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EXISTENCE OF SOLUTIONS TO A SELF-REFERRED AND HEREDITARY SYSTEM OF DIFFERENTIAL EQUATIONS

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Abstract. We establish the existence and uniqueness of a local solution for the system of differential equations

$$\frac{\partial}{\partial t}u(x,t) = u\Big(v\Big(\int_0^t u(x,s)ds, t\Big), t\Big)$$
$$\frac{\partial}{\partial t}v(x,t) = v\Big(u\Big(\int_0^t v(x,s)ds, t\Big), t\Big).$$

with given initial conditions $u(x,0) = u_0(x)$ and $v(x,0) = v_0(x)$.

1. Introduction

Equations representing self-reference phenomena have been written of the form

$$Au(x,t) = u(Bu(x,t),t), \tag{1.1}$$

where A, B are functionals on a real function space. The existence and uniqueness of solutions to this equation have been studied by several authors. The particular case when the variable x does not appear explicitly was studied in [1, 2, 3]. More general cases have been studien in [4, 5, 6]. In [4],

$$Au(x,t) = \frac{\partial}{\partial t}u(x,t) \quad \text{and} \qquad Bu(x,t) = \int_0^t u(x,s)ds,$$

where B can be interpreted as a "memory" functional. In [6], we have considered the equation

$$\frac{\partial^2}{\partial t^2}u(x,t) = k_1 u\Big(\frac{\partial^2}{\partial t^2}u(x,t) + k_2 u(x,t), t\Big)$$

where k_i are nonnegative real numbers, or bounded regular real functions. In this paper we establish the existence and uniqueness of local solutions for the system of functional differential equations

$$\frac{\partial}{\partial t}u(x,t) = u\Big(v\Big(\int_0^t u(x,s)ds, t\Big), t\Big)$$
$$\frac{\partial}{\partial t}v(x,t) = v\Big(u\Big(\int_0^t v(x,s)ds, t\Big), t\Big).$$

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This system can be considered a model for the evolution of two reasonings, as follows: If x is an event, t is the time, and $u(x,t), v(x,t)$ are two reasonings about x at time t, then the term $v(\int_0^t u(x,s)ds, t)$ can be considered as a "criticism" of v over all previous reasonings of u on x, up to time t.

2. The main result

In this section we prove the following theorem.

Theorem 2.1. *Let $u_0, v_0 : \mathbb{R} \to \mathbb{R}$ be bounded and Lipschitz continuous. Then, there exist $T_0 > 0$ and two real bounded and Lipschitz continuous functions $u_\infty, v_\infty : \mathbb{R} \times [0, T_0] \to \mathbb{R}$ such that*

$$
\begin{gathered}
\frac{\partial}{\partial t} u_\infty(x,t) = u_\infty\Big(v_\infty\Big(\int_0^t u_\infty(x,\tau)d\tau, t\Big), t\Big) \\
\frac{\partial}{\partial t} v_\infty(x,t) = v_\infty\Big(u_\infty\Big(\int_0^t v_\infty(x,\tau)d\tau, t\Big), t\Big) \\
u_\infty(x,0) = u_0(x), \quad v_\infty(x,0) = v_0(x)
\end{gathered}
$$

for all $x \in \mathbb{R}$ and all $t \in [0, T_0]$. Moreover the functions u_∞, v_∞ are unique.

Proof. Let u_0, v_0 be given, and let $L_0, M_0 > 0$ be such that

$$
|u_0(x) - u_0(y)| \le L_0|x-y|, \quad |v_0(x) - v_0(y)| \le M_0|x-y|.
$$

for all $x, y \in \mathbb{R}$. Define the sequences of functions $(u_n)_n, (v_n)_n$, for all $x \in R$ and $t > 0$, as follows:

$$
\begin{gathered}
u_1(x,t) = u_0(x) + \int_0^t u_0\big(v_0(u_0(x)\tau)\big)d\tau, \\
v_1(x,t) = v_0(x) + \int_0^t v_0\big(u_0(v_0(x)\tau)\big)d\tau, \\
u_{n+1}(x,t) = u_0(x) + \int_0^t u_n\Big(v_n\Big(\int_0^\tau u_n(x,s)ds, \tau\Big), \tau\Big)d\tau, \\
v_{n+1}(x,t) = v_0(x) + \int_0^t v_n\Big(u_n\Big(\int_0^\tau v_n(x,s)ds, \tau\Big), \tau\Big)d\tau.
\end{gathered}
$$

Notice that

$$
|u_1(x,t) - u_0(x)| \le \|u_0\|_\infty t \equiv A_1(t) \tag{2.1}
$$
$$
|v_1(x,t) - v_0(x)| \le \|v_0\|_\infty t \equiv B_1(t), \tag{2.2}
$$

for all $x \in \mathbb{R},\ t > 0$. Moreover, using (2.1), (2.2) we have

$$\begin{aligned}
&|u_2(x,t) - u_1(x,t)| \\
&\le \Big| \int_0^t u_1(v_1(\int_0^\tau u_1(x,s)ds, \tau)\tau)d\tau - \int_0^t u_0(v_0(u_0(x)\tau))d\tau \Big| \\
&\le \Big| \int_0^t u_1(v_1(\int_0^\tau u_1(x,s)ds, \tau)\tau)d\tau - \int_0^t u_0(v_1(\int_0^\tau u_1(x,s)ds, \tau))d\tau \Big| \\
&\quad + \Big| \int_0^t u_0(v_1(\int_0^\tau u_1(x,s)ds, \tau))d\tau - \int_0^t u_0(v_0(u_0(x)\tau))d\tau \Big| \\
&\le \int_0^t \|u_0\|_\infty \tau d\tau + \int_0^t L_0 \Big| v_1(\int_0^\tau u_1(x,s)ds, \tau) - v_0(u_0(x)\tau) \Big| d\tau \\
&\le \int_0^t \|u_0\|_\infty \tau d\tau + \int_0^t L_0 \Big[\Big| v_1(\int_0^\tau u_1(x,s)ds, \tau) - v_0(\int_0^\tau u_1(x,s)ds) \Big| \\
&\quad + \Big| v_0(\int_0^\tau u_1(x,s)ds) - v_0(u_0(x)\tau) \Big| \Big] d\tau \\
&\le \int_0^t \Big(\|u_0\|_\infty \tau + L_0 \Big[\|v_0\|_\infty \tau + M_0 \int_0^\tau \|u_0\|_\infty s ds \Big] \Big) d\tau \\
&= \int_0^t \Big(A_1(\tau) + L_0 \Big[B_1(\tau) + M_0 \int_0^\tau A_1(s)ds \Big] \Big) d\tau
\end{aligned}$$

for all $x \in \mathbb{R}$, and all $t > 0$. In a similar way we prove

$$|v_2(x,t) - v_1(x,t)| \le \int_0^t \Big(B_1(\tau) + M_0[A_1(\tau) + L_0 \int_0^\tau B_1(s)ds] \Big) d\tau$$

for all $x \in \mathbb{R}$, and all $t > 0$. We have also

$$\begin{aligned}
|u_1(x,t) - u_1(y,t)| &\le L_0|x-y| + \int_0^t L_0^2 M_0 |x-y| \tau d\tau \\
&= \Big(L_0 + \int_0^t L_0^2 M_0 \tau d\tau \Big) |x-y| \equiv L_1(t)|x-y|; \\
|v_1(x,t) - v_1(y,t)| &\le M_0|x-y| + \int_0^t L_0 M_0^2 |x-y| \tau d\tau \\
&= \Big(M_0 + \int_0^t L_0 M_0^2 \tau d\tau \Big) |x-y| \equiv M_1(t)|x-y|.
\end{aligned}$$

It is easy to prove the inequality

$$|u_2(x,t) - u_2(y,t)| \le \Big[L_0 + \int_0^t M_1(\tau) \frac{1}{2} \frac{d}{d\tau} (\int_0^\tau L_1(s)ds)^2 d\tau \Big] |x-y| \,.$$

Set now

$$L_2(t) \equiv L_0 + \int_0^t M_1(\tau) \frac{1}{2} \frac{d}{d\tau} \Big(\int_0^\tau L_1(s)ds \Big)^2 d\tau.$$

Moreover, we remark that

$$|v_2(x,t) - v_2(y,t)| \le \Big[M_0 + \int_0^t L_1(\tau) \frac{1}{2} \frac{d}{d\tau} \Big(\int_0^\tau M_1(s)ds \Big)^2 d\tau \Big] |x-y|,$$

and set

$$M_2(t) \equiv M_0 + \int_0^t L_1(\tau)\frac{1}{2}\frac{d}{d\tau}\Big(\int_0^\tau M_1(s)ds\Big)^2 d\tau.$$

We define for all n and $t > 0$:

$$\begin{aligned}
A_{n+1}(t) &= \int_0^t \Big(A_n(\tau) + L_{n-1}(\tau)[B_n(\tau) + M_{n-1}(\tau)\int_0^\tau A_n(s)ds]\Big)d\tau;\\
B_{n+1}(t) &= \int_0^t \Big(B_n(\tau) + M_{n-1}(\tau)[A_n(\tau) + L_{n-1}(\tau)\int_0^\tau B_n(s)ds]\Big)d\tau;\\
L_{n+1}(t) &= L_0 + \int_0^t M_n(\tau)\frac{1}{2}\frac{d}{d\tau}(\int_0^\tau L_n(s)ds)^2 d\tau;\\
M_{n+1}(t) &= M_0 + \int_0^t L_n(\tau)\frac{1}{2}\frac{d}{d\tau}(\int_0^\tau M_n(s)ds)^2 d\tau.
\end{aligned}$$

By induction, it is easily to prove that for all $x \in \mathbb{R},\ t > 0$,

$$|u_{n+1}(x,t) - u_n(x,t)| \le A_{n+1}(t) \tag{2.3}$$

$$|v_{n+1}(x,t) - v_n(x,t)| \le B_{n+1}(t) \tag{2.4}$$

and, for all $x, y \in \mathbb{R},\ t > 0$,

$$|u_{n+1}(x,t) - u_{n+1}(y,t)| \le L_{n+1}(t)|x-y| \tag{2.5}$$

$$|v_{n+1}(x,t) - v_{n+1}(y,t)| \le M_{n+1}(t)|x-y| \tag{2.6}$$

In a very simple way we can prove also that for all $x \in \mathbb{R},\ t > 0$,

$$|u_{n+1}(x,t)| \le e^t\|u_0\|_\infty \tag{2.7}$$

$$|v_{n+1}(x,t)| \le e^t\|v_0\|_\infty \tag{2.8}$$

Since

$$0 \le L_1(t) = L_0 + M_0L_0^2t^2/2$$
$$0 \le M_1(t) = M_0 + L_0M_0^2t^2/2,$$

we can choose $T_0 > 0$ and $h > 0$ such that $2h < 1$ and for all $t \in [0, T_0]$:

$$L_0^2\frac{t^2}{2} \le 1,$$
$$M_0^2\frac{t^2}{2} \le 1,$$
$$(M_0+L_0)^3\frac{t^2}{2} \le M_0 \wedge L_0,$$
$$0 \le (M_0+L_0+1)t + (M_0+L_0)^2\frac{t^2}{2} \le h.$$

Then $0 \le L_1(t),\ M_1(t) \le M_0 + L_0 \equiv K_0$ for all $t \in [0, T_0]$.

From the previous definitions we deduce:

$$0 \le L_2(t) \le L_0 + \int_0^t M_1(\tau)\frac{1}{2}\frac{d}{d\tau}\Big(\int_0^\tau L_1(s)ds\Big)^2 d\tau \le L_0 + K_0^3\frac{t^2}{2},$$
$$0 \le M_2(t) \le M_0 + \int_0^t L_1(\tau)\frac{1}{2}\frac{d}{d\tau}(\int_0^\tau M_1(s)ds)^2 d\tau \le M_0 + K_0^3\frac{t^2}{2}.$$

Then we have

$$0 \le L_2(t), \quad M_2(t) \le K_0 \quad \forall t \in [0, T_0],$$

and hence, by induction,

$$0 \le L_n(t), \quad M_n(t) \le M_0 + L_0 \equiv K_0 \quad \forall t \in [0, T_0]. \tag{2.9}$$

From the definitions of A_n e B_n, we deduce

$$0 \le A_{n+1}(t) \le \int_0^t \Big(A_n(\tau) + K_0 B_n(\tau) + K_0^2 \int_0^\tau A_n(s) ds \Big) d\tau;$$
$$0 \le B_{n+1}(t) \le \int_0^t \Big(A_n(\tau) + K_0 A_n(\tau) + K_0^2 \int_0^\tau B_n(s) ds \Big) d\tau.$$

For the continuity of A_n and B_n in $[0, T_0]$, we deduce:

$$0 \le A_{n+1}(t) \le \|A_n\|_\infty \Big(t + K_0^2 \frac{t^2}{2} \Big) + K_0 t \|B_n\|_\infty;$$
$$0 \le B_{n+1}(t) \le \|B_n\|_\infty \Big(t + K_0^2 \frac{t^2}{2} \Big) + K_0 t \|A_n\|_\infty.$$

Now, for all $t \in [0, T_0]$,

$$0 \le A_{n+1}(t); \quad B_{n+1}(t) \le h(\|A_n\|_\infty + \|B_n\|_\infty)$$

Hence, taking the supremum over t and adding the inequalities, we deduce that the series

$$\sum (\|A_n\|_\infty + \|B_n\|_\infty)$$

is convergent; then the same holds for both the series $\sum \|A_n\|_\infty$ and $\sum \|B_n\|_\infty$.

We remember that $L^\infty(\mathbb{R} \times [0, T_0]; \mathbb{R})$ is a complete metric space with respect to lagrangian metric; then from the inequalities (2.3), (2.4), applying the Banach-Caccioppoli theorem, we have that $(u_n)_n$ and $(v_n)_n$ are Cauchy sequences. Hence there exist two real functions u^* and v^*, defined in $\mathbb{R} \times [0, T_0]$ such that: $(u_n)_n$ is uniformly convergent to u^* and $(v_n)_n$ is uniformly convergent to v^* in $\mathbb{R} \times [0, T_0]$; moreover, from (2.7),(2.8) and (2.9), u^* and v^* are Lipschitz continuous in all the variables.

We remark that, for all $n \in N$, $x \in \mathbb{R}$, $t \in [0, T_0]$:

$$\begin{aligned} &\Big| u_n\Big(v_n\Big(\int_0^t u_n(x,\tau) d\tau, t \Big), t \Big) - u^*\Big(v^*\Big(\int_0^t u^*(x,\tau) d\tau, t \Big), t \Big) \Big| \\ &\le \|u_n - u^*\|_\infty + K_0 \|v_n - v^*\|_\infty + K_0^2 t \|u_n - u^*\|_\infty. \end{aligned}$$

Then u^* and v^* verify that for all $x \in \mathbb{R}$ and $t \in [0, T_0]$:

$$u^*(x,t) = u_0(x) + \int_0^t u^*\Big(v^*(\int_0^\tau u^*(x,s) ds, \tau), \tau \Big) d\tau,$$
$$v^*(x,t) = v_0(x) + \int_0^t v^*\Big(u^*(\int_0^\tau v^*(x,s) ds, \tau), \tau \Big) d\tau,$$

respectively. Let us now prove the uniqueness. Let (u_*, v_*) be another pair of solutions and remark that:

$$
\begin{aligned}
&\Big|u^*\Big(v^*(\int_0^\tau u^*(x,s)ds,\tau),\tau\Big) - u_*\Big(v_*(\int_0^\tau u_*(x,s)ds,\tau),\tau\Big)\Big| \\
&\le K_0\Big|v^*(\int_0^\tau u^*(x,s)ds,\tau) - v_*(\int_0^\tau u_*(x,s)ds,\tau)\Big| \\
&\quad + \Big|u^*\Big(v_*(\int_0^\tau u_*(x,s)ds,\tau),\tau\Big) - u_*\Big(v_*(\int_0^\tau u_*(x,s)ds,\tau),\tau\Big)\Big| \\
&\le K_0\Big(K_0\Big|\int_0^\tau u^*(x,s)ds - \int_0^\tau u_*(x,s)ds\Big| \\
&\quad + \Big|v^*(\int_0^\tau u^*(x,s)ds,\tau) - v_*(\int_0^\tau u^*(x,s)ds,\tau))\Big| + \|u^* - u_*\|_\infty \\
&\le (1 + K_0^2 t)\|u^* - u_*\|_\infty + K_0\|v^* - v_*\|_\infty.
\end{aligned}
$$

Therefore,

$$
|u^*(x,\tau) - u_*(x,\tau)| \le (t + K_0^2\frac{t^2}{2})\|u^* - u_*\|_\infty + K_0 t\|v^* - v_*\|_\infty.
$$

In a similar way we can prove the estimates:

$$
\begin{aligned}
|u^*(x,\tau) - u_*(x,\tau)| &\le (t + K_0^2\frac{t^2}{2})\|u^* - u_*\|_\infty + K_0 t\|v^* - v_*\|_\infty, \\
|v^*(x,\tau) - v_*(x,\tau)| &\le (t + K_0^2\frac{t^2}{2})\|v^* - v_*\|_\infty + K_0 t\|u^* - u_*\|_\infty.
\end{aligned}
$$

Then

$$
|u^*(x,\tau) - u_*(x,\tau)| \le (t + K_0^2\frac{t^2}{2} + K_0 t)\max(\|u^* - u_*\|_\infty; \|v^* - v_*\|_\infty),
$$

and

$$
|v^*(x,\tau) - v_*(x,\tau)| \le (t + K_0^2\frac{t^2}{2} + K_0 t)\max(\|v^* - v_*\|_\infty; \|u^* - u_*\|_\infty).
$$

From $(t + K_0^2\frac{t^2}{2} + K_0 t) \le h < 1$, we have

$$
\max(\|u^* - u_*\|_\infty; \|v^* - v_*\|_\infty) < h\max(\|u^* - u_*\|_\infty; \|v^* - v_*\|_\infty).
$$

Then the uniqueness follows and the proof is complete. □

3. Some Open Problems

The previous results and the proposed type of systems can be investigated and generalized in many different directions. In what follows, we give some of the problems whose investigation seems to be interesting.

(A) The first problem is to investigate the existence of global solutions, also for Lipschitzian and bounded initial data.

(B) It could be more difficult to establish existence and uniqueness for data u_0, v_0 bounded and uniformly continuous (or simply continuous). Moreover, when the global existence is guaranteed, an interesting problem can be to give particular condition on data u_0, v_0 such that there exists $T^* > 0$ for which $u(x,t) = v(x,t)$ for all $x \in \mathbb{R}$ and $t \ge T^*$.

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