_1$ .Then

$$\begin{aligned}
\sum_j \left(f_{ji} * f_{hj} \circ S_1^{-1}\right)\left(S_1(a)\right) &= \sum_j \left(f_{ji} \otimes f_{hj} \circ S_1^{-1}\right) \Delta_{1,1}\left(S_1(a)\right) \\
&= \sum_j \left(f_{ji} \otimes f_{hj} \circ S_1^{-1}\right) \sigma_{A_1,A_1}\left(S_1 \otimes S_1\right) \Delta_{1,1}(a) \\
&= \sum_j \left(f_{hj} \otimes f_{ji}\right)\left(id \otimes S_1\right) \Delta_{1,1}(a) \\
&= \sum_j f_{hi} m_1\left(id \otimes S_1\right) \Delta_{1,1}(a) \\
&= \sum_j f_{hi}\left(\varepsilon(a) 1_1\right) \\
&= \sum_j f_{hi}\left(\varepsilon\left(S_1(a)\right) 1_1\right) \\
&= \sum_j f_{hi}\left(1_1\right) \varepsilon\left(S_1(a)\right) \\
&= \delta_{hi} \varepsilon\left(S_1(a)\right)
\end{aligned}$$

i.e.

$$\sum_j f_{ji} * \left(f_{hj} \circ S_1^{-1}\right) = \delta_{ih} \varepsilon$$

Similarly , one can check that

$$\sum_j \left(f_{jh} \circ S_1^{-1}\right) * f_{ij} = \delta_{hi} \varepsilon \tag{3.38}$$

From equation 3.28 we have that for any $\alpha \in \pi$, $b \in A_\alpha, \omega_j \in_{inv} \Gamma_\alpha$

$$\omega_j b = \sum_h \left(f_{jh} * b\right) \omega_h$$

Inserting in this equation $b = \left(f_{jh} \circ S_1^{-1}\right) * a$ for some $a \in A_\alpha$ and summing over j we obtain

$$\begin{aligned}
\sum_j \omega_j \left(f_{jh} \circ S_1^{-1}\right) * a &= \sum_{j,h} \left(f_{jh} * \left(\left(f_{jh} \circ S_1^{-1}\right) * a\right)\right) \omega_h \\
&= \sum_{j,h} \left(\left(f_{jh} * \left(f_{jh} \circ S_1^{-1}\right)\right) * a\right) \omega_h \\
&= \sum_{j,h} \delta_{ih} \left(\varepsilon * a\right) \omega_h \\
&= a \omega_i
\end{aligned}$$

Recall that $\varepsilon * a = (id \otimes \varepsilon) \Delta_{\alpha,1}(a) = a$, and hence 3.29 follows.

To prove 2: For any $\alpha \in \pi, \rho \in \Gamma_\alpha$,we have from statement 1 and formula 3.29 that

$$\begin{aligned}\rho &= \sum_i a_i \omega_i, \qquad a_i \in A_\alpha,\ \omega_i \in_{inv} \Gamma_\alpha,\ i \in I \\ &= \sum_{i,j} \omega_j \left(\left(f_{ij} \circ S_1^{-1}\right) * a_i\right) \\ &= \sum_j \omega_j b_j,\end{aligned}$$

where

$$b_j = \sum_i \left(f_{ij} \circ S_1^{-1}\right) * a_i \in A_\alpha, \quad \forall j \in I.$$

For uniqueness:

Assume that for some b_i ($i \in I$ only finite number of $b_i's$ are different from zero)we have:

$$\sum_i \omega_i b_i = 0$$

We have to show that all $b_i's = 0\,(i \in I\ )$ .Using the uniqueness of decomposition 3.26 we have

$$\sum_i \omega_i b_i = 0$$

Then

$$\sum_{i,j} (f_{i,j} * b_i)\,\omega_j = 0$$

$$\sum_i (f_{i,j} * b_i) = 0 \qquad \forall j \in I$$

Computing the convolution product with $f_{jh} \circ S_1^{-1}$ summing over j and using 3.38

$$\begin{aligned}0 &= \sum_{i,j} \left(f_{jh} \circ S_1^{-1}\right) * (f_{i,j} * b_i) \\ &= \sum_{i,j} \left(\left(f_{jh} \circ S_1^{-1}\right) * f_{i,j}\right) * b_i \\ &= \sum_{i,j} \delta_{hi}\,(\varepsilon * b_i) \\ &= b_i\end{aligned}$$

i.e. $b_i = 0$ for each $i \in I$. □

Theorem 3.3 gives the complete description of left covariant $\pi-$graded bimodules . Using 3.28 , 3.1 we have

$$\left(\sum_i a_i\omega_i\right)b=\sum_i a_i\left(\omega_i b\right)=\sum_{i,.j} a_i\left(f_{ij}*b\right)\omega_j \tag{3.39}$$

$$\Delta^l_{\alpha,\beta}\left(\sum_i a_i\omega_i\right)=\sum_i \Delta_{\alpha,\beta}\left(a_i\right)\Delta^l_{\alpha,\beta}\left(\omega_i\right)=\sum_i \Delta_{\alpha,\beta}\left(a_i\right)\left(1_\alpha\otimes\xi_i\right),\xi_i\in_{inv}\Gamma_\beta \tag{3.40}$$

If $(f_{ij})_{i,j\in I}$ is a family of linear functionals in $A^{'}=\oplus_{\alpha\in\pi}A^{'}_{\alpha}$ satisfying relations 3.30, 3.31, then considering the left module $\Gamma=\{\Gamma_\alpha\}_{\alpha\in\pi}$ generated by $\omega^\alpha_i,\alpha\in\pi,i\in I$, and using the above formulae to introduce the right multiplication by elements of A , and the left action of A we obtain a left covariant $\pi-$graded bimodule .

Definition 3.5. Let (Γ,Δ^r) be a right covariant π-graded bimodule over A . An element $\eta\in\Gamma_\alpha$ is said to be right invariant if

$$\Delta^r_{\alpha,1}\left(\eta\right)=\eta\otimes 1_1 \tag{3.41}$$

Denote by $\Gamma_{inv}=\{\Gamma^\alpha_{inv}\}$ the set of all left invariant elements of Γ . Clearly,Γ^α_{inv} is a linear subspace of Γ_α for each $\alpha\in\pi$.

Theorem 3.4. *Let* $\Gamma=\left(\{\Gamma^\alpha\}_{\alpha\in\pi},\Delta^r\right)$ *be a right covariant* $\pi-$*graded bimodule over* A,$\{\eta^\alpha_i\}_{\alpha\in\pi}$*be a basis of* Γ^α_{inv} *of all right invariant elements of* Γ_α *for each* $\alpha\in\pi$ *.Then*

1. *For any* $\alpha\in\pi$ *,any element* $\varrho\in\Gamma_\alpha$ *is of the form*

$$\varrho=\sum_i a_i\eta_i \tag{3.42}$$

where a_i *'s* $\in A_\alpha$ *are uniquely determined ,* η_i*'s* $\in\Gamma^\alpha_{inv}$*,for any* $\alpha\in\pi$ *.*

2. *For any* $\alpha\in\pi$ *,any element* $\rho\in\Gamma_\alpha$ *is of the form*

$$\varrho=\sum_i \eta_i b_i \tag{3.43}$$

where b_i *'s* $\in A_\alpha$ *are uniquely determined ,* η_i*'s* $\in\Gamma^\alpha_{inv}$*,for any* $\alpha\in\pi$ *.*

3. *There exists linear functionals* $g_{ij}\in A^{'}=\oplus_{\alpha\in\pi}A^{'}_\alpha$ *such that for any* $\alpha\in\pi$

$$\eta_i b=\sum_j\left(b*g_{ij}\right)\eta_j \tag{3.44}$$

$$a\eta_i = \sum_j \eta_j \left(a * \left(g_{ij} \circ S_1^{-1}\right)\right) \tag{3.45}$$

where $a, b \in A_\alpha, \eta_i^{,}s, \eta_j^{,}s \in \Gamma_{inv}^{\alpha}$ *.These functionals are uniquely determined by* 3.44.*They satisfy the following relations*

$$g_{ij}(ab) = \sum_k g_{ik}(a) g_{kj}(b) \tag{3.46}$$

for any $i, j \in I$, $a, b \in A_\alpha$ *.Moreover*

$$g_{ij}(1_\alpha) = \delta_{ij} \tag{3.47}$$

The proof is similar to that of theorem 3.3.

Remark 3.2. Any functional $g_{ij} \in A' = \oplus_{\alpha\in\pi} A'_\alpha$ is of the form $g_{ij} = \sum_\alpha g_{ij}{}^{\alpha}$ where

$$g_{ij}^{\alpha}(a) = 0 \quad\quad if\ a \notin A_\alpha$$

Theorem 3.5. *Let* $\Gamma = \left(\{\Gamma_\alpha\}_{\alpha\in\pi}, \Delta^l, \Delta^r\right)$ *be a* $\pi-$*graded bicovariant bimodule over* A,$\left\{(\omega_i^\alpha)_{i\in\ I}\right\}_{\alpha\in\pi}$ *be a basis of* ${}_{inv}\Gamma = \{{}_{inv}\Gamma_\alpha\}_{\alpha\in\pi}$ *of all left invariant elements of* Γ *.Then*

1. *For any* $i \in I, \alpha, \beta \in \pi, \omega_i^{\alpha\beta} \in \Gamma_{\alpha\beta}$

$$\Delta_{\alpha,\beta}^{r}\left(\omega_i^{\alpha\beta}\right) = \sum_j \omega_j^\alpha \otimes R_{ji} \tag{3.48}$$

where $i, j \in \pi, R_{ji} \in A_\beta$ *satisfy the following relation*

$$\Delta_{\alpha,\beta}\left(R_{ji}\right) = \sum_h R_{jh} \otimes R_{hi} \tag{3.49}$$

and for $R_{ji} \in A_1$

$$\varepsilon\left(R_{ji}\right) = \delta_{ji} \tag{3.50}$$

2. *For each* $\alpha \in \pi$ *there exists a basis* $(\eta_i)_{i\in I}$ *of all right invariant elements of* Γ_α *such that for* $\omega_i \in \Gamma_\alpha$

$$\omega_i = \sum_j \eta_j R_{ji} \quad\quad \forall i \in I \tag{3.51}$$

3. *For any* $j, h \in I$, $a \in A_\alpha$

$$R_{ij}\left(a * f_{ih}\right) = \left(g_{ji} * a\right) R_{hi}, \quad i, j \in I \tag{3.52}$$

where f_{ij}, g_{ij} *are functionals introduced in theorems 3.3 , 3.4*

Proof. Using equation 3.9 for any $\alpha, \beta, \gamma \in \pi$ we have

$$\left(\Delta^l_{\alpha,\beta} \otimes id\right) \Delta^r_{\alpha\beta,\gamma} = \left(id \otimes \Delta^r_{\beta,\gamma}\right) \Delta^l_{\alpha,\beta\gamma}$$

Let $\omega_i^{\alpha\beta\gamma} \in \Gamma_{\alpha\beta\gamma}$

$$\begin{aligned}\left(\Delta^l_{\alpha,\beta} \otimes id\right) \Delta^r_{\alpha\beta,\gamma}\left(\omega_i^{\alpha\beta\gamma}\right) &= \left(id \otimes \Delta^r_{\beta,\gamma}\right) \Delta^l_{\alpha,\beta\gamma}\left(\omega_i^{\alpha\beta\gamma}\right) \\ &= 1_\alpha \otimes \Delta^r_{\beta,\gamma}\left(\omega_i^{\beta\gamma}\right)\end{aligned}$$

i.e.

$$\Delta^r_{\alpha\beta,\gamma}\left(\omega_i\right) \in_{inv} \Gamma_{\alpha\beta} \otimes A_\gamma$$

Then for $\omega_i^{\alpha\beta\gamma} \in \Gamma_{\alpha\beta\gamma}$

$$\Delta^r_{\alpha,\beta\gamma}\left(\omega_i^{\alpha\beta\gamma}\right) = \sum_j \omega_j^\alpha \otimes R_{ji}$$

Applying $(id \otimes \Delta_{\beta,\gamma})$to both sides of the above equation

$$\begin{aligned}\sum_j \omega_j^\alpha \otimes \Delta_{\beta,\gamma}\left(R_{ji}\right) &= \left(id \otimes \Delta_{\beta,\gamma}\right) \Delta^r_{\alpha,\beta\gamma}\left(\omega_i^{\alpha\beta\gamma}\right) \\ &= \left(\Delta^r_{\alpha,\beta} \otimes id\right) \Delta^r_{\alpha\beta,\gamma}\left(\omega_i^{\alpha\beta\gamma}\right) \\ &= \left(\Delta^r_{\alpha,\beta} \otimes id\right)\left(\sum_h \omega_h^{\alpha\beta} \otimes R_{hi}\right) \\ &= \sum_{j,h} \omega_j^\alpha \otimes R_{jh} \otimes R_{hi}\end{aligned}$$

Comparing both sides of the above equation

$$\Delta_{\beta,\gamma}\left(R_{ji}\right) = \sum_h R_{jh} \otimes R_{hi}$$

and hence 3.49 is proven. Let $\omega_i^\alpha \in \Gamma_\alpha$

$$\Delta_{\alpha,1}\left(\omega_i^\alpha\right) = \sum_j \omega_j^\alpha \otimes R_{ji} \quad , R_{ji} \in A_1$$

Applying $(id \otimes \varepsilon)$ to both sides of the above equation

$$\begin{aligned}\left(id \otimes \varepsilon\right) \Delta_{\alpha,1}\left(\omega_i^\alpha\right) &= \omega_i^\alpha \\ &= \left(id \otimes \varepsilon\right)\left(\sum_j \omega_j^\alpha \otimes R_{ji}\right) \\ &= \sum_j \omega_j^\alpha \otimes \varepsilon\left(R_{ji}\right)\end{aligned}$$

$\Longrightarrow$

$$\varepsilon\left(R_{ji}\right)=\delta_{ji}$$

To prove statement 2: First we have that for $R_{ij}\in\ A_1,\alpha\in\pi$

$$m_\alpha\left(id\otimes S_{\alpha^{-1}}\right)\Delta_{\alpha,\alpha^{-1}}=m_\alpha\left(S_{\alpha^{-1}}\otimes id\right)\Delta_{\alpha^{-1},\alpha}=\varepsilon 1_\alpha$$

By using 3.49 ,3.50 we obtain

$$\sum_h S_{\alpha^{-1}}\left(R_{ih}\right)R_{hj}=\delta_{ij}1_\alpha \tag{3.53}$$

$$\sum_h R_{ih}S_{\alpha^{-1}}\left(R_{hj}\right)=\delta_{ij}1_\alpha \tag{3.54}$$

For any $\alpha\in\pi, j\in I$, let

$$\eta_j=\sum_i\omega_i S_{\alpha^{-1}}\left(R_{ij}\right) \tag{3.55}$$

Multiplying both sides of 3.55 by R_{ji} and summing over j then using 3.53 we obtain

$$\begin{aligned}\sum_j\eta_j R_{ji} &= \sum_{i,j}\omega_i S_{\alpha^{-1}}\left(R_{ij}\right)R_{ji}\\ &= \omega_i\end{aligned}$$

and 3.51 follows. It remains to show that η_j defined in 3.55 is right invariant

Let $\eta_j\in\Gamma_\alpha$, $\eta_j=\sum_i\omega_i S_{\alpha^{-1}}\left(R_{ij}\right),\omega_i\in_{inv}\Gamma_\alpha, R_{ij}\in A_{\alpha^{-1}}$.

$$\begin{aligned}\Delta^r_{\alpha,1}\left(\eta_j\right) &= \Delta^r_{\alpha,1}\left(\sum_i\omega_i S_{\alpha^{-1}}\left(R_{ij}\right)\right)\\ &= \sum_i\Delta^r_{\alpha,1}\left(\omega_i\right)\Delta_{\alpha,1}\left(S_{\alpha^{-1}}\left(R_{ij}\right)\right)\\ &= \sum_i\Delta^r_{\alpha,1}\left(\omega_i\right)\sigma_{A_\alpha,A_1}\left(S_1\otimes S_{\alpha^{-1}}\right)\Delta_{1,\alpha^{-1}}\left(R_{ij}\right)\\ &= \sum_{i,h,k}\left(\omega_h\otimes R_{hi}\right)\left(S_{\alpha^{-1}}\left(R_{kj}\right)\otimes S_1\left(R_{ik}\right)\right)\\ &= \sum_{i,h,k}\omega_h S_{\alpha^{-1}}\left(R_{kj}\right)\otimes R_{hi}S_1\left(R_{ik}\right)\\ &= \sum_{h,k}\omega_h S_{\alpha^{-1}}\left(R_{kj}\right)\otimes\delta_{h,k}1_1\\ &= \sum_k\omega_k S_{\alpha^{-1}}\left(R_{kj}\right)\otimes 1_1\\ &= \eta_j\otimes 1_1\end{aligned}$$

For any $\alpha \in \pi$,let $\eta \in \Gamma_\alpha$ be a right invariant element .According to theorem 3.3.2

$$\begin{aligned}\eta &= \sum_i \omega_i c_i \quad , c_i \in A_\alpha \\ &= \sum_{i,j} \eta_j R_{ji} c_i \quad , R_{ji} \in A_\alpha\end{aligned}$$

then

$$\eta = \sum_j \eta_j b_j, \quad b_j \in A_\alpha \tag{3.56}$$

If $\eta = \sum_i \omega_i S_{\alpha^{-1}} (R_{ij})$,then using 3.55 we have

$$\sum_{i,j} \omega_i S_{\alpha^{-1}} (R_{ij}) \, b_j = 0$$

Using 3.3.1 we have

$$\sum_j S_{\alpha^{-1}} (R_{ij}) \, b_j = 0 \quad for\ each\ i \in I$$

Multiplying both sides of the above equation by R_{ji} we obtain

$$b_j = 0$$

for any $j \in I$.

This means that the decomposition 3.56 is unique.

Applying $\Delta^r_{1,\alpha}$ to both sides of 3.56

$$\begin{aligned}\Delta^r_{1,\alpha} (\eta) &= \Delta^r_{1,\alpha} \left(\sum_j \eta_j b_j \right) \\ \xi^1 \otimes 1_\alpha &= \sum_j \left(\xi^1_j \otimes 1_\alpha \right) \Delta_{1,\alpha} (b_j)\end{aligned}$$

Comparing this formula with 3.56 we get that

$$\Delta_{1,\alpha} (b_j) = b_{j(1,1)} \otimes 1_\alpha$$

Applying $\varepsilon \otimes id$ we get that $b_j = \varepsilon \left(b_{j(1,1)} \right) 1_\alpha$.This way we proved that for any $\alpha \in \pi$,any $\eta \in \Gamma^\alpha_{inv}$ is unique linear combination of $\eta_j \, (j \in I)$ .Therefore ,$(\eta_j)_{j \in I}$ is a basis in Γ^{inv}_α and statement 2 is proven.

To prove statement 3 :

Using 3.45 we have for any $\alpha \in \pi, a \in A_\alpha, \eta_j \in \Gamma^\alpha_{inv}$

$$a\eta_j = \sum_i \eta_i \left(a * \left(g_{ji} \circ S_1^{-1}\right)\right)$$

Using 3.55 we get

$$\sum_i a\omega_i S_{\alpha^{-1}}(R_{ij}) = \sum_{i,h} \omega_h S_{\alpha^{-1}}(R_{hi}) \left(a * \left(g_{ji} \circ S_1^{-1}\right)\right)$$

Using 3.31 we get

$$\sum_{i,h} \omega_h \left(\left(f_{ih} \circ S_1^{-1}\right) * a\right) S_{\alpha^{-1}}(R_{ij}) = \sum_{i,h} \omega_h S_{\alpha^{-1}}(R_{hi}) \left(a * \left(g_{ji} \circ S_1^{-1}\right)\right)$$

Using 3.3.1 we get

$$\sum_i \left(\left(f_{ih} \circ S_1^{-1}\right) * a\right) S_{\alpha^{-1}}(R_{ij}) = \sum_i S_{\alpha^{-1}}(R_{hi}) \left(a * \left(g_{ji} \circ S_1^{-1}\right)\right)$$

Applying $S_\alpha \left(S_\alpha = S_{\alpha^{-1}}^{-1}\right)$ to both sides of this equation ,using that S_α is antimultiplicative we get:

$$\sum_i R_{ij} S_\alpha \left(\left(f_{ih} \circ S_1^{-1}\right) * a\right) = \sum_i S_\alpha \left(a * \left(g_{ji} \circ S_1^{-1}\right)\right) R_{hi} \tag{3.57}$$

We compute

$$\begin{aligned}
S_\alpha \left(\left(f_{ih} \circ S_1^{-1}\right) * a\right) &= S_{\alpha^{-1}}^{-1} \left(id \otimes \left(f_{ih} \circ S_1^{-1}\right)\right) \Delta_{\alpha,1}(a) \\
&= (id \otimes f_{ih}) \left(S_{\alpha^{-1}}^{-1} \otimes S_1^{-1}\right) \Delta_{\alpha,1}(a) \\
&= (f_{ih} \otimes id) \Delta_{1,\alpha^{-1}} \left(S_{\alpha^{-1}}^{-1}(a)\right) \\
&= S_{\alpha^{-1}}^{-1}(a) * f_{ih} \\
S_\alpha \left(a * \left(g_{ji} \circ S_1^{-1}\right)\right) &= S_{\alpha^{-1}}^{-1} \left(\left(g_{ji} \circ S_1^{-1}\right) \otimes id\right) \Delta_{1,\alpha}(a) \\
&= (g_{ji} \otimes id) \left(S_1^{-1} \otimes S_{\alpha^{-1}}^{-1}\right) \Delta_{1,\alpha}(a) \\
&= (id \otimes g_{ji}) \Delta_{\alpha^{-1},1} \left(S_{\alpha^{-1}}^{-1}(a)\right) \\
&= g_{ji} * S_{\alpha^{-1}}^{-1}(a)
\end{aligned}$$

i.e.

$$\sum_i R_{ij} \left(S_{\alpha^{-1}}^{-1}(a) * f_{ih}\right) = \sum_i \left(g_{ji} * S_{\alpha^{-1}}^{-1}(a)\right) R_{hi}$$

Replacing a by $S_{\alpha^{-1}}^{-1}(a)$ we obtain

$$\sum_i R_{ij} (a * f_{ih}) = \sum_i (g_{ji} * a) R_{hi}$$

And 3.52 follows. Note that if $a, R_{ij}, R_{hi} \in A_1$, then applying ε to both sides of 3.52 and using 3.50 we obtain:

$$\begin{aligned}
\varepsilon\left(\sum_i R_{ij}\left(a * f_{ih}\right)\right) &= \varepsilon\left(\sum_i \left(g_{ji} * a\right) R_{hi}\right) \\
\sum_i \varepsilon\left(R_{ij}\right) \varepsilon\left(a * f_{ih}\right) &= \sum_i \varepsilon\left(g_{ji} * a\right) \varepsilon\left(R_{hi}\right) \\
\sum_i \delta_{ij} \varepsilon\left(a * f_{ih}\right) &= \sum_i \varepsilon\left(g_{ji} * a\right) \delta_{hi}
\end{aligned}$$

But

$$\begin{aligned}
\varepsilon\left(a * f_{ih}\right) &= \left(f_{ih} \otimes id\right) \Delta_{1,1}(a) \\
&= f_{ih}\left(a_{(1,1)}\right) \varepsilon\left(a_{(2,1)}\right) \\
&= f_{ih}(a)
\end{aligned}$$

Similarly: $\varepsilon\left(g_{ji} * a\right) = g_{ji}(a)$ i.e. $f_{ij}(a) = g_{ij}(a)$, for any $a \in A_1$. From which we get that

$$\sum_i R_{ij}\left(a * f_{ih}\right) = \sum_i \left(f_{ji} * a\right) R_{hi} \tag{3.58}$$

□

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For any $\alpha, \beta \in \pi, \eta_j \in \Gamma_{inv}^{\alpha\beta}$, applying $\Delta_{\alpha,\beta}^l$ to both sides of equation 3.55 we obtain:

$$\begin{aligned}
\Delta_{\alpha,\beta}^l\left(\eta_j\right) &= \Delta_{\alpha,\beta}^l\left(\sum_h \omega_h S_{(\alpha\beta)^{-1}}\left(R_{hj}\right)\right) \\
&= \sum_h \Delta_{\alpha,\beta}^l\left(\omega_h\right) \Delta_{\alpha,\beta}\left(S_{(\alpha\beta)^{-1}}\left(R_{hj}\right)\right) \\
&= \sum_{i,h}\left(1_\alpha \otimes \omega_h^\beta\right)\left(\sigma_{A_\beta, A_\alpha}\left(S_{\beta^{-1}} \otimes S_{\alpha^{-1}}\right) \Delta_{\beta^{-1},\alpha^{-1}}\left(R_{hj}\right)\right) \\
&= \sum_{i,h}\left(1_\alpha \otimes \omega_h^\beta\right)\left(S_{\alpha^{-1}}\left(R_{ij}\right) \otimes S_{\beta^{-1}}\left(R_{hi}\right)\right) \\
&= \sum_{i,h}\left(S_{\alpha^{-1}}\left(R_{ij}\right) \otimes \omega_h^\beta S_{\beta^{-1}}\left(R_{hi}\right)\right)
\end{aligned}$$

i.e.

$$\Delta_{\alpha,\beta}^l\left(\eta_j\right) = \sum_i S_{\alpha^{-1}}\left(R_{ij}\right) \otimes \eta_i^\beta \tag{3.59}$$

Using 3.5 ,3.48

$$\Delta_{\alpha,\beta}^r\left(\sum_i a_i \omega_i\right) = \sum_i \Delta_{\alpha,\beta}\left(a_i\right)\left(\omega_j^\alpha \otimes \ R_{ij}\right) \tag{3.60}$$

Theorem 3.6. *Let $(f_{ij})_{i,j\in I}$ be the family of functionals defined on A satisfying relations 3.30, 3.31, $\left(R_{ij}^{\alpha}\right)_{i,j\in I}$be a family of elements of $A=\{A_\alpha\}_{\alpha\in\pi}$ satisfying relations 3.49, 3.50, 3.58 for each $\alpha\in\pi$. Consider the left module $\Gamma=\{\Gamma_\alpha\}_{\alpha\in\pi}$ over $A=\{A_\alpha\}_{\alpha\in\pi}$ generated by ω_i^{α} , $i\in I$, $\alpha\in\pi$ for each $\alpha\in\pi$, and using formulae 3.39, 3.40, 3.60 to introduce right multiplication by elements of A, left and right actions of A on Γ then $\Gamma=\left(\{\Gamma_\alpha\}_{\alpha\in\pi},\Delta^l,\Delta^r\right)$ is a $\pi-$graded bicovariant bimodule over A.*

Proof. Using formula 3.39 to introduce right multiplication by elements of A ,one can easily check that Γ is also a $\pi-$graded right module over A. , i.e.$\Gamma=\{\Gamma_\alpha\}_{\alpha\in\pi}$ is a $\pi-$graded bimodule over A .

Using 3.40 to define a left action of A on Γ ,taking into consideration 3.3.1 we find that 3.1 ,3.2 are satisfied for let $\rho\in\Gamma_{\alpha\beta}$, $b\in A_{\alpha\beta},\alpha,\beta\in\pi$, using 3.3.1 $\rho=\sum_i a_i\omega_i^{\alpha\beta}$, $a_i\in A_{\alpha\beta}$, $\omega_i^{\alpha\beta}\in_{inv}\Gamma_{\alpha\beta}$

$$\begin{aligned}
\Delta_{\alpha,\beta}^{l}(b\rho) &= \Delta_{\alpha,\beta}^{l}\left(b\sum_i a_i\omega_i^{\alpha\beta}\right)\\
&= \Delta_{\alpha,\beta}^{l}\left(\sum_i (ba_i)\,\omega_i^{\alpha\beta}\right)\\
&= \sum_i \Delta_{\alpha,\beta}(ba_i)\,\Delta_{\alpha,\beta}^{l}\left(\omega_i^{\alpha\beta}\right)\\
&= \sum_i \Delta_{\alpha,\beta}(b)\,\Delta_{\alpha,\beta}(a_i)\,\Delta_{\alpha,\beta}^{l}\left(\omega_i^{\alpha\beta}\right)\\
&= \Delta_{\alpha,\beta}(b)\sum_i \Delta_{\alpha,\beta}(a_i)\,\Delta_{\alpha,\beta}^{l}\left(\omega_i^{\alpha\beta}\right)\\
&= \Delta_{\alpha,\beta}(b)\,\Delta_{\alpha,\beta}^{l}\left(\sum_i a_i\omega_i^{\alpha\beta}\right)\\
&= \Delta_{\alpha,\beta}(b)\,\Delta_{\alpha,\beta}^{l}(\rho)
\end{aligned}$$

And

$$\begin{aligned}
\Delta_{\alpha,\beta}^{l}(\rho b) &= \Delta_{\alpha,\beta}^{l}\left(\left(\sum_i a_i\omega_i^{\alpha\beta}\right)b\right)\\
&= \Delta_{\alpha,\beta}^{l}\left(\sum_i a_i\left(\omega_i^{\alpha\beta}b\right)\right)\\
&= \Delta_{\alpha,\beta}^{l}\left(\sum_{i,j} a_i\left((f_{ij}*b)\,\omega_j^{\alpha\beta}\right)\right)\\
&= \sum_{i,j}\Delta_{\alpha,\beta}^{l}\left((a_i\,(f_{ij}*b))\,\omega_j^{\alpha\beta}\right)
\end{aligned}$$

$$
\begin{aligned}
&= \sum_{i,j} \Delta_{\alpha,\beta}\left(a_i\left(f_{ij} * b\right)\right) \Delta^l_{\alpha,\beta}\left(\omega_j^{\alpha\beta}\right) \\
&= \sum_{i,j} \left(\Delta_{\alpha,\beta}\left(a_i\right) \Delta_{\alpha,\beta}\left(f_{ij} * b\right)\right)\left(1_\alpha \otimes \omega_j^{\beta}\right) \\
&= \sum_{i,j} \left(\Delta_{\alpha,\beta}\left(a_i\right) \Delta_{\alpha,\beta}\left(b_{(1,\alpha\beta)} f_{ij}\left(b_{(2,1)}\right)\right)\right)\left(1_\alpha \otimes \omega_j^{\beta}\right) \\
&= \sum_{i,j} \left(\Delta_{\alpha,\beta}\left(a_i\right)\left(b_{(1,\alpha)} \otimes f_{ij}\left(b_{(3,1)}\right) b_{(2,\beta)}\right)\right)\left(1_\alpha \otimes \omega_j^{\beta}\right) \\
&= \sum_{i,j} \Delta_{\alpha,\beta}\left(a_i\right)\left(b_{(1,\alpha)} \otimes f_{ij}\left(b_{(3,1)}\right) b_{(2,\beta)} \omega_j^{\beta}\right) \\
&= \sum_{i,j} \Delta_{\alpha,\beta}\left(a_i\right)\left(b_{(1,\alpha)} \otimes \left(f_{ij} * b_{(2,\beta)}\right) \omega_j^{\beta}\right) \\
&= \sum_{i} \Delta_{\alpha,\beta}\left(a_i\right)\left(b_{(1,\alpha)} \otimes \omega_i^{\beta} b_{(2,\beta)}\right) \\
&= \sum_{i} \Delta_{\alpha,\beta}\left(a_i\right)\left(1_\alpha \otimes \omega_i^{\beta}\right)\left(b_{(1,\alpha)} \otimes b_{(2,\beta)}\right) \\
&= \sum_{i} \Delta_{\alpha,\beta}\left(a_i\right) \Delta^l_{\alpha,\beta}\left(\omega_i\right) \Delta_{\alpha,\beta}(b) \\
&= \Delta^l_{\alpha,\beta}\left(\left(\sum_i a_i \omega_i\right)\right) \Delta_{\alpha,\beta}(b) \\
&= \Delta^l_{\alpha,\beta}(\rho) \Delta_{\alpha,\beta}(b)
\end{aligned}
$$

Moreover , using 3.3.1 for any α , β , $\gamma \in \pi$, $\rho \in \Gamma_{\alpha\beta\gamma}$, $\rho = \sum_i a_i\omega_i$ where $a_i \in A_{\alpha\beta\gamma}$, $\omega_i \in_{inv} \Gamma_{\alpha\beta\gamma}$ we have:

$$
\begin{aligned}
\left(\Delta_{\alpha,\beta} \otimes id\right) \Delta^l_{\alpha\beta,\gamma}(\rho) &= \left(\Delta_{\alpha,\beta} \otimes id\right) \Delta^l_{\alpha\beta,\gamma}\left(\sum_i a_i \omega_i^{\alpha\beta\gamma}\right) \\
&= \sum_i \left(\Delta_{\alpha,\beta} \otimes id\right) \Delta_{\alpha\beta,\gamma}\left(a_i\right) \Delta^l_{\alpha\beta,\gamma}\left(\omega_i^{\alpha\beta\gamma}\right) \\
&= \sum_i \left(\Delta_{\alpha,\beta} \otimes id\right)\left(\Delta_{\alpha\beta,\gamma}\left(a_i\right)\left(1_{\alpha\beta} \otimes \omega_i^{\gamma}\right)\right) \\
&= \sum_i \left(\Delta_{\alpha,\beta} \otimes id\right) \Delta_{\alpha\beta,\gamma}\left(a_i\right)\left(\Delta_{\alpha,\beta} \otimes id\right)\left(1_{\alpha\beta} \otimes \omega_i^{\gamma}\right) \\
&= \sum_i \left(\left(\Delta_{\alpha,\beta} \otimes id\right) \Delta_{\alpha\beta,\gamma}\left(a_i\right)\right)\left(1_\alpha \otimes 1_\beta \otimes \omega_i^{\gamma}\right) \\
\left(id \otimes \Delta^l_{\beta,\gamma}\right) \Delta^l_{\alpha,\beta\gamma}(\rho) &= \left(id \otimes \Delta^l_{\beta,\gamma}\right) \Delta^l_{\alpha,\beta\gamma}\left(\sum_i a_i \omega_i^{\alpha\beta\gamma}\right)
\end{aligned}
$$

$$
\begin{aligned}
&= \sum_i \left(id \otimes \Delta^l_{\beta,\gamma}\right)\left(\Delta_{\alpha,\beta\gamma}\left(a_i\right)\Delta^l_{\alpha,\beta\gamma}\left(\omega_i^{\alpha\beta\gamma}\right)\right) \\
&= \sum_i \left(id \otimes \Delta_{\beta,\gamma}\right)\left(\Delta_{\alpha,\beta\gamma}\left(a_i\right)\right)\left(id \otimes \Delta^l_{\beta,\gamma}\right)\left(\Delta^l_{\alpha,\beta\gamma}\left(\omega_i^{\alpha\beta\gamma}\right)\right) \\
&= \sum_i \left(id \otimes \Delta_{\beta,\gamma}\right)\left(\Delta_{\alpha,\beta\gamma}\left(a_i\right)\right)\left(id \otimes \Delta^l_{\beta,\gamma}\right)\left(1_\alpha \otimes \omega_i^{\beta\gamma}\right) \\
&= \sum_i \left(id \otimes \Delta_{\beta,\gamma}\right)\left(\Delta_{\alpha,\beta\gamma}\left(a_i\right)\right)\left(1_\alpha \otimes 1_\beta \otimes \omega_i^{\gamma}\right) \\
&= \sum_i \left(\Delta_{\alpha,\beta} \otimes id\right)\Delta_{\alpha\beta,\gamma}\left(a_i\right)\left(1_\alpha \otimes 1_\beta \otimes \omega_i^{\gamma}\right)
\end{aligned}
$$

i.e.

$$
\left(\Delta_{\alpha,\beta} \otimes id\right)\Delta^l_{\alpha\beta,\gamma} = \left(id \otimes \Delta^l_{\beta,\gamma}\right)\Delta^l_{\alpha,\beta\gamma}
$$

which means that 3.3 is satisfied.

Finally , for any $\alpha \in \pi$, letting $\rho \in \Gamma_\alpha$, then using 3.3.1 $\rho = \sum_i a_i\omega_i^\alpha, a_i \in A_\alpha, \omega_i \in_{inv} \Gamma_\alpha$

$$
\left(\varepsilon \otimes id\right)\Delta^l_{1,\alpha}\left(\rho\right) = \left(\varepsilon \otimes id\right)\Delta^l_{1,\alpha}\left(\sum_i a_i\omega_i\right)
$$

$$
\begin{aligned}
&= \sum_i \left(\varepsilon \otimes id\right)\Delta_{1,\alpha}\left(a_i\right)\Delta^l_{1,\alpha}\left(\omega_i\right) \\
&= \sum_i \left(\varepsilon \otimes id\right)\Delta_{1,\alpha}\left(a_i\right)\Delta^l_{1,\alpha}\left(\omega_i\right) \\
&= \sum_i \left(\varepsilon \otimes id\right)\left(\Delta_{1,\alpha}\left(a_i\right)\right)\left(\varepsilon \otimes id\right)\left(\Delta^l_{1,\alpha}\left(\omega_i\right)\right) \\
&= \sum_i a_i\omega_i \\
&= \rho
\end{aligned}
$$

i.e. 3.4 is satisfied.

This means that $\Gamma = \left(\{\Gamma_\alpha\}_{\alpha\in\pi}, \Delta^l\right)$ is a $\pi-$graded left covariant bimodule over A.

Using formula 3.60 to introduce right action of A on Γ one can easily check that $\Gamma = \left(\{\Gamma_\alpha\}_{\alpha\in\pi}, \Delta^r\right)$ is a $\pi-$graded right covariant bimodule over A, for let $\rho \in \Gamma_{\alpha\beta}$, $b \in A_{\alpha\beta}, \alpha, \beta \in \pi$, using 3.3.1

$\rho = \sum_i a_i\omega_i,\ a_i \in A_{\alpha\beta},\ \omega_i \in_{inv} \Gamma_{\alpha\beta}$

$$\begin{aligned}
\Delta^r_{\alpha,\beta}(b\rho) &= \Delta^r_{\alpha,\beta}\left(b\sum_i a_i\omega_i\right) \\
&= \Delta^r_{\alpha,\beta}\left(\sum_i (ba_i)\,\omega_i\right) \\
&= \sum_i \Delta_{\alpha,\beta}(ba_i)\,\Delta^r_{\alpha,\beta}(\omega_i) \\
&= \sum_i \Delta_{\alpha,\beta}(b)\,\Delta_{\alpha,\beta}(a_i)\,\Delta^r_{\alpha,\beta}(\omega_i) \\
&= \Delta_{\alpha,\beta}(b)\sum_i \Delta_{\alpha,\beta}(a_i)\,\Delta^r_{\alpha,\beta}(\omega_i) \\
&= \Delta_{\alpha,\beta}(b)\,\Delta^r_{\alpha,\beta}\left(\sum_i a_i\omega_i\right) \\
&= \Delta_{\alpha,\beta}(b)\,\Delta^r_{\alpha,\beta}(\rho)
\end{aligned}$$

Again , for $\rho \in \Gamma_{\alpha\beta}$, $b \in A_{\alpha\beta}, \alpha, \beta \in \pi$, using 3.3.1 $\rho = \sum_i a_i\omega_i$, $a_i \in A_{\alpha\beta}$, $\omega_i \in_{inv} \Gamma_{\alpha\beta}$. Using 3.44 we get

$$\begin{aligned}
\Delta^r_{\alpha,\beta}(\rho b) &= \Delta^r_{\alpha,\beta}\left(\left(\sum_i a_i\omega_i\right)b\right) \\
&= \Delta^r_{\alpha,\beta}\left(\sum_i a_i(\omega_i b)\right) \\
&= \Delta^r_{\alpha,\beta}\left(\sum_{i,j} a_i\left((f_{ij} * b)\,\omega_j\right)\right) \\
&= \Delta^r_{\alpha,\beta}\left(\sum_{i,j} \left(a_i(f_{ij} * b)\right)\omega_j\right) \\
&= \sum_{i,j} \Delta_{\alpha,\beta}\left(a_i(f_{ij} * b)\right)\Delta^r_{\alpha,\beta}(\omega_j) \\
&= \sum_{i,j} \Delta_{\alpha,\beta}(a_i)\,\Delta_{\alpha,\beta}(f_{ij} * b)\,\Delta^r_{\alpha,\beta}(\omega_j) \\
&= \sum_{i,j} \Delta_{\alpha,\beta}(a_i)\,\Delta_{\alpha,\beta}\big(b_{(1,\alpha\beta)}f_{ij}(b_{2,1})\big)(\omega_k \otimes R_{kj}) \\
&= \sum_{i,j,k} \Delta_{\alpha,\beta}(a_i)\big(b_{(1,\alpha)} \otimes b_{(2,\beta)}f_{ij}(b_{3,1})\big)(\omega_k \otimes R_{kj}) \\
&= \sum_{i,j,k} \Delta_{\alpha,\beta}(a_i)\big(b_{(1,\alpha)}\omega_k \otimes b_{(2,\beta)}f_{ij}(b_{3,1})\,R_{kj}\big)
\end{aligned}$$

$$
\begin{aligned}
&= \sum_{i,j,k} \Delta_{\alpha,\beta}(a_i)\left(b_{(1,\alpha)}\omega_k \otimes \left(f_{ij} * b_{(2,\beta)}\right) R_{kj}\right) \\
&= \sum_{i,j,k,l} \Delta_{\alpha,\beta}(a_i)\left(\omega_l\left(\left(f_{kl}\circ S_1^{-1}\right) * b_{(1,\alpha)}\right) \otimes R_{ji}\left(b_{(2,\beta)} * f_{jk}\right)\right) \\
&= \sum_{i,j,k,l} \Delta_{\alpha,\beta}(a_i)\left((\omega_l \otimes R_{ji})\left(\left(f_{kl}\circ S_1^{-1}\right) * b_{(1,\alpha)} \otimes b_{(2,\beta)} * f_{jk}\right)\right) \\
&= \sum_{i,j,k,l} \Delta_{\alpha,\beta}(a_i)\left((\omega_l \otimes R_{ji})\left(b_{(1,\alpha)} f_{kl}\left(S_1^{-1}\left(b_{(2,1)}\right)\right) \otimes f_{jk}\left(b_{(3,1)}\right) b_{(4,\beta)}\right)\right) \\
&= \sum_{i,j,l} \Delta_{\alpha,\beta}(a_i)\left((\omega_l \otimes R_{ji})\left(b_{(1,\alpha)} \otimes f_{jl}\left(b_{(3,1)} S_1^{-1}\left(b_{(2,1)}\right)\right) b_{(4,\beta)}\right)\right) \\
&= \sum_{i,j,l} \Delta_{\alpha,\beta}(a_i)\left((\omega_l \otimes R_{ji})\left(b_{(1,\alpha)} \otimes f_{jl}\left(\varepsilon\left(b_{(2,1)}\right) 1_1\right) b_{(3,\beta)}\right)\right) \\
&= \sum_{i,j,l} \Delta_{\alpha,\beta}(a_i)\left((\omega_l \otimes R_{ji})\left(b_{(1,\alpha)} \otimes \varepsilon\left(b_{(2,1)}\right) \delta_{jl} b_{(3,\beta)}\right)\right) \\
&= \sum_{i,j} \Delta_{\alpha,\beta}(a_i)\left((\omega_j \otimes R_{ji})\left(b_{(1,\alpha)} \otimes b_{(2,\beta)}\right)\right) \\
&= \sum_{i} \Delta_{\alpha,\beta}(a_i)\, \Delta^r_{\alpha,\beta}(\omega_i)\, \Delta_{\alpha,\beta}(b) \\
&= \Delta^r_{\alpha,\beta}(\rho)\, \Delta_{\alpha,\beta}(b)
\end{aligned}
$$

and thus 3.5, 3.6 are satisfied. Moreover , using 3.3.1 for any α , β , $\gamma \in \pi$, $\rho \in \Gamma_{\alpha\beta\gamma}$, $\rho = \sum_i a_i \omega_i^{\alpha\beta\gamma}$ where $a_i \in A_{\alpha\beta\gamma}$, $\omega_i^{\alpha\beta\gamma} \in_{inv} \Gamma_{\alpha\beta\gamma}$ we have:

$$
\begin{aligned}
\left(\Delta^r_{\alpha,\beta} \otimes id\right) \Delta^r_{\alpha\beta,\gamma}(\rho) &= \left(\Delta^r_{\alpha,\beta} \otimes id\right) \Delta^r_{\alpha\beta,\gamma}\left(\sum_i a_i \omega_i\right) \\
&= \sum_i \left(\Delta^r_{\alpha,\beta} \otimes id\right)\left(\Delta_{\alpha\beta,\gamma}(a_i)\, \Delta^r_{\alpha\beta,\gamma}(\omega_i)\right) \\
&= \sum_i \left(\Delta_{\alpha,\beta} \otimes id\right)\left(\Delta_{\alpha\beta,\gamma}(a_i)\right)\left(\Delta^r_{\alpha,\beta} \otimes id\right)\left(\Delta^r_{\alpha\beta,\gamma}(\omega_i)\right) \\
&= \sum_{i,j} \left(\Delta_{\alpha,\beta} \otimes id\right)\left(\Delta_{\alpha\beta,\gamma}(a_i)\right)\left(\Delta^r_{\alpha,\beta} \otimes id\right)\left(\omega_j^{\alpha\beta} \otimes R_{ji}\right) \\
&= \sum_{i,j,k} \left(\Delta_{\alpha,\beta} \otimes id\right)\left(\Delta_{\alpha\beta,\gamma}(a_i)\right)\left(\omega_k^{\alpha} \otimes R_{kj} \otimes R_{ji}\right) \\
\left(id \otimes \Delta_{\beta,\gamma}\right) \Delta^r_{\alpha,\beta\gamma}(\rho) &= \left(id \otimes \Delta_{\beta,\gamma}\right) \Delta^r_{\alpha,\beta\gamma}\left(\sum_i a_i \omega_i\right) \\
&= \sum_i \left(id \otimes \Delta_{\beta,\gamma}\right)\left(\Delta_{\alpha,\beta\gamma}(a_i)\, \Delta^r_{\alpha,\beta\gamma}(\omega_i)\right)
\end{aligned}
$$

$$
\begin{aligned}
&= \sum_i \left(id\otimes\Delta_{\beta,\gamma}\right)\left(\Delta_{\alpha,\beta\gamma}\left(a_i\right)\right)\left(id\otimes\Delta_{\beta,\gamma}\right)\left(\Delta^r_{\alpha,\beta\gamma}\left(\omega_i\right)\right)\\
&= \sum_{i,k} \left(id\otimes\Delta_{\beta,\gamma}\right)\left(\Delta_{\alpha,\beta\gamma}\left(a_i\right)\right)\\
&= \sum_{i,j,k} \left(id\otimes\Delta_{\beta,\gamma}\right)\left(\Delta_{\alpha,\beta\gamma}\left(a_i\right)\right)\left(\omega^\alpha_k\otimes R_{kj}\otimes R_{ji}\right)
\end{aligned}
$$

i.e. $\left(\Delta^r_{\alpha,\beta}\otimes id\right)\Delta^r_{\alpha\beta,\gamma} = \left(id\otimes\Delta_{\beta,\gamma}\right)\Delta^r_{\alpha,\beta\gamma}$ which means that 3.7 is satisfied. Finally, for any $\alpha\in\pi$, letting $\rho\in\Gamma_\alpha$, then using 3.4.1 $\rho\in\Gamma_\alpha, \rho=\sum_i a_i\omega_i$ where $a_i\in A_\alpha$, $\omega_i\in_{inv}\Gamma_\alpha$

$$
\begin{aligned}
\left(id\otimes\varepsilon\right)\Delta^r_{\alpha,1}\left(\rho\right) &= \left(id\otimes\varepsilon\right)\Delta^r_{\alpha,1}\left(\sum_i a_i\omega_i\right)\\
&= \sum_i \left(id\otimes\varepsilon\right)\left(\Delta_{\alpha,1}\left(a_i\right)\Delta^r_{\alpha,1}\left(\omega_i\right)\right)\\
&= \sum_i \left(id\otimes\varepsilon\right)\left(\Delta_{\alpha,1}\left(a_i\right)\right)\left(id\otimes\varepsilon\right)\left(\Delta^r_{\alpha,1}\left(\omega_i\right)\right)\\
&= \sum_i a_i\omega_i\\
&= \rho
\end{aligned}
$$

i.e. 3.8 is satisfied . This means that $\Gamma=\left(\{\Gamma_\alpha\}_{\alpha\in\pi},\Delta^r\right)$ is a $\pi-$graded right covariant bimodule over A. To prove the bicovariance conditions , for any $\alpha,\beta,\gamma\in\pi$,$\rho\in\Gamma_{\alpha\beta\gamma}$, using 3.3.1 $\rho=\sum_i a_i\omega^{\alpha\beta\gamma}_i$, $a_i\in A_{\alpha\beta\gamma}$, $\omega^{\alpha\beta\gamma}_i\in_{inv}\Gamma_{\alpha\beta\gamma}$ we compute

$$
\begin{aligned}
\left(id\otimes\Delta^r_{\beta,\gamma}\right)\Delta^l_{\alpha,\beta\gamma}\left(\rho\right) &= \left(id\otimes\Delta^r_{\beta,\gamma}\right)\Delta^l_{\alpha,\beta\gamma}\left(\sum_i a_i\omega^{\alpha\beta\gamma}_i\right)\\
&= \sum_i \left(id\otimes\Delta^r_{\beta,\gamma}\right)\left(\Delta_{\alpha,\beta\gamma}\left(a_i\right)\Delta^l_{\alpha,\beta\gamma}\left(\omega^{\alpha\beta\gamma}_i\right)\right)\\
&= \sum_i \left(id\otimes\Delta_{\beta,\gamma}\right)\left(\Delta_{\alpha,\beta\gamma}\left(a_i\right)\right)\left(id\otimes\Delta^r_{\beta,\gamma}\right)\left(\Delta^l_{\alpha,\beta\gamma}\left(\omega_. i^{\alpha\beta\gamma}\right)\right)\\
&= \sum_i \left(id\otimes\Delta_{\beta,\gamma}\right)\left(\Delta_{\alpha,\beta\gamma}\left(a_i\right)\right)\left(1_\alpha\otimes\Delta^r_{\beta,\gamma}\left(\omega^{\beta\gamma}_i\right)\right)\\
&= \sum_{i,j} \left(id\otimes\Delta_{\beta,\gamma}\right)\left(\Delta_{\alpha,\beta\gamma}\left(a_i\right)\right)\left(1_\alpha\otimes\omega^\beta_j\otimes R_{ji}\right)
\end{aligned}
$$

$$\begin{aligned}
\left(\Delta^l_{\alpha,\beta}\otimes id\right)\Delta^r_{\alpha\beta,\gamma}(\rho) &= \left(\Delta^l_{\alpha,\beta}\otimes id\right)\Delta^r_{\alpha\beta,\gamma}\left(\sum_i a_i\omega_i^{\alpha\beta\gamma}\right)\\
&= \sum_i \left(\Delta^l_{\alpha,\beta}\otimes id\right)\left(\Delta_{\alpha\beta,\gamma}(a_i)\,\Delta^r_{\alpha\beta,\gamma}\left(\omega_i^{\alpha\beta\gamma}\right)\right)\\
&= \sum_i \left(\Delta_{\alpha,\beta}\otimes id\right)\left(\Delta_{\alpha\beta,\gamma}(a_i)\right)\left(\Delta^l_{\alpha,\beta}\otimes id\right)\left(\Delta^r_{\alpha\beta,\gamma}\left(\omega_i^{\alpha\beta\gamma}\right)\right)\\
&= \sum_{i,j} \left(\Delta_{\alpha,\beta}\otimes id\right)\left(\Delta_{\alpha\beta,\gamma}(a_i)\right)\left(\Delta^l_{\alpha,\beta}\otimes id\right)\left(\omega_j^{\alpha\beta}\otimes R_{ji}\right)\\
&= \sum_{i,j} \left(\Delta_{\alpha,\beta}\otimes id\right)\left(\Delta_{\alpha\beta,\gamma}(a_i)\right)\left(1_\alpha\otimes\omega_j^{\beta}\otimes R_{ji}\right)
\end{aligned}$$

and hence 3.9 is proved and $\Gamma = \left(\{\Gamma_\alpha\}_{\alpha\in\pi}, \Delta^l, \Delta^r\right)$is a $\pi-$graded bicovariant bimodule over A. □

4 first order differential calculus on Hopf Group Coalgebras

Let $A^2 = \{A^2_\alpha\}_{\alpha\in\pi}$ be the $\pi-$graded bimodule introduced in section 2.We introduce left and right actions of A on A^2 .For any $\alpha,\beta\in\pi$ let $q\in A_{\alpha\beta}\otimes A_{\alpha\beta}$,and $\left(\Delta_{\alpha,\beta}\otimes\Delta_{\alpha,\beta}\right)(q) = \sum_k a_k\otimes b_k\otimes c_k\otimes d_k$, where $a_k, c_k\in A_\alpha,\ b_k, d_k\in A_\beta,\ k = 1,2,...,n$.We set

$$\Phi^l_{\alpha,\beta}(q) = \sum_k a_kc_k\otimes b_k\otimes d_k \tag{4.1}$$

$$\Phi^r_{\alpha,\beta}(q) = \sum_k a_k\otimes c_k\otimes b_kd_k \tag{4.2}$$

We compute

$$\begin{aligned}
(id\otimes m_\beta)\left(\Phi^l_{\alpha,\beta}(q)\right) &= (id\otimes m_\beta)\left(\sum_k a_kc_k\otimes b_k\otimes d_k\right)\\
&= \sum_k a_kc_k\otimes b_kd_k\\
&= \Delta_{\alpha,\beta}\left(m_{\alpha\beta}(q)\right)\\
&= 0
\end{aligned}$$

Similarly we have

$$(m_\alpha\otimes id)\left(\Phi^r_{\alpha,\beta}(q)\right) = (m_\alpha\otimes id)\left(\sum_k a_k\otimes c_k\otimes b_kd_k\right)$$

$$= \sum_k a_k c_k \otimes b_k d_k$$
$$= \Delta_{\alpha,\beta}\left(m_{\alpha\beta}\left(q\right)\right)$$
$$= 0$$

Therfore,

$$\Phi^l_{\alpha,\beta} : A^2_{\alpha\beta} \longrightarrow A_\alpha \otimes A^2_\beta \tag{4.3}$$

and

$$\Phi^r_{\alpha,\beta} : A^2_{\alpha\beta} \longrightarrow A^2_\alpha \otimes A_\beta \tag{4.4}$$

Clearly, both are linear map. We will show that $A^2 = \left(\left\{A^2_\alpha\right\}_{\alpha\in\pi}, \Phi^l, \Phi^r\right)$ is a $\pi-$ graded bicovariant bimodule over A.

First , we will prove that $A^2 = \left(\left\{A^2_\alpha\right\}_{\alpha\in\pi}, \Phi^l, \Phi^r\right)$ is a $\pi-$ graded left covariant bimodule over A.

Let $\alpha, \beta \in \pi, q \in A^2_{\alpha\beta}, q = b \otimes c$, then

$$\begin{aligned}
\Phi^l_{\alpha,\beta}\left(aq\right) &= \Phi^l_{\alpha,\beta}\left(ab \otimes c\right) \\
&= a_{(1,\alpha)} b_{(1,\alpha)} c_{(1,\alpha)} \otimes a_{(2,\beta)} b_{(2,\beta)} \otimes c_{(2,\beta)} \\
&= \left(a_{(1,\alpha)} \otimes a_{(2,\beta)}\right) \cdot \left(b_{(1,\alpha)} c_{(1,\alpha)} \otimes b_{(2,\beta)} \otimes c_{(2,\beta)}\right) \\
&= \Delta_{\alpha,\beta}\left(a\right) \cdot \Phi^l_{\alpha,\beta}\left(b \otimes c\right) \\
&= \Delta_{\alpha,\beta}\left(a\right) \cdot \Phi^l_{\alpha,\beta}\left(q\right)
\end{aligned}$$

and

$$\begin{aligned}
\Phi^l_{\alpha,\beta}\left(qa\right) &= \Phi^l_{\alpha,\beta}\left(b \otimes ca\right) \\
&= b_{(1,\alpha)} c_{(1,\alpha)} a_{(1,\alpha)} \otimes b_{(2,\beta)} \otimes c_{(2,\beta)} a_{(2,\beta)} \\
&= \left(b_{(1,\alpha)} c_{(1,\alpha)} \otimes b_{(2,\beta)} \otimes c_{(2,\beta)}\right) . \left(a_{(1,\alpha)} \otimes a_{(2,\beta)}\right) \\
&= \Phi^l_{\alpha,\beta}\left(b \otimes c\right) \cdot \Delta_{\alpha,\beta}\left(a\right) \\
&= \Phi^l_{\alpha,\beta}\left(q\right) \cdot \Delta_{\alpha,\beta}\left(a\right)
\end{aligned}$$

Moreover , for any $\alpha, \beta, \gamma \in \pi, q \in A^2_{\alpha\beta\gamma}, q = a \otimes b$ we compute

$$\begin{aligned}
(\Delta_{\alpha,\beta} \otimes id)\, \Phi^l_{\alpha\beta,\gamma}(q) &= (\Delta_{\alpha,\beta} \otimes id)\, \Phi^l_{\alpha\beta,\gamma}(a \otimes b) \\
&= (\Delta_{\alpha,\beta} \otimes id)\left(a_{(1,\alpha\beta)} b_{(1,\alpha\beta)} \otimes a_{(2,\gamma)} \otimes b_{(2,\gamma)}\right) \\
&= a_{(1,\alpha)} b_{(1,\alpha)} \otimes a_{(2,\beta)} b_{(2,\beta)} \otimes a_{(3,\gamma)} \otimes b_{(3,\gamma)}
\end{aligned}$$

$$\begin{aligned}
\left(id \otimes \Phi^l_{\beta,\gamma}\right) \Phi^l_{\alpha,\beta\gamma}(q) &= \left(id \otimes \Phi^l_{\beta,\gamma}\right) \Phi^l_{\alpha,\beta\gamma}(a \otimes b) \\
&= \left(id \otimes \Phi^l_{\beta,\gamma}\right)\left(a_{(1,\alpha)} b_{(1,\alpha)} \otimes a_{(2,\beta\gamma)} \otimes b_{(2,\beta\gamma)}\right) \\
&= a_{(1,\alpha)} b_{(1,\alpha)} \otimes a_{(2,\beta)} b_{(2,\beta)} \otimes a_{(3,\gamma)} \otimes b_{(3,\gamma)}
\end{aligned}$$

i.e.

$$(\Delta_{\alpha,\beta} \otimes id)\, \Phi^l_{\alpha,\beta\gamma} = \left(id \otimes \Phi^l_{\beta,\gamma}\right) \Phi^l_{\alpha\beta,\gamma}$$

Finally , for any $\alpha \in \pi, q \in A^2_\alpha, q = a \otimes b$

$$\begin{aligned}
(\varepsilon \otimes id)\, \Phi^l_{1,\alpha}(q) &= (\varepsilon \otimes id)\, \Phi^l_{1,\alpha}(a \otimes b) \\
&= (\varepsilon \otimes id)\left(a_{(1,1)} b_{(1,1)} \otimes a_{(2,\alpha)} \otimes b_{(2,\alpha)}\right) \\
&= \varepsilon\left(a_{(1,1)} b_{(1,1)}\right) a_{(2,\alpha)} \otimes b_{(2,\alpha)} \\
&= \varepsilon\left(a_{(1,1)}\right) \varepsilon\left(b_{(1,1)}\right) a_{(2,\alpha)} \otimes b_{(2,\alpha)} \\
&= a \otimes b \\
&= q
\end{aligned}$$

and thus the conditions of definition 3.1.1 are fulfilled and $A^2 = \left(\{A^2_\alpha\}_{\alpha\in\pi}, \Phi^l\right)$ is a $\pi-$ graded left covariant bimodule over A. Similarly , one can check that $A^2 = \left(\{A^2_\alpha\}_{\alpha\in\pi}, \Phi^r\right)$ is a $\pi-$ graded right covariant bimodule over A. Finally , we check the bicovariance condition

For any $\alpha, \beta, \gamma \in \pi, q \in A^2_{\alpha\beta\gamma}, q = a \otimes b$ we compute

$$\begin{aligned}
\left(id \otimes \Phi^r_{\beta,\gamma}\right) \Phi^l_{\alpha,\beta\gamma}(q) &= \left(id \otimes \Phi^r_{\beta,\gamma}\right) \Phi^l_{\alpha,\beta\gamma}(a \otimes b) \\
&= \left(id \otimes \Phi^r_{\beta,\gamma}\right)\left(a_{(1,\alpha)} b_{(1,\alpha)} \otimes a_{(2,\beta\gamma)} \otimes b_{(2,\beta\gamma)}\right) \\
&= a_{(1,\alpha)} b_{(1,\alpha)} \otimes a_{(2,\beta)} \otimes b_{(2,\beta)} \otimes a_{(3,\gamma)} b_{(3,\gamma)}
\end{aligned}$$

$$\begin{aligned}\left(\Phi^l_{\alpha,\beta}\otimes id\right)\Phi^r_{\alpha\beta,\gamma}(q) &= \left(\Phi^l_{\alpha,\beta}\otimes id\right)\Phi^r_{\alpha\beta,\gamma}(a\otimes b)\\ &= \left(\Phi^l_{\alpha,\beta}\otimes id\right)\left(a_{(1,\alpha\beta)}\otimes b_{(1,\alpha\beta)}\otimes a_{(2,\gamma)}b_{(2,\gamma)}\right)\\ &= a_{(1,\alpha)}b_{(1,\alpha)}\otimes a_{(2,\beta)}\otimes b_{(2,\beta)}\otimes a_{(3,\gamma)}b_{(3,\gamma)}\end{aligned}$$

which proves that $A^2 = \left(\{A^2_\alpha\}_{\alpha\in\pi}, \Phi^l, \Phi^r\right)$ is a $\pi-$ graded bicovariant bimodule over A.

On $A\otimes A = \{A_\alpha \otimes A_\alpha\}_{\alpha\in\pi}$ we define two families of linear mappings

$r = \{r_\alpha : A_\alpha\otimes A_\alpha \longrightarrow A_\alpha\otimes A_1\}_{\alpha\in\pi}$

$t = \{t_\alpha : A_\alpha\otimes A_\alpha \longrightarrow A_1\otimes A_\alpha\}_{\alpha\in\pi}$

For any $\alpha\in\pi, a, b\in A_\alpha$ we set

$$r_\alpha(a\otimes b) = (a\otimes 1_1)\,\Delta_{\alpha,1}(b) \tag{4.5}$$

$$t_\alpha(a\otimes b) = (1_1\otimes a)\,\Delta_{1,\alpha}(b) \tag{4.6}$$

It is clear that r_α, t_α are bijections for each $\alpha\in\pi$ for example for $a\in A_\alpha, b\in A_1$ the inverse of r_α is given by

$$r^{-1}_\alpha(a\otimes b) = (a\otimes 1_\alpha)\,(S_{\alpha^{-1}}\otimes id)\,\Delta_{\alpha^{-1},\alpha}(b) \tag{4.7}$$

Similarly , for $a\in A_1, b\in A_\alpha$ the inverse of t_α is given by

$$t^{-1}_\alpha(a\otimes b) = (b\otimes 1_\alpha)\,(S_{\alpha^{-1}}\otimes id)\,\sigma_{A_{\alpha^{-1}},A_\alpha}\Delta_{\alpha,\alpha^{-1}}(b) \tag{4.8}$$

One can easily show that for each $\alpha\in\pi$, $r_\alpha(A^2_\alpha) = A_\alpha\otimes\ker\varepsilon$, for let $\alpha\in\pi$, $a\in A_\alpha, b\in\ker\varepsilon$

$$\begin{aligned} m_\alpha r^{-1}_\alpha(a\otimes b) &= m_\alpha\left((a\otimes 1_\alpha)(S_{\alpha^{-1}}\otimes id)\,\Delta_{\alpha^{-1},\alpha}(b)\right)\\ &= aS_{\alpha^{-1}}\left(b_{(1,\alpha^{-1})}\right)b_{(2,\alpha)}\\ &= a\varepsilon(b)\,1_\alpha\\ &= 0\end{aligned}$$

From which we get $r^{-1}_\alpha(A_\alpha\otimes\ker\varepsilon) = A^2_\alpha$ i.e.

$$r_\alpha\left(A^2_\alpha\right) = A_\alpha\otimes\ker\varepsilon \tag{4.9}$$

Similarly , one can prove that

$$t_\alpha\left(A^2_\alpha\right) = \ker\varepsilon\otimes A_\alpha \tag{4.10}$$

Proposition 4.1. *For any* $\alpha, \beta, \gamma \in \pi$

$$(\Delta_{\alpha,\beta} \otimes id)\, r_{\alpha\beta} = (id \otimes r_\beta)\, \Phi^l_{\alpha,\beta} \tag{4.11}$$

$$(id \otimes \Delta_{\alpha,\beta})\, t_{\alpha\beta} = (t_\alpha \otimes id)\, \Phi^r_{\alpha,\beta} \tag{4.12}$$

Proof. We will prove that for any $\alpha \in \pi$

$$r_\alpha = (id \otimes \varepsilon \otimes id)\, \Phi^l_{\alpha,1} \tag{4.13}$$

$$t_\alpha = (\varepsilon \otimes id \otimes id)\, \Phi^r_{1,\alpha} \tag{4.14}$$

For any $\alpha \in \pi, a, b \in A_\alpha, a \otimes b \in A_\alpha \otimes A_\alpha$

$$\begin{aligned}
(id \otimes \varepsilon \otimes id)\, \Phi^l_{\alpha,1}\,(a \otimes b) &= (id \otimes \varepsilon \otimes id)\left(a_{(1,\alpha)} b_{(1,\alpha)} \otimes a_{(2,1)} \otimes b_{(2,1)}\right) \\
&= a_{(1,\alpha)} b_{(1,\alpha)} \varepsilon\left(a_{(2,1)}\right) \otimes b_{(2,1)} \\
&= a b_{(1,\alpha)} \otimes b_{(2,1)} \\
&= (a \otimes 1_1)\, \Delta_{\alpha,1}(b) \\
&= r_\alpha\,(a \otimes b)
\end{aligned}$$

$$\begin{aligned}
(\varepsilon \otimes id \otimes id)\, \Phi^r_{1,\alpha}\,(a \otimes b) &= (\varepsilon \otimes id \otimes id)\left(a_{(1,1)} \otimes b_{(1,1)} \otimes a_{(1,\alpha)} b_{(1,\alpha)}\right) \\
&= \varepsilon\left(a_{(1,1)}\right) b_{(1,1)} \otimes a_{(1,\alpha)} b_{(1,\alpha)} \\
&= b_{(1,1)} \otimes a b_{(1,\alpha)} \\
&= (1_1 \otimes a)\, \Delta_{1,\alpha}(b) \\
&= t_\alpha\,(a \otimes b)
\end{aligned}$$

To prove 4.11

$$\begin{aligned}
(\Delta_{\alpha,\beta} \otimes id)\, r_{\alpha\beta} &= (\Delta_{\alpha,\beta} \otimes id)\,(id \otimes \varepsilon \otimes id)\, \Phi^l_{\alpha\beta,1} \\
&= (id \otimes id \otimes \varepsilon \otimes id)\,(\Delta_{\alpha,\beta} \otimes id)\, \Phi^l_{\alpha\beta,1} \\
&= (id \otimes id \otimes \varepsilon \otimes id)\left(id \otimes \Phi^l_{\beta,1}\right) \Phi^l_{\alpha,\beta} \\
&= \left(id \otimes (id \otimes \varepsilon \otimes id)\, \Phi^l_{\beta,1}\right) \Phi^l_{\alpha,\beta} \\
&= (id \otimes r_\beta)\, \Phi^l_{\alpha,\beta}
\end{aligned}$$

Similarly ,one can prove 4.12.

□

Proposition 4.2. *For any $\alpha \in \pi$ an element of A^2_α is left -(right- respectively)invariant if and only if it is of the form* $r_\alpha^{-1}(1_\alpha \otimes x)$ $(t_\alpha^{-1}(y \otimes 1_\alpha)$ *respectively) where* $x \in \ker\varepsilon$ $(y \in \ker\varepsilon$ *respectively).*

Proof. For any $\alpha \in \pi$, let $x \in \ker\varepsilon$ .We compute

$$\begin{aligned}
\Phi^l_{1,\alpha}\left(r_\alpha^{-1}(1_\alpha \otimes x)\right) &= \Phi^l_{1,\alpha}\left(S_{\alpha^{-1}}\left(x_{(1,\alpha^{-1})}\right) \otimes \ x_{(2,\alpha)}\right) \\
&= S_1\left(x_{(2,1)}\right) x_{(3,1)} \otimes S_{\alpha^{-1}}\left(x_{(1,\alpha^{-1})}\right) \otimes x_{(4,\alpha)} \\
&= 1_1 \otimes \varepsilon\left(x_{(2,1)}\right) S_{\alpha^{-1}}\left(x_{(1,\alpha^{-1})}\right) \otimes x_{(3,\alpha)} \\
&= 1_1 \otimes S_{\alpha^{-1}}\left(x_{(1,\alpha^{-1})}\right) \otimes x_{(2,\alpha)} \\
&= 1_1 \otimes r_\alpha^{-1}(1_\alpha \otimes x)
\end{aligned}$$

i.e. $r_\alpha^{-1}(1_\alpha \otimes x)$ is left -invariant element.

Conversly,if $r_\alpha^{-1}(1_\alpha \otimes x)$ is left -invariant element for some $\alpha \in \pi$, let $x \in \ker\varepsilon$, $a \in A_\alpha$. Equation 4.11 implies that

$$(id \otimes r_\alpha)\,\Phi^l_{1,\alpha}\left(r_\alpha^{-1}(a \otimes x)\right) = (\Delta_{1,\alpha} \otimes id)\, r_\alpha\left(r_\alpha^{-1}(a \otimes x)\right)$$

From which we obtain

$$1_1 \otimes a \otimes x = \Delta_{1,\alpha}(a) \otimes x$$

i.e.

$$\Delta_{1,\alpha}(a) = 1_1 \otimes a$$

From which we obtain

$$a = 1_\alpha.$$

□

Theorem 4.3. *Let R be a right ideal of A_1 contained in $ker\varepsilon$, $N = \{N_\alpha\}_{\alpha\in\pi}$, where for each $\alpha \in \pi$, $N_\alpha = r_\alpha^{-1}(A_\alpha \otimes R)$ is a sub-bimodule of $A^2 = \{A^2_\alpha\}_{\alpha\in\pi}$. Moreover , let $\Gamma = \{\Gamma_\alpha\}_{\alpha\in\pi}$, $\Gamma_\alpha = A^2_\alpha/N_\alpha, \Pi = \{\Pi_\alpha : A^2_\alpha \longrightarrow A^2_\alpha/N_\alpha\}$ be the family of canonical epimorphisms, $d = \{d_\alpha : d_\alpha = \Pi_\alpha \circ D_\alpha\}$ .Then the $\pi-$graded first order differential calculus $\Gamma = \left(\{\Gamma_\alpha\}_{\alpha\in\pi}, d\right)$ is left covariant . Any $\pi-$graded left covariant first order differential calculus on A can be obtained in this way.*

Proof. For any $\alpha \in \pi$,let R be a right ideal of A_1 contained in $ker\varepsilon$. We shall prove that $r_\alpha^{-1}(A_\alpha \otimes R)$ is a sub-bimodule of A_α^2.For any $\alpha \in \pi$,let $q \in r_\alpha^{-1}(A_\alpha \otimes R)$,i.e. $q = r_\alpha^{-1}(b \otimes c)$, $b \in A_\alpha, c \in R$.For $a \in A_\alpha$

$$\begin{aligned}
a \cdot q &= (a \otimes 1_\alpha) q \\
&= r_\alpha^{-1}(r_\alpha((a \otimes 1_\alpha) q)) \\
&= r_\alpha^{-1}(r_\alpha(a \otimes 1_\alpha) r_\alpha(q)) \\
&= r_\alpha^{-1}((a \otimes 1_1) r_\alpha(q)) \\
&= r_\alpha^{-1}((a \otimes 1_1)(b \otimes c)) \\
&= r_\alpha^{-1}(ab \otimes c) \\
&\in r_\alpha^{-1}(A_\alpha \otimes R)
\end{aligned}$$

$$\begin{aligned}
q \cdot a &= q(1_\alpha \otimes a) \\
&= r_\alpha^{-1}(r_\alpha(q(1_\alpha \otimes a))) \\
&= r_\alpha^{-1}(r_\alpha(q)\Delta_{\alpha,1}(a)) \\
&= r_\alpha^{-1}((b \otimes c)\Delta_{\alpha,1}(a)) \\
&= r_\alpha^{-1}\left(ba_{(1,\alpha)} \otimes ca_{(2,1)}\right) \\
&\in r_\alpha^{-1}(A_\alpha \otimes R)
\end{aligned}$$

Which proves that $N_\alpha = r_\alpha^{-1}(A_\alpha \otimes R)$ is a sub-bimodule of A_α^2.

To prove that it is left covariant we have to prove that for any $\alpha, \beta \in \pi$, $\Phi_{\alpha,\beta}^l(N_{\alpha\beta}) \subset A_\alpha \otimes N_\beta$.

Using 4.11 we have

$$(id \otimes r_\beta)\Phi_{\alpha,\beta}^l = (\Delta_{\alpha,\beta} \otimes id) r_{\alpha\beta}$$

i.e.

$$\Phi_{\alpha,\beta}^l = \left(id \otimes r_\beta^{-1}\right)(\Delta_{\alpha,\beta} \otimes id) r_{\alpha\beta}$$

Now ,for any $\alpha, \beta \in \pi$, consider $N_{\alpha\beta} = r_{\alpha\beta}^{-1}(A_{\alpha\beta} \otimes R)$

$$\begin{aligned}
\Phi_{\alpha,\beta}^l(N_{\alpha\beta}) &= \left(id \otimes r_\beta^{-1}\right)(\Delta_{\alpha,\beta} \otimes id) r_{\alpha\beta}(N_{\alpha\beta}) \\
&= \left(id \otimes r_\beta^{-1}\right)(\Delta_{\alpha,\beta} \otimes id) r_{\alpha\beta}\left(r_{\alpha\beta}^{-1}(A_{\alpha\beta} \otimes R)\right) \\
&= \left(id \otimes r_\beta^{-1}\right)(\Delta_{\alpha,\beta} \otimes id)(A_{\alpha\beta} \otimes R)
\end{aligned}$$

$$\begin{aligned}
&= \left(id \otimes r_\beta^{-1}\right)\left(\Delta_{\alpha,\beta}\left(A_{\alpha\beta}\right) \otimes R\right) \\
&\subset \left(id \otimes r_\beta^{-1}\right)\left(A_\alpha \otimes A_\beta \otimes R\right) \\
&= A_\alpha \otimes r_\beta^{-1}\left(A_\beta \otimes R\right) \\
&= A_\alpha \otimes N_\beta
\end{aligned}$$

Conversly , if $N = \left(\{N_\alpha\}_{\alpha\in\pi}, \Phi^l\right)$ is a left covariant bimodule , then , using theorem 3.3.1 and proposition 4.2 there exsists a family $(x_i)_{i\in I}$ of elements of $ker\varepsilon$ such that for any $\alpha \in \pi, q \in N_\alpha$ can be written as $q = \sum_i a_i \cdot r_\alpha^{-1}\left(1_\alpha \otimes x_i\right), a_i \in A_\alpha$. But for each $i \in I$ we have

$$\begin{aligned}
a_i \cdot r_\alpha^{-1}\left(1_\alpha \otimes x_i\right) &= \left(a_i \otimes 1_\alpha\right) r_\alpha^{-1}\left(1_\alpha \otimes x_i\right) \\
&= r_\alpha^{-1}\left(r_\alpha\left(a_i \otimes 1_\alpha\right)\left(1_\alpha \otimes x_i\right)\right) \\
&= r_\alpha^{-1}\left(\left(a_i \otimes 1_\alpha\right)\left(1_\alpha \otimes x_i\right)\right) \\
&= r_\alpha^{-1}\left(a_i \otimes x_i\right)
\end{aligned}$$

Denoting by R_α the linear span of all $x_i^{,}s$ we obtain that $N_\alpha = r_\alpha^{-1}\left(A_\alpha \otimes R_\alpha\right)$

We shall show that all $R_\alpha^{,}s$ coincide with R_1. From proposition 4.2 we have

$${}_{inv}N_\alpha = r_\alpha^{-1}\left(1_\alpha \otimes R_\alpha\right)$$

and since N_α is a left covariant bimodule we have

$$\begin{aligned}
\Phi_{\alpha,1}^l\left({}_{inv}N_\alpha\right) &= 1_\alpha \otimes_{inv} N_1 \\
&= 1_\alpha \otimes r_1^{-1}\left(1_1 \otimes R_1\right)
\end{aligned}$$

Now let $r_\alpha^{-1}\left(1_\alpha \otimes x_i\right) \in_{inv} N_\alpha$, $x_i \in R_\alpha$

$$\begin{aligned}
\Phi_{\alpha,1}^l\left(r_\alpha^{-1}\left(1_\alpha \otimes x_i\right)\right) &= \Phi_{\alpha,1}^l\left(S_{\alpha^{-1}}\left(x_{i(1,\alpha^{-1})}\right) \otimes x_{i(2,\alpha)}\right) \\
&= S_{\alpha^{-1}}\left(x_{i(2,\alpha^{-1})}\right) x_{i(3,\alpha)} \otimes S_{1^{-1}}\left(x_{i(1,1)}\right) \otimes x_{i(4,1)} \\
&= 1_\alpha \varepsilon\left(x_{i(2,1)}\right) \otimes S_{1^{-1}}\left(x_{i(1,1)}\right) \otimes x_{i(3,1)} \\
&= 1_\alpha \otimes S_{1^{-1}}\left(x_{i(1,1)}\right) \otimes x_{i(2,1)} \\
&= 1_\alpha \otimes r_1^{-1}\left(1_1 \otimes x_i\right)
\end{aligned}$$

i.e.

$$x_i \in R_1$$

$$\Longrightarrow R_\alpha \subseteq R_1$$

.

Similarly we can show that $R_1 \subseteq R_\alpha$, and hence $R_\alpha = R_1$ for each $\alpha \in \pi$. Denote by R to any of the $R_\alpha^{'}s$, then

$$N_\alpha = r_\alpha^{-1}\left(A_\alpha \otimes R\right)$$

It remains to show that R is a right ideal of A_1. Let $x \in R, a \in A_1$, then $r_1^{-1}\left(1_1 \otimes x\right) \in N_1$.

$$\begin{aligned} r_1^{-1}\left(1_1 \otimes x\right) \cdot a &= r_1^{-1}\left(1_1 \otimes x\right)\left(1_1 \otimes a\right) \\ &= r_1^{-1}\left(\left(1_1 \otimes x\right) r_1 \left(1_1 \otimes a\right)\right) \\ &\in N_1 = r_1^{-1}\left(A_1 \otimes R\right) \\ &\quad \left(N_1 \text{ is a bimodule}\right) \end{aligned}$$

i.e

$$\left(1_1 \otimes x\right) r_1 \left(1_1 \otimes a\right) \in A_1 \otimes R$$

therefore

$$\begin{aligned} \left(1_1 \otimes x\right) r_1 \left(1_1 \otimes a\right) &= r_1\left(r_1^{-1}\left(1_1 \otimes x\right)\left(1_1 \otimes a\right)\right) \\ &= r_1\left(r_1^{-1}\left(\left(1_1 \otimes x\right)\Delta_{1,1}(a)\right)\right) \\ &= \left(1_1 \otimes x\right)\Delta_{1,1}(a) \in A_1 \otimes R \end{aligned}$$

and $\left(\varepsilon \otimes id\right)\left(\left(1_1 \otimes x\right)\Delta_{1,1}(a)\right) = xa \in R$ □

Theorem 4.4. *Let R be a right ideal of A_1 contained in $ker\varepsilon$, $N = \{N_\alpha\}_{\alpha\in\pi}$,where for each $\alpha \in \pi$, $N_\alpha = t_\alpha^{-1}\left(A_\alpha \otimes R\right)$ is a sub-bimodule of $A^2 = \{A_\alpha^2\}_{\alpha\in\pi}$. Moreover ,let $\Gamma = \{\Gamma_\alpha\}_{\alpha\in\pi}$, $\Gamma_\alpha = A_\alpha^2/N_\alpha, \Pi = \{\Pi_\alpha : A_\alpha^2 \longrightarrow A_\alpha^2/N_\alpha\}$ be the family of canonical epimorphisms , $d = \{d_\alpha : d_\alpha = \Pi_\alpha \circ D_\alpha\}$ .Then the first order differential calculus $\Gamma = \left(\{\Gamma_\alpha\}_{\alpha\in\pi}, d\right)$ is right covariant . Any right covariant first order differential calculus on A can be obtained in this way.*

Proof. For any $\alpha \in \pi$,let R be a right ideal of A_1 contained in $ker\varepsilon$. We shall prove that $t_\alpha^{-1}(R \otimes A_\alpha)$ is a sub-bimodule of A_α^2.For any $\alpha \in \pi$,let $q \in t_\alpha^{-1}(R \otimes A_\alpha)$,i.e. $q = t_\alpha^{-1}(d \otimes e)$, $d \in R, e \in A_\alpha$.For $a \in A_\alpha$

$$\begin{aligned} a \cdot q &= (a \otimes 1_\alpha) q \\ &= t_\alpha^{-1}(t_\alpha((a \otimes 1_\alpha) q)) \\ &= t_\alpha^{-1}(t_\alpha(a \otimes 1_\alpha) t_\alpha(q)) \\ &= t_\alpha^{-1}((1_1 \otimes a) t_\alpha(q)) \\ &= t_\alpha^{-1}((1_1 \otimes a)(d \otimes e)) \\ &= t_\alpha^{-1}(d \otimes ae) \\ &\in t_\alpha^{-1}(R \otimes A_\alpha) \end{aligned}$$

$$\begin{aligned} q \cdot a &= q(1_\alpha \otimes a) \\ &= t_\alpha^{-1}(t_\alpha(q(1_\alpha \otimes a))) \\ &= t_\alpha^{-1}(t_\alpha(q)\Delta_{1,\alpha}(a)) \\ &= r_\alpha^{-1}((d \otimes e)\Delta_{1,\alpha}(a)) \\ &= t_\alpha^{-1}\left(da_{(1,1)} \otimes ca_{(2,\alpha)}\right) \\ &\in t_\alpha^{-1}(R \otimes A_\alpha) \end{aligned}$$

which proves that $N_\alpha = t_\alpha^{-1}(R \otimes A_\alpha)$ is a sub-bimodule of A_α^2. To prove that it is right covariant we have to prove that for any $\alpha, \beta \in \pi$, $\Phi_{\alpha,\beta}^l(N_{\alpha\beta}) \subset N_\alpha \otimes A_\beta$. Using 4.13 we have

$$(t_\alpha \otimes id)\Phi_{\alpha,\beta}^r = (id \otimes \Delta_{\alpha,\beta}) t_{\alpha\beta}$$

i.e.

$$\Phi_{\alpha,\beta}^r = \left(t_\alpha^{-1} \otimes id\right)(id \otimes \Delta_{\alpha,\beta}) t_{\alpha\beta}$$

Now ,for any $\alpha,\beta\in\pi$,consider $N_{\alpha\beta}=t_{\alpha\beta}^{-1}\left(R\otimes A_{\alpha\beta}\right)$

$$\begin{aligned}
\Phi^{l}_{\alpha,\beta}\left(N_{\alpha\beta}\right) &= \left(t_{\alpha}^{-1}\otimes id\right)\left(id\otimes\Delta_{\alpha,\beta}\right)t_{\alpha\beta}\left(N_{\alpha\beta}\right)\\
&= \left(t_{\alpha}^{-1}\otimes id\right)\left(id\otimes\Delta_{\alpha,\beta}\right)t_{\alpha\beta}\left(t_{\alpha\beta}^{-1}\left(R\otimes A_{\alpha\beta}\right)\right)\\
&= \left(t_{\alpha}^{-1}\otimes id\right)\left(id\otimes\Delta_{\alpha,\beta}\right)\left(R\otimes A_{\alpha\beta}\right)\\
&\subset \left(t_{\alpha}^{-1}\otimes id\right)\left(R\otimes A_{\alpha}\otimes A_{\beta}\right)\\
&= t_{\alpha}^{-1}\left(R\otimes A_{\alpha}\right)\otimes A_{\beta}\\
&= N_{\alpha}\otimes A_{\beta}
\end{aligned}$$

Conversly , if $N=\left(\{N_{\alpha}\}_{\alpha\in\pi},\Phi^{r}\right)$ is a right covariant bimodule then , using theorem 3.4.1 and proposition 4.2 there exsists a family $(y_i)_{i\in I}$ of elements of $ker\varepsilon$ such that for any $\alpha\in\pi, q\in N_{\alpha}$ can be written as $q=\sum_i a_i\cdot\, t_{\alpha}^{-1}\left(y_i\otimes 1_{\alpha}\right), a_i\in A_{\alpha}$. But for each $i\in I$ we have

$$\begin{aligned}
a_i\cdot r_{\alpha}^{-1}\left(y_i\otimes 1_{\alpha}\right) &= \left(a_i\otimes 1_{\alpha}\right)t_{\alpha}^{-1}\left(y_i\otimes 1_{\alpha}\right)\\
&= t_{\alpha}^{-1}\left(t_{\alpha}\left(1_{\alpha}\otimes a_i\right)\left(y_i\otimes 1_{\alpha}\right)\right)\\
&= t_{\alpha}^{-1}\left(\left(1_{\alpha}\otimes a_i\right)\left(y_i\otimes 1_{\alpha}\right)\right)\\
&= t_{\alpha}^{-1}\left(y_i\otimes a_i\right)
\end{aligned}$$

Denoting by R_{α} the linear span of all $x_i^{,}s$ we obtain that

$$N_{\alpha}=t_{\alpha}^{-1}\left(R\otimes A_{\alpha}\right)$$

We shall show that all $R_{\alpha}^{,}s$ coincide with R_1. From proposition 4.2 we have

$$N_{\alpha}^{inv}=t_{\alpha}^{-1}\left(1_{\alpha}\otimes A_{\alpha}\right)$$

and since N_{α} is a left covariant bimodule we have

$$\begin{aligned}
\Phi^{r}_{1,\alpha}\left(N_{\alpha}^{inv}\right) &= N_1^{inv}\otimes 1_{\alpha}\\
&= t_1^{-1}\left(R\otimes A_1\right)\otimes 1_{\alpha}
\end{aligned}$$

Now let $t_\alpha^{-1}(y_i \otimes 1_\alpha) \in N_\alpha^{inv}$, $y_i \in R_\alpha$

$$\begin{aligned}
\Phi^r_{1,\alpha}\left(t_\alpha^{-1}(y_i \otimes 1_\alpha)\right) &= \Phi^r_{1,\alpha}\left(S_{\alpha^{-1}}\left(y_{i(2,\alpha^{-1})}\right) \otimes y_{i(1,1)}\right) \\
&= S_{1^{-1}}\left(y_{i(4,1)}\right) \otimes y_{i(1,1)} \otimes S_{\alpha^{-1}}\left(y_{i(3,\alpha^{-1})}\right) y_{i(2,\alpha)} \\
&= S_{1^{-1}}\left(y_{i(3,1)}\right) \otimes y_{i(1,1)} \otimes 1_\alpha \varepsilon\left(y_{i(2,1)}\right) \\
&= S_{1^{-1}}\left(y_{i(2,1)}\right) \otimes y_{i(1,1)} \otimes 1_\alpha \\
&= t_1^{-1}(y_i \otimes 1_1) \otimes 1_\alpha
\end{aligned}$$

i.e.

$$y_i \in R_1$$

$$\Longrightarrow R_\alpha \subseteq R_1$$

Similarly we can show that $R_1 \subseteq R_\alpha$, and hence $R_\alpha = R_1$ for each $\alpha \in \pi$.

Denote by R to any of the $R_\alpha^{,}s$, then $N_\alpha = t_\alpha^{-1}(R \otimes A_\alpha)$

It remains to show that R is a right ideal of A_1.

Let $y \in R, a \in A_1$, A then $t_1^{-1}(y \otimes 1_1) \in N_1$.

$$\begin{aligned}
t_1^{-1}(y \otimes 1_1) \cdot a &= t_1^{-1}(y \otimes 1_1)(1_1 \otimes a) \\
&= t_1^{-1}((y \otimes 1_1) t_1 (1_1 \otimes a)) \in N_1 \\
&= t_\alpha^{-1}(R \otimes A_1) \\
&\quad (N_1 \text{ is a bimodule})
\end{aligned}$$

i.e $(y \otimes 1_1) t_1 (1_1 \otimes a) \in R \otimes A_1$

therefore

$$\begin{aligned}
(y \otimes 1_1) t_1 (1_1 \otimes a) &= t_1\left(t_1^{-1}(y \otimes 1_1)(1_1 \otimes a)\right) \\
&= t_1\left(t_1^{-1}((y \otimes 1_1) \Delta_{1,1}(a))\right) \\
&= (y \otimes 1_1) \Delta_{1,1}(a) \in A_1 \otimes R
\end{aligned}$$

and $(id \otimes \varepsilon)((y \otimes 1_1) \Delta_{1,1}(a)) = ya \in R$

□

We shall now formulate the concept of $ad-$invariance . Let

$$ad_\alpha : A_1 \longrightarrow A_1 \otimes A_\alpha$$

be such that for any $a \in A_1$

$$ad_\alpha(a) = t_\alpha\left(r_\alpha^{-1}(1_\alpha \otimes a)\right) \tag{4.15}$$

i.e.

$$ad_\alpha(a) = a_{(2,1)} \otimes S_{\alpha^{-1}}\left(a_{(1,\alpha^{-1})}\right) a_{(3,\alpha)}$$

where

$$(id \otimes \Delta_{1,\alpha})\,\Delta_{\alpha^{-1},\alpha}(a) = a_{(1,\alpha^{-1})} \otimes a_{(2,1)} \otimes a_{(3,\alpha)} \tag{4.16}$$

such that

$$(ad_\alpha \otimes id)\,ad_\beta(a) = (id \otimes \Delta_{\alpha,\beta})\,ad_{\alpha\beta} \tag{4.17}$$

Using 4.15 ,and the standard properties of comultiplication and coinverse one can prove 4.17, for let $a \in A_1$.For any $\alpha, \beta \in \pi$

$$\begin{aligned}(ad_\alpha \otimes id)\,ad_\beta(a) &= (ad_\alpha \otimes id)\left(a_{(2,1)} \otimes S_{\beta^{-1}}\left(a_{(1,\beta^{-1})}\right) a_{(3,\beta)}\right)\\ &= a_{(3,1)} \otimes S_{\alpha^{-1}}\left(a_{(2,\alpha^{-1})}\right) a_{(4,\alpha)} \otimes S_{\beta^{-1}}\left(a_{(1,\beta^{-1})}\right) a_{(5,\beta)}\end{aligned}$$

$$\begin{aligned}(id \otimes \Delta_{\alpha,\beta})\,ad_{\alpha\beta} &= (id \otimes \Delta_{\alpha,\beta})\left(a_{(2,1)} \otimes S_{(\alpha\beta)^{-1}}\left(a_{\left(1,(\alpha\beta)^{-1}\right)}\right) a_{(3,\alpha\beta)}\right)\\ &= a_{(3,1)} \otimes S_{\alpha^{-1}}\left(a_{(2,\alpha^{-1})}\right) a_{(4,\alpha)} \otimes S_{\beta^{-1}}\left(a_{(1,\beta^{-1})}\right) a_{(5,\beta)}\end{aligned}$$

which proves equation 4.17. □

A linear subset $T \subset A_1$ is $\pi - ad$ invariant if $ad_\alpha(T) \subset T \otimes A$ for any $\alpha \in \pi$.

Lemma 4.5. *Let T be $\pi - ad$ invariant subset of A_1, R be a right ideal of A_1 generated by T. Then R is $\pi - ad$ invariant.*

Proof. Let $a, b \in A_1$,we will prove that for any $\alpha \in \pi$

$$ad_\alpha (ab) = \left(1_1 \otimes S_{\alpha^{-1}} \left(b_{(1,\alpha^{-1})}\right)\right) ad_\alpha (a) \Delta_{1,\alpha} \left(b_{(2,\alpha)}\right) \tag{4.18}$$

$$\begin{aligned}
r_\alpha^{-1} (1_\alpha \otimes ab) &= (1_\alpha \otimes 1_\alpha) (S_{\alpha^{-1}} \otimes id) \Delta_{\alpha^{-1},\alpha} (ab) \\
&= S_{\alpha^{-1}} \left(a_{(1,\alpha^{-1})} b_{(1,\alpha^{-1})}\right) \otimes a_{(2,\alpha)} b_{(2,\alpha)} \\
&= S_{\alpha^{-1}} \left(b_{(1,\alpha^{-1})}\right) S_{\alpha^{-1}} \left(a_{(1,\alpha^{-1})}\right) \otimes a_{(2,\alpha)} b_{(2,\alpha)} \\
&= \left(S_{\alpha^{-1}} \left(b_{(1,\alpha^{-1})}\right) \otimes 1_\alpha\right) \left(S_{\alpha^{-1}} \left(a_{(1,\alpha^{-1})}\right) \otimes a_{(2,\alpha)}\right) \left(1_\alpha \otimes b_{(2,\alpha)}\right) \\
&= \left(S_{\alpha^{-1}} \left(b_{(1,\alpha^{-1})}\right) \otimes 1_\alpha\right) r_\alpha^{-1} (a) \left(1_\alpha \otimes b_{(2,\alpha)}\right)
\end{aligned}$$

Applying t_α to both sides of the above equation we get

$$t_\alpha r_\alpha^{-1} (1_\alpha \otimes ab) = t_\alpha \left(S_{\alpha^{-1}} \left(b_{(1,\alpha^{-1})}\right) \otimes 1_\alpha\right) t_\alpha r_\alpha^{-1} (a)\, t_\alpha \left(1_\alpha \otimes b_{(2,\alpha)}\right) \tag{4.19}$$

$$ad_\alpha (ab) = \left(1_\alpha \otimes S_{\alpha^{-1}} \left(b_{(1,\alpha^{-1})}\right)\right) ad_\alpha (a) \Delta_{1,\alpha} \left(b_{(2,\alpha)}\right) \tag{4.20}$$

Thus for $a, b \in T, a, b, ab \in R, R$ being an ideal in A_1, T being $\pi - ad$ invariant we find that

$$ad_\alpha (ab) \in R \otimes A_\alpha$$

i.e.

$$ad_\alpha (R) \subset R \otimes A_\alpha$$

which means that R is $\pi - ad$ invariant. □

Let $A^2 = \left(\{A_\alpha^2\}_{\alpha\in\pi}, \Phi^l, \Phi^r\right)$ is a $\pi-$ graded bicovariant bimodule over A.By virtue of condition 3 of definition 3.3 we have for any $\alpha, \beta, \gamma \in \pi$ $\left(\Phi_{\alpha,1}^l \otimes id\right) \Phi_{\alpha,\beta}^r = \left(id \otimes \Phi_{1,\beta}^r\right) \Phi_{\alpha,\beta}^l$

Applying $id \otimes \varepsilon \otimes id \otimes id$ to both sides of the above equation we get

$$\left((id \otimes \varepsilon \otimes id) \Phi_{\alpha,1}^l \otimes id\right) \Phi_{\alpha,\beta}^r = \left(id \otimes (\varepsilon \otimes id \otimes id) \Phi_{1,\beta}^r\right) \Phi_{\alpha,\beta}^l$$

Using 4.13, 4.14

$$(r_\alpha \otimes id) \Phi_{\alpha,\beta}^r = (id \otimes t_\beta) \Phi_{\alpha,\beta}^l$$

i.e.

$$\Phi_{\alpha,\beta}^r = \left(r_\alpha^{-1} \otimes id\right) (id \otimes t_\beta) \Phi_{\alpha,\beta}^l$$

Now let $x \in \ker \varepsilon$. From proposition 4.2 for any $\alpha \in \pi$ we have $r_{\alpha}^{-1}(1_{\alpha} \otimes x)$ is a left invariant element then

$$\begin{aligned}
\Phi_{\alpha,\beta}^{r}\left(r_{\alpha\beta}^{-1}(1_{\alpha\beta}\otimes x)\right) &= \left(r_{\alpha}^{-1}\otimes id\right)(id\otimes t_{\beta})\,\Phi_{\alpha,\beta}^{l}\left(r_{\alpha\beta}^{-1}(1_{\alpha\beta}\otimes x)\right) \\
&= \left(r_{\alpha}^{-1}\otimes id\right)(id\otimes t_{\beta})\left(1_{\alpha}\otimes r_{\beta}^{-1}(1_{\beta}\otimes x)\right) \\
&= \left(r_{\alpha}^{-1}\otimes id\right)\left(1_{\alpha}\otimes t_{\beta}r_{\beta}^{-1}(1_{\beta}\otimes x)\right) \\
&= \left(r_{\alpha}^{-1}\otimes id\right)(1_{\alpha}\otimes ad_{\beta}(x))
\end{aligned}$$

Theorem 4.6. *Let R be a right ideal of A_1 contained in $\ker\varepsilon$ and $\Gamma = \left(\{\Gamma_{\alpha}\}_{\alpha\in\pi}, d\right)$ be the $\pi-$graded left covariant first order differential calculus described in 4.3 .Then $\Gamma = \left(\{\Gamma_{\alpha}\}_{\alpha\in\pi}, d\right)$ is bicovariant if and only if R is $\pi-ad$ invariant.*

Proof. Let for any $\alpha \in \pi$ R be a right ideal of A_1such that $R \subset \ker\varepsilon$ and $N_{\alpha} = r_{\alpha}^{-1}(A_{\alpha}\otimes R)$ .Using theorem 4.3 we see that $N = \left(\{N_{\alpha}\}_{\alpha\in\pi}, \Phi^{l}\right)$ is a $\pi-$graded left covariant bimodule .Assume that R is $\pi-$ad invariant , let ${}_{inv}N_{\alpha}$ be the set of all left invariant elements of N_{α} for each $\alpha \in \pi$. Then formula 3.78 shows that for any $\alpha, \beta \in \pi$

$$\Phi_{\alpha,\beta}^{r}\left({}_{inv}N_{\alpha\beta}\right) \subset_{inv} N_{\alpha}\otimes A_{\beta}$$

Now decomposition 3.26 shows that $\Phi_{\alpha,\beta}^{r}(N_{\alpha\beta}) \subset N_{\alpha}\otimes A_{\beta}$,and this means that implication 2.14 holds.

Conversly,assume that $N = \{N_{\alpha}\}_{\alpha\in\pi}$ is a $\pi-$graded bicovariant bimodule .This means that 2.14 holds.Then (see proof 4.4) for each $\alpha\in\pi$, $N_{\alpha} = t_{\alpha}^{-1}(R'\otimes A_{\alpha})$ where R' be a right ideal of A_1 such that $R' \subset \ker\varepsilon$.In particular, $N_1 = t_1^{-1}(R'\otimes A_1)$ .Using 3.78 and that $(\varepsilon\otimes id)\,t_1^{-1}(a\otimes b) = a\varepsilon(b)$,and $(id\otimes\varepsilon)\,r_1^{-1}(a\otimes b) = aS_1(b)$,one can easily checks that $R = R'$.

So we have for any $\alpha\in\pi$

$$r_{\alpha}^{-1}(A_{\alpha}\otimes R) = t_{\alpha}^{-1}(R\otimes A_{\alpha})$$

$$t_{\alpha}r_{\alpha}^{-1}(A_{\alpha}\otimes R) = R\otimes A_{\alpha}$$

therefore $ad_{\alpha}(R) = t_{\alpha}r_{\alpha}^{-1}(1_{\alpha}\otimes R)$
$\subset t_{\alpha}r_{\alpha}^{-1}(A_{\alpha}\otimes R) = R\otimes A_{\alpha}$
therefore R is $\pi-$ad invariant. □

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