Finite Proofs for Infinitary Formulas

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Abstract. Recent work has shown that the infinitary logic of here-and-there is closely related to the input language of the ASP grounder GRINGO. A formal system axiomatizing that logic exists, but a proof in that system may include infinitely many formulas. In this note, we define a correspondence between the validity of infinitary formulas in the logic of here-and-there and the provability of formulas in some finite deductive systems. On the basis of this correspondence, we can use finite proofs to justify the validity of infinitary formulas.

1 Introduction

The semantics of ASP programs can be defined using a translation that turns programs into sets of infinitary propositional formulas [2]. To prove properties of ASP programs we need then to reason about stable models of infinitary formulas in the sense of Truszczynski [9]. In particular, we often need to know which transformations of infinitary formulas do not affect their stable models. It is useful to know for instance, that stable models of infinitary formulas are not affected by applying the infinitary De Morgan's laws

$$\bigvee_{\alpha \in A} \neg F_\alpha \leftrightarrow \neg \bigwedge_{\alpha \in A} F_\alpha \tag{1}$$

and

$$\bigwedge_{\alpha \in A} \neg F_\alpha \leftrightarrow \neg \bigvee_{\alpha \in A} F_\alpha, \tag{2}$$

where A may be infinite. "Strongly equivalent" transformations of this kind are used in the proof of the interchangeability of the cardinality constraint $\{p(X)\}0$ and the conditional literal $\bot : p(X)$ [4, Example 7], as well as the proof of correctness of the n-queens program given in the electronic appendix of [2].

Strongly equivalent transformations of infinitary formulas are characterized by the infinitary logic of here-and-there [3]. The set of theorems in the sense of that paper coincides with the set of all infinitary formulas that are "HT-valid"—satisfied by all interpretations in the sense of the logic of here-and-there.

The set of theorems is defined in [3] in terms of closure under a set of inference rules; there is no definition of a proof. It is possible to reformulate that definition in terms of proofs, but those proofs would consist generally of infinitely many formulas, because some of the inference rules introduced there have infinitely many premises. In formalized mathematics, proofs are useful in that they are finite syntactic objects that can establish the validity of assertions about infinite domains. "Infinite proofs", on the other hand, do not have this property.

Can we use finite syntactic objects of some kind to establish that an infinitary formula is HT-valid, at least in some cases?

The definition of an instance of a propositional formula introduced in [4] may help us answer this question. Propositions 1 and 3 in that paper show that substituting infinitary formulas for atoms in a finite intuitionistically provable formula results in an HT-valid formula. For example, the formula

$$((p \wedge q) \vee p) \leftrightarrow p \tag{3}$$

is intuitionistically provable;[1] it follows that for any infinitary formulas F and G, the infinitary formula

$$((F \wedge G) \vee F) \leftrightarrow F \tag{4}$$

is HT-valid. We can think of a proof of (3) as a finite proof of (4) with respect to the substitution that maps p to F and q to G. In a similar way, we can talk about finite proofs of the formula

$$\left(\left(\bigwedge_{\alpha \in A} F_\alpha\right) \vee F_{\alpha_0}\right) \leftrightarrow F_{\alpha_0} \tag{5}$$

for any finite family $(F_\alpha)_{\alpha \in A}$ of infinitary formulas and any α_0 from A.

In this note, we show how the idea of an infinitary instance of a finite formula can be applied in a more general setting. We will define instances for first-order formulas, and that will allow us, for example, to talk about finite proofs of (5) even when A is infinite. Consider the signature that has (symbols for) the elements of A as object constants and a single unary predicate constant P. We will see that (5) is the instance of the first-order formula

$$(\forall x P(x) \vee P(\alpha_0)) \leftrightarrow P(\alpha_0) \tag{6}$$

corresponding to the substitution that maps $P(\alpha)$ to F_α. This formula is intuitionistically provable, and according to the main theorem of this note it follows that (5) is HT-valid.

After a review of the infinitary logic of here-and-there in Section 2, we define instances of a first-order formula in Section 3, and state the main theorem in Section 4. Two other useful forms of the main theorem are discussed in Section 6. The proof of the main theorem is outlined in Section 7.

2 Infinitary Logic of Here-and-There

This review follows [3] and [4].

[1] Formalizations of propositional intuitionistic logic can be found, for instance, in Chapters 2 and 8 of [8]. Formalizations of first-order intuitionistic logic can be found in Chapters 13 and 15 of that book.

2.1 Infinitary Formulas

Throughout this note, we will use σ to denote a propositional signature, that is, a set of propositional atoms. For every nonnegative integer r, *(infinitary propositional) formulas (over σ) of rank r* are defined recursively, as follows:

- every atom from σ is a formula of rank 0;
- if $\mathcal{H}$ is a set of formulas, and r is the smallest nonnegative integer that is greater than the ranks of all elements of $\mathcal{H}$, then $\mathcal{H}^\wedge$ and $\mathcal{H}^\vee$ are formulas of rank r;
- if F and G are formulas, and r is the smallest nonnegative integer that is greater than the ranks of F and G, then $F \rightarrow G$ is a formula of rank r.

We will write $\{F, G\}^\wedge$ as $F \wedge G$, and $\{F, G\}^\vee$ as $F \vee G$. The symbols $\top$ and $\bot$ will be understood as abbreviations for $\emptyset^\vee$ and for $\emptyset^\wedge$ respectively; $\neg F$ and $F \leftrightarrow G$ are understood as abbreviations in the usual way.

A set or family of formulas is *bounded* if the ranks of its members are bounded from above. For any bounded family $(F_\alpha)_{\alpha \in A}$ of formulas, we denote the formula $\{F_\alpha : \alpha \in A\}^\wedge$ by $\bigwedge_{\alpha \in A} F_\alpha$, and similarly for disjunctions.

2.2 HT-Interpretations

An *HT-interpretation* of σ is an ordered pair $\langle I^h, I^t \rangle$ of subsets of σ such that $I^h \subseteq I^t$. The symbols h, t are called *worlds*; respectively *here* and *there*. They are ordered by the relation $h < t$. HT-interpretations are the special case of Kripke models for intuitionistic logic[2] with only two worlds.

The satisfaction relation between an HT-interpretation $I = \langle I^h, I^t \rangle$, a world w, and a formula is defined recursively, as follows:

- $I, w \models p$ if $p \in I^w$;
- $I, w \models \mathcal{H}^\wedge$ if for every formula F in $\mathcal{H}$, $I, w \models F$;
- $I, w \models \mathcal{H}^\vee$ if there is a formula F in $\mathcal{H}$ such that $I, w \models F$;
- $I, w \models F \rightarrow G$ if, for every world w' such that $w \leq w'$, $I, w' \not\models F$ or $I, w' \models G$.

We write $I \models F$ if $I, h \models F$. A formula is *HT-valid* if it is satisfied by all HT-interpretations.

3 Substitutions and Instances

By Σ we denote an arbitrary signature in the sense of first-order logic that contains at least one object constant. The signature may include propositional constants (viewed as predicate constants of arity 0). Object constants will be viewed as function constants of arity 0. In first-order formulas over Σ, we treat the binary connectives $\wedge$, $\vee$, and $\rightarrow$ and the 0-place connective $\bot$ as primitive; $\top$, $\neg$, and $\leftrightarrow$ are the usual abbreviations from propositional logic.

[2] `http://plato.stanford.edu/entries/logic-intuitionistic/#KriSemForIntLog`

A *substitution* is a function ψ that maps each closed atomic formula over Σ to an infinitary formula over σ, such that its range is bounded. A substitution ψ is extended from closed atomic formulas to arbitrary closed first-order formulas over Σ as follows:

- $\psi\bot$ is $\bot$;
- $\psi(\alpha_1 = \alpha_2)$, where α_1, α_2 are ground terms, is $\top$ if α_1 is α_2 and $\bot$ otherwise;
- $\psi(F \odot G)$, where $\odot$ is a binary connective, is $\psi F \odot \psi G$;
- $\psi\forall vF$ is $\bigwedge_\alpha \psi F^v_\alpha$, where α ranges over the ground terms of Σ;[3]
- $\psi\exists vF$ is $\bigvee_\alpha \psi F^v_\alpha$, where α ranges over the ground terms of Σ.

The formula ψF will be called the *instance of F with respect to ψ*.

For example, if Σ includes the elements of A as object constants, but no other function constants, then it is clear that (5) is the instance of (6) with respect to the substitution ψ defined as follows:

$$\psi(P(\alpha)) = F_\alpha.$$

If the function constants of Σ are the object constant a and the unary function constant s, then any infinite conjunction of the form

$$\bigwedge_{i\geq 0}(F_i \to G_i),$$

where F_i, G_i are infinitary formulas, is the instance of the first-order formula

$$\forall x(P(x) \to Q(x))$$

with respect to the substitution ψ defined as follows:

$$\psi(P(s^i(a))) = F_i,$$

$$\psi(Q(s^i(a))) = G_i.$$

4 Main Theorem

The main theorem stated below shows that if a closed first-order formula is intuitionistically provable then all its instances are HT-valid. The theorem is actually more general because it refers to a deductive system that includes, in addition to the axioms and inference rules of first-order intuitionistic logic with equality, some additional axioms. We can add, first of all, the axiom schema

$$F \vee (F \to G) \vee \neg G \tag{7}$$

from [5; 10], the axiom schema

$$\exists x(F \to \forall xF) \tag{8}$$

[3] By F^v_α we denote the result of substituting α for all free occurrences of v in F.

from [7], and the "decidable equality" axiom

$$x = y \vee x \neq y. \tag{9}$$

We can also include the axioms of the Clark Equality Theory [1]:

$$f(x_1, \ldots, x_n) \neq g(y_1, \ldots, y_m) \tag{10}$$

for all pairs of distinct function constants f, g from Σ;

$$f(x_1, \ldots, x_n) = f(y_1, \ldots, y_n) \to (x_1 = y_1 \wedge \cdots \wedge x_n = y_n) \tag{11}$$

for all function constants f from Σ of arity greater than 0; and

$$t(x) \neq x \tag{12}$$

for all terms $t(x)$ that contain x but are different from x.

The deductive system obtained from first-order intuitionistic logic with equality by adding axioms (7)–(12) will be denoted by **HHT** ("Herbrand logic of here-and-there").

Main Theorem. *If a closed first-order formula F is provable in* **HHT** *then any instance of F is HT-valid.*

Example 1. The infinitary De Morgan's laws (1) and (2) with non-empty A are HT-valid because they are instances of the first-order formulas

$$\exists x \neg P(x) \leftrightarrow \neg \forall x P(x)$$

and

$$\forall x \neg P(x) \leftrightarrow \neg \exists x P(x)$$

respectively, and these formulas are provable in **HHT**. (To prove the first equivalence right-to-left, use (8) with $P(x)$ as F.)

Example 2. Any equivalence of the form

$$\left(\left(\bigvee_{\alpha \in A} F_\alpha \right) \to G \right) \leftrightarrow \bigwedge_{\alpha \in A} (F_\alpha \to G)$$

with non-empty A (Example 2 from [4]) is HT-valid because it is an instance of the intuitionistically provable formula

$$(\exists x P(x) \to Q) \leftrightarrow \forall x (P(x) \to Q).$$

Example 3. Any formula of the form

$$\bigvee_{\alpha \in A} \left(F_\alpha \to \bigwedge_{\beta \in A} F_\beta \right),$$

where A is non-empty, is HT-valid because it is an instance of the axiom schema (8).

5 Including Restrictors

In this section, we assume that some unary predicate symbols of the signature Σ may be designated as *restrictors*. The role of restrictors will be somewhat similar to the role of sorts in a many-sorted signature. A *generalized variable* is defined as a variable or an expression of the form

$$(x_1 : R_1, \ldots, x_n : R_n) \tag{13}$$

where $x_1, \ldots, x_n$ ($n \geq 1$) are distinct variables, and $R_1, \ldots, R_n$ are restrictors. *Formulas with restrictors* are defined recursively in the same way as first-order formulas over Σ but with the additional case that a quantifier may be followed by a generalized variable. For instance, if Σ includes the unary predicate constants R_1, R_2, P, Q, and the first two are restrictors, then

$$\exists(x : R_1)P(x) \wedge \exists(y : R_2)Q(y) \leftrightarrow \exists(x : R_1, y : R_2)(P(x) \wedge Q(y)) \tag{14}$$

is a formula with restrictors.

The definition of an instance in Section 3 has the limitation that the infinitary conjunctions and disjunctions corresponding to all quantifiers have the same indexing set. Restrictors allow us to overcome that limitation.

Generalized variables (13) can be eliminated from a formula with restrictors by replacing subformulas of the form

$$\forall(x_1 : R_1, \ldots, x_n : R_n)F$$

with

$$\forall x_1 \ldots x_n(R_1(x_1) \wedge \cdots \wedge R_n(x_n) \to F),$$

and subformulas of the form

$$\exists(x_1 : R_1, \ldots, x_n : R_n)F$$

with

$$\exists x_1 \ldots x_n(R_1(x_1) \wedge \cdots \wedge R_n(x_n) \wedge F).$$

To prove a formula with restrictors means to prove the first-order formula over Σ obtained by this transformation. For instance, we can say that formula (14) is provable in the intuitionistic predicate calculus because the formula

$$\exists x(R_1(x) \wedge P(x)) \wedge \exists y(R_2(y) \wedge Q(y)) \leftrightarrow \exists xy(R_1(x) \wedge R_2(y) \wedge P(x) \wedge Q(y))$$

is provable in that deductive system. Satisfaction of formulas with restrictors is defined similarly.

In the presence of restrictors, a *substitution* is defined as a function ψ that maps each closed atomic formula F over Σ to one of the formulas $\top$, $\bot$, if F begins with a restrictor, and to an element of a bounded set of infinitary formulas over σ otherwise. A substitution ψ is extended to first-order formulas over Σ with restrictors in the same way as for first-order formulas as in Section 3, with the additional cases of universal and existential quantification over generalized variables:

– $\psi\forall(x_1\!:\!R_1,\ldots,x_n\!:\!R_n)F$ is

$$\bigwedge_{\alpha_1,\ldots,\alpha_n\ :\ \psi R_1(\alpha_1)=\cdots=\psi R_n(\alpha_n)=\top} \psi F^{x_1\cdots x_n}_{\alpha_1\cdots\alpha_n},$$

– $\psi\exists(x_1\!:\!R_1,\ldots,x_n\!:\!R_n)F$ is

$$\bigvee_{\alpha_1,\ldots,\alpha_n\ :\ \psi R_1(\alpha_1)=\cdots=\psi R_n(\alpha_n)=\top} \psi F^{x_1\cdots x_n}_{\alpha_i\cdots\alpha_n}.$$

Example 4. Any formula of the form

$$\bigvee_{\alpha\in A} F_\alpha \wedge \bigvee_{\beta\in B} G_\beta \leftrightarrow \bigvee_{(\alpha,\beta)\in A\times B} (F_\alpha \wedge G_\beta) \tag{15}$$

is an instance of (14): include the elements of $A\cup B$ among the object constants of σ and choose ψ so that

$$\begin{array}{l}\psi R_1(\alpha)=\top \text{ iff } \alpha\in A,\\ \psi R_2(\alpha)=\top \text{ iff } \alpha\in B,\\ \psi P(\alpha)=F_\alpha \text{ for all } \alpha\in A,\\ \psi P(\alpha)=G_\alpha \text{ for all } \alpha\in B.\end{array}$$

We may extend the main theorem to formulas with restrictors. In light of this extension, we know for example that any formula of the form (15) is HT-valid by virtue of being an instance of (14).

Main Theorem for Formulas with Restrictors. *If a closed first-order formula F (with restrictors) is provable in* **HHT** *then any instance of F is HT-valid.*

6 Including Second-Order Axioms

We will define now an extension $\mathbf{HHT}^2$ of **HHT** where predicate and function variables of arbitrary arity are included in the language, as in [6, Section 1.2.3]. The set of axioms and inference rules of **HHT** is extended by adding the usual postulates for order second-order quantifiers, the axiom schema of comprehension

$$\exists p\forall x_1\ldots x_n(p(x_1,\ldots,x_n)\leftrightarrow F) \tag{16}$$

($n\geq 0$), where the predicate variable p is not free in F, and the axiom of choice

$$\begin{aligned}&\forall x_1\ldots x_n\exists x_{n+1}\, p(x_1,\ldots,x_{n+1})\rightarrow\\ &\qquad\exists f\forall x_1\ldots x_n(p(x_1,\ldots,x_n,f(x_1,\ldots,x_n)))\end{aligned} \tag{17}$$

($n>0$). The main theorem can be extended as follows.

Main Theorem for $\mathbf{HHT^2}$. *If a closed first-order formula F (possibly with restrictors) is provable in $\mathbf{HHT^2}$ then any instance of F is HT-valid.*

In the special case when the signature Σ contains finitely many function constants, by DCA we denote the domain closure axiom:

$$\forall p \left(\bigwedge C_f(p) \;\rightarrow\; \forall x\; p(x) \right)$$

where the conjunction extends over all function constants f from Σ, and $C_f(p)$ ("set p is closed under f") stands for the formula

$$\forall x_1 \dots x_n (p(x_1) \wedge \dots \wedge p(x_n) \rightarrow p(f(x_1, \dots, x_n)).$$

(In the presence of DCA, axioms (9) and (12) become redundant.) For instance, if Σ contains an object constant a and unary function constant s and no other function constants, then DCA turns into the second-order axiom of induction

$$\forall p\,(p(a) \wedge \forall x\,(p(x) \rightarrow p\,(s(x))) \rightarrow \forall x\; p(x))\,, \tag{18}$$

and $\mathbf{HHT}^2$+ DCA becomes an extension of second-order intuitionistic arithmetic.

In the following version of the main theorem, the signature Σ is assumed to contain finitely many function constants.

Main Theorem for $\mathbf{HHT}^2$+ DCA. *If a closed first-order formula F (possibly with restrictors) is provable in $\mathbf{HHT}^2$+ DCA then any instance of F is HT-valid.*

Note that both versions of the main theorem stated in this section refer to first-order formulas provable using second-order axioms. The notion of a substitution is not defined here for second-order formulas.

Example 5. Any equivalence of the form

$$\left(F_0 \wedge \bigwedge_{i \geq 0} (F_i \rightarrow F_{i+1}) \right) \leftrightarrow \bigwedge_{i \geq 0} F_i$$

(Example 1 from [4]) is HT-valid. Indeed, with the appropriate choice of the signature Σ, it is an instance of the formula

$$P(a) \wedge \forall x (P(x) \rightarrow P(s(x))) \leftrightarrow \forall x P(x).$$

This formula is provable in $\mathbf{HHT}^2$+ DCA. (The implication left-to-right is given by axiom (18).)

7 Proof of Main Theorem

The proof of the theorem makes use of "Herbrand HT-interpretations"—Kripke models with two worlds and with the universe consisting of all ground terms of the signature Σ. We will see that all theorems of **HHT** (and its extensions discussed in the previous section) are satisfied by all Herbrand HT-interpretations.

On the other hand, for any substitution ψ and any HT-interpretation I of σ, we can find an Herbrand HT-interpretation J such that J satisfies a closed first-order formula F if and only if I satisfies ψF. The main theorem will directly follow from these two facts.

An *Herbrand HT-interpretation* of a first-order signature Σ is a pair $\langle J^h, J^t \rangle$ of subsets of the Herbrand base of Σ (that is, the set of all ground atomic formulas over Σ that do not include equality) such that $J^h \subseteq J^t$. By $\mathcal{U}$ we denote the Herbrand universe of Σ, that is, the set of all ground terms over Σ.

For each function $\mathfrak{f}$ of arity $n > 0$ that maps from $\mathcal{U}^n$ to $\mathcal{U}$ we introduce a function constant $\mathfrak{f}^*$ of arity n, called the *function name* of $\mathfrak{f}$. For each pair $\mathfrak{p} = (\mathfrak{p}_h, \mathfrak{p}_t)$ of subsets of $\mathcal{U}^n$ such that $\mathfrak{p}_h \subseteq \mathfrak{p}_t$, we introduce an n-ary predicate constant $\mathfrak{p}^*$, called the *predicate name* of $(\mathfrak{p}_h, \mathfrak{p}_t)$. By Σ^* we denote the signature obtained by adding all function and predicate names to Σ, and by $\mathcal{U}^*$ we denote the Herbrand universe of Σ^*. Then for each term $\alpha \in \mathcal{U}^*$, we define the term $\widehat{\alpha} \in \mathcal{U}$ recursively as follows:

- if α is an object constant from $\mathcal{U}$ then $\widehat{\alpha}$ is α;
- if α is of the form $f(\alpha_1, \dots, \alpha_n)$ where f is a function constant from Σ, then $\widehat{\alpha}$ is $f(\widehat{\alpha_1}, \dots, \widehat{\alpha_n})$;
- if α is of the form $\mathfrak{f}^*(\alpha_1, \dots, \alpha_n)$ where $\mathfrak{f}^*$ is a function name, then $\widehat{\alpha}$ is the element of $\mathcal{U}$ obtained by applying $\mathfrak{f}$ to $\langle \widehat{\alpha_1}, \dots, \widehat{\alpha_n} \rangle$.

The satisfaction relation between an Herbrand HT-interpretation $J = \langle J^h, J^t \rangle$, a world w, and a closed second-order formula F over Σ is defined recursively, as follows:

(i) $J, w \not\models \bot$.
(ii) $J, w \models \alpha_1 = \alpha_2$ if $\widehat{\alpha_1}$ is $\widehat{\alpha_2}$.
(iii) $J, w \models P(\alpha_1, \dots, \alpha_n)$ if $P(\widehat{\alpha_1}, \dots, \widehat{\alpha_n}) \in J^w$.
(iv) $J, w \models \mathfrak{p}^*(\alpha_1, \dots, \alpha_n)$ if $\langle \widehat{\alpha_1}, \dots, \widehat{\alpha_n} \rangle \in \mathfrak{p}_w$.
(v) $J, w \models F \wedge G$ if $J, w \models F$ and $J, w \models G$; similarly for $\vee$.
(vi) $J, w \models F \rightarrow G$ if for every world w' such that $w \leq w'$, $J, w' \not\models F$ or $J, w' \models G$.
(vii) $J, w \models \forall v F$, where v is an object variable, if for each ground term α over Σ, $J, w \models F^v_\alpha$; similarly for $\exists$.
(viii) $J, w \models \forall v F$, where v is a function variable, if for each function name $\mathfrak{f}^*$ of the same arity as v, $J, w \models F^v_{\mathfrak{f}^*}$; similarly for $\exists$.[4]
(ix) $J, w \models \forall v F$, where v is a predicate variable, if for each predicate name $\mathfrak{p}^*$ of the same arity as v, $J, w \models F^v_{\mathfrak{p}^*}$; similarly for $\exists$.

A closed second-order formula F over Σ^* is *HHT-valid* if $J, h \models F$ for every Herbrand HT-interpretation J.

[4] The notation for substituting a function name or function variable for a function variable is the same as that of substitution a term for an object variable; similarly for predicate names and predicate variables.

Soundness Lemma.

(a) If a second-order formula F over Σ^* is provable in $\mathbf{HHT}^2$ *then the universal closure of F is HHT-valid.*
(b) For any first-order signature Σ containing finitely many function constants, if a second-order formula F over Σ^* is provable in $\mathbf{HHT}^2$*+ DCA then the universal closure of F is HHT-valid.*

The lemma is proved by induction on the derivation of F.

Lifting Lemma. *Let I be an HT-interpretation of a propositional signature σ, ψ be a substitution from a first-order signature Σ (possibly containing restrictors) to σ, and J be the Herbrand HT-interpretation defined by the condition: for every world w*

$$J, w \models P(\alpha_1, \dots, \alpha_n) \text{ iff } I, w \models \psi P(\alpha_1, \dots \alpha_n).$$

Then for any closed first-order formula F (possibly with restrictors)

$$J, w \models F \text{ iff } I, w \models \psi F.$$

The lemma is proved by strong induction on the total number of connectives and quantifiers in F. If F is atomic, then the assertion of the lemma is immediate from the definition of J. Here two of the other cases.

Case $\forall v F$:

$$\begin{array}{ll} & J, w \models \forall v F \\ \text{iff} & \text{for each ground term } \alpha,\ J, w \models F^v_\alpha \\ \text{iff} & \text{for each ground term } \alpha,\ I, w \models \psi F^v_\alpha \\ \text{iff} & I, w \models \bigwedge_\alpha \psi F^v_\alpha \\ \text{iff} & I, w \models \psi\left(\bigwedge_\alpha F^v_\alpha\right). \end{array}$$

Case $\forall(x_1{:}R_1, \dots, x_n{:}R_n)F$: We need to show that

$$J, w \models \forall(x_1{:}R_1, \dots, x_n{:}R_n)F$$

iff

$$I, w \models \bigwedge_{\alpha_1,\dots,\alpha_n:\ \psi R_1(\alpha_1)=\dots=\psi R_n(\alpha_n)=\top} \psi F^{x_1,\cdots,x_n}_{\alpha_1,\cdots,\alpha_n}. \tag{19}$$

Indeed,

$$\begin{array}{ll} & J, w \models \forall(x_1{:}R_1, \dots, x_n{:}R_n)F \\ \text{iff} & J, w \models \forall x_1, \dots, x_n(R_1(x_1) \wedge \dots \wedge R_n(x_n) \to F) \\ \text{iff} & J, w' \models F^{x_1,\cdots,x_n}_{\alpha_1,\cdots,\alpha_n} \text{ in every world } w' \geq w \text{ and for each tuple of ground terms} \\ & \alpha_1, \dots, \alpha_n \text{ such that } J, w' \models R_1(\alpha_1) \wedge \dots \wedge R_n(\alpha_n) \\ \text{iff} & I, w' \models \psi F^{x_1,\cdots,x_n}_{\alpha_1,\cdots,\alpha_n} \text{ in every world } w' \geq w \text{ and for each tuple of ground terms} \\ & \alpha_1, \dots, \alpha_n \text{ such that } I, w' \models \psi R_1(\alpha_1) \wedge \dots \wedge \psi R_n(\alpha_n) \end{array}$$

iff $I, w' \models \psi F^{x_1,\cdots,x_n}_{\alpha_1,\cdots,\alpha_n}$ in every world $w' \geq w$ and for each tuple of ground terms $\alpha_1, \ldots, \alpha_n$ such that $\psi R_1(\alpha_1) = \cdots = \psi R_n(\alpha_n) = \top$

iff in every world $w' \geq w$,
$I, w' \models \bigwedge_{\alpha_1,\ldots,\alpha_n:\ \psi R_1(\alpha_1)=\cdots=\psi R_n(\alpha_n)=\top} \psi F^{x_1,\cdots,x_n}_{\alpha_1,\cdots,\alpha_n}$.

The condition above is equivalent to (19) by the monotonicity property of the satisfaction relation in the logic of here-and-there.

The main theorem is immediate from the two lemmas stated above.

8 Future Work

The system $\mathbf{HHT}^2$+ DCA is very strong. It includes, for instance, inutitionistic second-order arithmetic with the axiom of choice as a special case. We doubt that we will come across a case where it is not sufficiently strong for the study of HT-valid infinitary formulas. On the other hand, as noted in Section 6, our notion of a substitution currently only applies to first-order formulas. We plan to extend the definition of a substitution, and the main theorem, to cover second-order formulas as well. We hope that such an extension will allow us, for instance, to give a finite proof of the equivalence

$$\bigwedge_{\substack{A\subseteq C\\ A\neq\emptyset}} \left(\bigwedge_{a\in A} P(a) \to \bigvee_{a\in C\setminus A} P(a) \right) \leftrightarrow \bigwedge_{a\in C} \neg P(a),$$

which establishes the interchangeability of the cardinality constraint and the conditional literal mentioned in the introduction.

9 Conclusion

The infinitary logic of here-and-there is of theoretical interest in the field of answer set programming because it can be used to study properties of programs in the input language of the ASP grounder GRINGO. In this note we defined a correspondence between the validity of infinitary formulas in the logic of here-and-there and the provability of formulas in some finite deductive systems involving first-order and second-order formulas. We showed that this correspondence can be used to justify the validity of infinitary formulas in the logic of here-and-there by means of finite proofs.

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References

1. Clark, K.: Negation as failure. In: Gallaire, H., Minker, J. (eds.) Logic and Data Bases, pp. 293–322. Plenum Press, New York (1978)
2. Gebser, M., Harrison, A., Kaminski, R., Lifschitz, V., Schaub, T.: Abstract Gringo. In: Proceedings of International Conference on Logic Programming (ICLP) (2015), `http://www.cs.utexas.edu/users/vl/papers/AG.pdf`; to appear
3. Harrison, A., Lifschitz, V., Pearce, D., Valverde, A.: Infinitary equilibrium logic and strong equivalence. In: Proceedings of International Conference on Logic Programming and Nonmonotonic Reasoning (LPNMR) (2015), `http://www.cs.utexas.edu/users/vl/papers/iel_lpnmr.pdf`; to appear
4. Harrison, A., Lifschitz, V., Truszczynski, M.: On equivalence of infinitary formulas under the stable model semantics. Theory and Practice of Logic Programming 15(1), 18–34 (2015)
5. Hosoi, T.: The axiomatization of the intermediate propositional systems S_n of Gödel. Journal of the Faculty of Science of the University of Tokyo 13, 183–187 (1966)
6. Lifschitz, V., Morgenstern, L., Plaisted, D.: Knowledge representation and classical logic. In: van Harmelen, F., Lifschitz, V., Porter, B. (eds.) Handbook of Knowledge Representation, pp. 3–88. Elsevier (2008)
7. Lifschitz, V., Pearce, D., Valverde, A.: A characterization of strong equivalence for logic programs with variables. In: Procedings of International Conference on Logic Programming and Nonmonotonic Reasoning (LPNMR). pp. 188–200 (2007)
8. Mints, G.: A Short Introduction to Intuitionistic Logic. Kluwer (2000)
9. Truszczynski, M.: Connecting first-order ASP and the logic FO(ID) through reducts. In: Erdem, E., Lee, J., Lierler, Y., Pearce, D. (eds.) Correct Reasoning: Essays on Logic-Based AI in Honor of Vladimir Lifschitz, pp. 543–559. Springer (2012)
10. Umezawa, T.: On intermediate many-valued logics. Journal of the Mathematical Society of Japan 11(2), 116–128 (1959)