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On Fuzzy Soft Semigroups

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Abstract: In this paper we have introduced the concept of fuzzy soft semigroup which is a generalization of soft semigroup and studied some basic properties. We have defined the product of two fuzzy soft sets over a semigroup, fuzzy soft semigroups and fuzzy soft ideals and investigated their basic properties. We have also discussed the notions of fuzzy soft quasi-ideals, bi-ideals, generalized bi-ideals and interior ideals defined over a semigroup.

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INTRODUCTION

L.A. Zadeh [1] introduced the notion of a fuzzy set in 1965. What Zadeh proposed is very much a paradigm shift that first gained acceptance in the Far East and its successful application has ensured the adoption around the world. Fuzzy set theory was guided by the assumption that classical sets were not natural, appropriate or useful notions in describing the real life problems, because every object encountered in this real physical world carries some degree of fuzziness. Zadeh's ideas marked a new direction and fuzzy set theoretic approach was adopted by Rosenfeld *et al.* [2] for groups. Rosenfeld's paper inspired the development of fuzzy abstract algebras. He also introduced fuzzy graphs, an area which has been growing actively since then. Later authors like N. Kuroki [3], J. Ahsan [4, 5] and many others applied the concept of fuzzy sets for studies in fuzzy semigroups, fuzzy groups, fuzzy rings, fuzzy ideals, fuzzy semirings and fuzzy near-rings and so on.

In 1999, Molodtsov [6] initiated a novel concept of soft set theory, which was a completely new approach for modeling uncertainty and had a rich potential for applications in several directions. Furthermore Maji *et al.* [7] worked on soft set theory in 2003. They also presented the definition of fuzzy soft set [8] in 2001. Feng Feng [9] initiated the study of soft semirings and soft ideals over semirings in 2008. M. I. Ali *et al.* [10, 11] introduced new operations on soft sets, soft semigroups and soft ideals and generalized fuzzy ideals of semigroups in 2009. Another paper on fuzzy soft semigroups by Cheng-fu Yang [12] has been recently published which partially covers the theory of ideals. In [12] the notions of union and intersection are used from [8]. Ali *et al.* have pointed out some basic problems with these operations and introduced new operations on fuzzy soft sets in [13]. In our present paper, we have used these new operations on fuzzy soft sets which avoid the limitations of working with the older ones. We have observed that the choice of fuzzy soft operations can be very significant in certain cases and at some points; we may have generalizations of analogous results in soft set theory, or, more specifically in the theory of semigroups. A number of new results are particularly obtained after the application of newly defined operations. It is tried to give a complete account of ideal theory in fuzzy soft semigroups.

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Firstly, we have presented preliminaries on the theory of fuzzy sets, soft sets, fuzzy soft sets and soft semigroups. Next, we have developed the theory of fuzzy soft semigroups. We have given the product of two fuzzy soft sets over a semigroup. There are two different types of products of two fuzzy soft sets; one is extended while the other is restricted. We have observed that the restricted product coincides with the extended product, if the parameter set is same for both fuzzy soft sets. After defining the products, we have studied their properties in fuzzy

soft structures. We see that, the products are not fully distributed over the fuzzy soft operations of union and intersection. Counter examples are given to show the improperness of expressions. We have established the respective laws for distributivity of products over the operations. Cartesian product of two fuzzy soft sets is also defined. Again, there are two types of Cartesian products i.e. Internal Cartesian product and External Cartesian product. Both products can be used according to the requirements of any given situation. Then we have defined fuzzy soft semigroup and fuzzy soft subsemigroup and proved some basic properties of fuzzy soft semigroups. At the end of this section, an example is given to show that the fuzzy soft unions of fuzzy soft semigroups may not be a fuzzy soft semigroup. The last section provides the elaborated notions of fuzzy soft ideals, fuzzy soft quasi-ideals, fuzzy soft bi-ideals, fuzzy soft generalized bi-ideals and fuzzy soft interior ideals over a semigroup. Many interesting results are derived through the theory development and a number of examples are constructed.

PRELIMINARIES

Let X be a set. A fuzzy subset of X is a function from X into the unit closed interval [0,1]. The set of all fuzzy subsets of X is called the fuzzy power set of X and is denoted by FP(X). Let μ and υ be two fuzzy subsets of a semigroup S. Then the product $\mu\circ\upsilon$ is defined by

$$(\mu\circ\upsilon)(x)=\begin{cases}\vee_{x=yz}\{\mu(y)\wedge\upsilon(z)\} & \text{if } \exists y, z\in S, \text{ such that } x=yz\\ 0 & \text{otherwise}\end{cases}$$

for all $x\in S$. The operation $\circ$ is associative.

Definition 1: [3] Let $\mu, \upsilon\in FP(X)$. If $\mu(x)\leq\nu(x)$ for all $x\in X$, then μ is said to be contained in υ and we write $\mu\subseteq\upsilon$(or $\upsilon\supseteq\mu$).

Clearly, the inclusion relation $\subseteq$ is a partial order on FP(X).

Definition 2: [3] Let $\mu, \upsilon\in FP(X)$. Then $\mu\vee\upsilon$ and $\mu\wedge\upsilon$ are fuzzy subsets of X, defined as follows:
For all $x\in X$,

$$(\mu\vee\upsilon)(x)=\mu(x)\vee\upsilon(x)$$
$$(\mu\wedge\upsilon)(x)=\mu(x)\wedge\upsilon(x)$$

The fuzzy subsets $\mu\vee\upsilon$ and $\mu\wedge\upsilon$ are called the union and intersection of μ and υ, respectively.

Definition 3: [3] A fuzzy subset μ of S is called a fuzzy subsemigroup of a semigroup S if

$$m(ab)\geq m(a)m(b)\ forall\ a,b\in S$$

Definition 4: [3] A fuzzy subset μ of a semigroup S is called a fuzzy left (right) ideal of S if

$$m(ab) \geq m(b)\big(m(ab) \geq m(a)\big) \; for all a, b \in S.$$

A fuzzy subset μ of a semigroup S is called a fuzzy ideal of S if it is both a fuzzy left and a fuzzy right ideal of S.

Definition 5: [3] A fuzzy subset μ of a semigroup S is called a fuzzy quasi-ideal of S if

$$(\mathbf{S} \circ \mu) \wedge (\mu \circ \mathbf{S}) \subseteq \mu$$

Definition 6: [3] A fuzzy subsemigroup μ of a semigroup S is called a fuzzy bi-ideal of S if

$$m(xyz) \geq m(x)m(z) \; for all x, y, z \in S.$$

Definition 7: [3] A fuzzy subset μ of a semigroup S is called a fuzzy generalized bi-ideal of S if

$$m(xyz) \geq m(x)m(z) \; for all x, y, z \in S.$$

Definition 8: [3] A fuzzy subset μ of a semigroup S is called a fuzzy interior ideal of S if

$$m(xay) \geq m(a) \; for all x, a, y \in S.$$

We refer to [1-5, 14] for further terms, notions and results on semigroups and fuzzy semigroups which are used in this paper. In the following, we present preliminaries on the theory of soft sets, fuzzy soft sets and soft semigroups which are taken from [7-11, 13].

Let U be an initial universe and E be a set of parameters. Let P(U) denotes the power set of U and A, B be non-empty subsets of E.

Definition 9: [6] A pair (F,A) is called a soft set over U, where F is a mapping given by $F: A \rightarrow \mathbf{P}(U)$.

In other words, a soft set over U is a parameterized family of subsets of the universe U. For e∈ A, F(e) may be considered as the set of e-approximate elements of the soft set (F,A). Clearly, a soft set is not a set.

Definition 10: [9] For two soft sets (F,A) and (G,B) over a common universe U, we say that (F,A) is a soft subset of (G,B) if

1) $A \subseteq B$ and
2) $F(e) \subseteq G(e)$ for all e∈ A.

We write $(F,A)\widehat{\subset}(G,B)$. (F,A) is said to be a soft super set of (G,B), if (G,B) is a soft subset of (F,A). We denote it by $(F,A)\widehat{\supset}(G,B)$.

Definition 11: [7] Two soft sets (F,A) and (G,B) over a common universe U are said to be soft equal if (F,A) is a soft subset of (G,B) and (G,B) is a soft subset of (F,A).
Let FP(U) denotes the set of all fuzzy subsets of U.

Definition 12: [8] A pair (F,A) is called a fuzzy soft set over U, where F is a mapping given by $F: A \rightarrow \mathbf{FP}(U)$.

Definition 13: [8] For two fuzzy soft sets (F,A) and (G,B) over a common universe U, we say that (F,A) is a fuzzy soft subset of (G,B) if

1) $A \subseteq B$;
2) for all e∈ A, F(e) is a fuzzy subset of G(e).

We write $(F,A)\widehat{\subset}(G,B)$. (F,A) is said to be a fuzzy soft super set of (G,B), if (G,B) is a fuzzy soft subset of (F,A). We denote it by $(F,A)\widehat{\supset}(G,B)$.

Definition 14: [8] Two fuzzy soft sets (F,A) and (G,B) over a common universe U are said to be fuzzy soft equal if (F,A) is a fuzzy soft subset of (G,B) and (G,B) is a fuzzy soft subset of (F,A).

Definition 15: [8] If (F,A) and (G,B) are two fuzzy soft sets over the same universe U then "(F,A) AND (G,B) " is a fuzzy soft set denoted by (F,A)$\wedge$(G,B) and is defined by $(F,A)(G,B)=(H,A\times B)$ where, $H((\alpha,\beta))=F(\alpha)\wedge G(\beta)$ for all $(\alpha,\beta)\in A\times B$. Here $\wedge$ is the operation of fuzzy intersection of two fuzzy sets.

Definition 16: [8] If (F,A) and (G,B) are two fuzzy soft sets over the same universe U then "(F,A) OR (G,B) " is a fuzzy soft set, denoted by (F,A)$\vee$(G,B) and is defined by $(F,A)\vee(G,B)=(O,A\times B)$ where, $O((\alpha,\beta))=F(\alpha)\vee G(\beta)$ for all $(\alpha,\beta)\in A\times B$. Here $\vee$ is the operation of fuzzy union of two fuzzy sets.

Definition 17: [8] Union of two fuzzy soft sets (F,A) and (G,B) over the common univers U is the fuzzy soft set (H,C), where C = A$\cup$B and for all e$\in$C,

$$H(e)=\begin{cases} F(e) & \text{if } e\in A-B \\ G(e) & \text{if } e\in B-A \\ F(e)\vee G(e) & \text{if } e\in A\cap B \end{cases}$$

We write $(F,A)\widetilde{\cup}(G,B)=(H,C)$.

Now we are considering the following definitions from [13].

Definition 18: [13] The extended intersection of two fuzzy soft sets (F,A) and (G,B) over a common universe U, is the fuzzy soft set (H,C) where C = A$\cup$B and for all e$\in$C,

$$H(e)=\begin{cases} F(e) & \text{if } e\in A-B \\ G(e) & \text{if } e\in B-A \\ F(e)\wedge G(e) & \text{if } e\in A\cap B \end{cases}$$

We write $(F,A)\cap_{\varepsilon}(G,B)=(H,C)$.

Definition 19: [13] If (F,A) and (G,B) are two fuzzy soft sets over a common universe U with $A\cap B\neq\varnothing$ then their restricted intersection is a fuzzy soft set (H, A$\cap$B) denoted by $(F,A)\cap_R(G,B)=(H,A\cap B)$ where, H is a function from A$\cap$B to FP(U) defined as $H(e)=F(e)\wedge G(e)$ for all $e\in A\cap B$.

Definition 20: [13] If (F,A) and (G,B) are two fuzzy soft sets over the same universe U such that $A\cap B\neq\varnothing$ then their restricted union is denoted by $(F,A)\cup_R(G,B)$ and is defined as $(F,A)\cup_R(G,B)=(H,C)$ where C = A$\cap$B and for all e$\in$C,

$$H(e)=F(e)\vee G(e)$$

Definition 21: [13] Let U be an initial universe set, E be the set of parameters and A, B be the non-empty subsets of E. Then (F,A) is called a relative null fuzzy soft set (with respect to the parameter set A), denoted by Φ_A, if $F(e)=\varnothing$ for all e$\in$A.

(G,A) is called a relative whole fuzzy soft set (with respect to the parameter seta A), denoted by U_A, if G(e) = U for all e$\in$A, where U is the fuzzy subset of U mapping every element of U on 1.

The relative whole fuzzy soft set U_E with respect to the universe set of parameters E is called the absolute fuzzy soft set over U. We also write it as (U,E).

Definition 22: [11] The restricted product (H,C) of two soft sets (F,A) and (G,B) over a semigroup S is defined as the soft set $(H,C) = (F,A)\hat{\circ}(G,B)$ where C = A$\cap$B is non-empty and H is a set valued function from C to P(S) defined as $H(c) = F(c)G(c)$ for all c$\in$ C.

Definition 23: [11] A soft set (F,A) over S is called a soft semigroup over S if

$$(F,A)\hat{\circ}(F,A)\widetilde{\subset}(F,A)$$

It is easy to see that a soft set (F,A) over S is a soft semigroup if and only if $\varnothing \neq F(a)$ is a subsemigroup of S, for all a$\in$ A. [11].

Definition 24: [11] A soft set (F,A) over a semigroup S is called a soft left (right) ideal overS if

$$(S,E)\hat{\circ}(F,A)\widetilde{\subset}(F,A)((F,A)\hat{\circ}(S,E)\widetilde{\subset}(F,A))$$

A soft set over S is a soft ideal if it is both a soft left and a soft right ideal over S.

FUZZY SOFT SEMIGROUPS

In this section, we introduce the fuzzy soft sets which are defined over a semigroup and study some of their basic properties. First we define the operations of multiplication for two fuzzy soft sets over a semigroup and then we discuss fuzzy soft semigroups and fuzzy soft subsemigroups. Fuzzy soft ideals are defined over a semigroup and ideals, quasi-ideals, bi-ideals, generalized bi-ideals and interior ideals are studied with the basic results for fuzzy soft sets.

Throughout this section S and T will denote semigroups unless stated otherwise. Let S be a semigroup and let A, B, C be non-empty subsets of E, where E is the set of parameters. Now we define the restricted, extended and Cartesian products of two fuzzy soft sets over a semigroup. We also establish the Associative laws for these operations and Distributive laws of restricted and extended multiplication, over the operations of union and intersection.

Definition 25: If $A \cap B \neq \varnothing$ then the restricted product (H,C) of two fuzzy soft sets (F,A) and (G,B) over a semigroup S is defined as the fuzzy soft set, $(H,A \cap B)$ denoted by $(F,A)\tilde{o}(G,B)$ where H is a function from A$\cap$B to FP(S) defined by, $H(e) = F(e) \circ G(e)$, for all $e \in A \cap B$. Here $F(e) \circ G(e)$ is the product of two fuzzy subsets of the semigroup S.

Proposition 1: The operation "$\tilde{o}$" is associative.

Proof: Straightforward.

Definition 26: The extended product of two fuzzy soft sets (F,A) and (G,B) over S is defined as the fuzzy soft set, $(H,C)=(F,A)\tilde{o}_{\varepsilon}(G,B)$ where C = A∪B and for all e∈C.

$$H(e)=\begin{cases} F(e) & \text{if } e\in A?B \\ G(e) & \text{if } e\in B?A \\ F(e)\circ G(e) & \text{if } e\in A\cap B \end{cases}$$

Here $F(e)\circ G(e)$ is the product of two fuzzy subsets of the semigroup S.

Proposition 2: The operation "$\tilde{o}_{\varepsilon}$ " is associative.

Proof: Straightforward.

If FSS(S)^E denotes the collection of all fuzzy soft sets which are defined over S, where E is the set of parameters, then (FSS(S)^E, $\tilde{o}_{\varepsilon}$) becomes a semigroup. We note that the operation "$\tilde{o}_{\varepsilon}$ " is only a partial operation on FSS(S)^E.

Theorem 1: Let (F,A) and (G,B) and (H,C) be any fuzzy soft sets over S where A, B and C are suitably chosen subsets of E. Then

$$(F,A)\tilde{o}((G,B)\overset{\frown}{\cup}(H,C))=((F,A)\tilde{o}(G,B))\overset{\frown}{\cup}((F,A)\tilde{o}(H,C));$$
$$(F,A)\tilde{o}((G,B)\cup_R(H,C))=((F,A)\tilde{o}(G,B))\cup_R((F,A)\tilde{o}(H,C));$$
$$(F,A)\tilde{o}_{\varepsilon}((G,B)\cup_R(H,C))\overset{\frown}{\subset}((F,A)\tilde{o}_{\varepsilon}(G,B))\cup_R((F,A)\tilde{o}_{\varepsilon}(H,C));$$
$$(F,A)\tilde{o}_{\varepsilon}((G,B)\overset{\frown}{\cup}(H,C))\overset{\frown}{\subset}((F,A)\tilde{o}_{\varepsilon}(G,B))\overset{\frown}{\cup}((F,A)\tilde{o}_{\varepsilon}(H,C));$$
$$(F,A)\overset{\frown}{\cup}(G,B))\tilde{o}(H,C)=((F,A)\tilde{o}(H,C))\overset{\frown}{\cup}((G,B)\tilde{o}(H,C));$$
$$(F,A)\cup_R(G,B))\tilde{o}(H,C)=((F,A)\tilde{o}(H,C))\cup_R((G,B)\tilde{o}(H,C));$$
$$(F,A)\cup_R(G,B))\tilde{o}_{\varepsilon}(H,C)\overset{\frown}{\subset}((F,A)\tilde{o}_{\varepsilon}(H,C))\cup_R((G,B)\tilde{o}_{\varepsilon}(H,C));$$
$$(F,A)\overset{\frown}{\cup}(G,B))\tilde{o}_{\varepsilon}(H,C)\overset{\frown}{\subset}((F,A)\tilde{o}_{\varepsilon}(H,C))\overset{\frown}{\cup}((G,B)\tilde{o}_{\varepsilon}(H,C)).$$

Proof: By applying the definitions of operations, we get the proof.

Example 1: Let S be a semigroup of four elements {a,b,c,d} with the following multiplication table

.	a	b	c	d
a	a	a	a	a
b	a	a	a	a
c	a	a	b	a
d	a	a	b	b

Let E = {e_1, e_2, e_3, e_4, e_5} be a set of parameters. Suppose that A = {e_1, e_2, e_3}, B = {e_2, e_3, e_4} and C = {e_3, e_4, e_5}. Let (F,A) and (G,B) and (H,C) be the fuzzy soft sets over S defined by the following tables

F	e_1	e_2	e_3
a	0.7	0.6	0.8
b	0	0	0.2
c	0.3	0.3	0.7
d	0	0.4	0.1

G	e_2	e_3	e_4
a	0.6	0.8	0.3
b	0	0.2	0.9
c	0.3	0.7	0.1
d	0.4	0.1	0

H	e_3	e_4	e_5
a	0.7	0.3	0.6
b	0	0.9	0
c	0.3	0.1	0.3
d	0	0	0.4

Let

$$(F,A)\tilde{o}_{\varepsilon}((G,B)\cup_R(H,C))=(F_1,A\cup(B\cap C))$$

and

$$((F,A)\tilde{o}_{\varepsilon}(G,B))\cup_R((F,A)\tilde{o}_{\varepsilon}(H,C))=(G_1,(A\cup B)\cap(A\cup C))$$

Then

F_1	e_1	e_2	e_3	e_4
a	0.7	0.6	0.8	0.3
b	0	0	0.7	0.9
c	0.3	0.3	0	0.1
d	0	0.4	0	0

,

G_1	e_1	e_2	e_3	e_4
a	0.7	0.6	0.8	0.3
b	0	0.4	0.7	0.9
c	0.3	0.3	0	0.1
d	0	0.4	0	0

From above tables we see that

$$F_1(e_1) = G_1(e_1),\ F_1(e_2) \subseteq G_1(e_2),\ F_1(e_3) = G_1(e_3) \text{ and } F_1(e_4) = G_1(e_4). \text{ Thus}$$

$$(F,A)\tilde{o}_{\varepsilon}((G,B) \cup_R (H,C)) \neq ((F,A)\tilde{o}_{\varepsilon}(G,B)) \cup_R ((F,A)\tilde{o}_{\varepsilon}(H,C))$$

Let

$$(F,A)\tilde{o}_{\varepsilon}((G,B)\overset{\frown}{\cup}(H,C)) = (F_2, A \cup (B \cup C))$$

and

$$((F,A)\tilde{o}_{\varepsilon}(G,B))\overset{\frown}{\cup}((F,A)\tilde{o}_{\varepsilon}(H,C)) = (G_2, (A \cup B) \cup (A \cup C))$$

Then

F_2	e_1	e_2	e_3	e_4	e_5
a	0.7	0.6	0.8	0.3	0.6
b	0	0.4	0.7	0.9	0
c	0.3	0	0	0.1	0.3
d	0	0	0	0	0.4

,

G_2	e_1	e_2	e_3	e_4	e_5
a	0.7	0.6	0.8	0.3	0.6
b	0	0.4	0.7	0.9	0
c	0.3	0.3	0	0.1	0.3
d	0	0.4	0	0	0.4

From above tables we see that

$$F_2(e_1) = G_2(e_1),\ F_2(e_2) \subseteq G_2(e_2),\ F_2(e_3) = G_2(e_3) \text{ and } F_2(e_4) = G_2(e_4).$$

Thus

$$(F,A)\tilde{o}_{\varepsilon}((G,B)\overset{\frown}{\cup}(H,C)) \neq ((F,A)\tilde{o}_{\varepsilon}(G,B))\overset{\frown}{\cup}((F,A)\tilde{o}_{\varepsilon}(H,C))$$

Proposition 3: Let (F,A) and (G,B) and (H,C) be any fuzzy soft sets over S where A, B and C are suitably chosen subsets of E. Then

1. $(F,A)\tilde{o}((G,B) \cap_R (H,C)) \overset{\frown}{\subset} ((F,A)\tilde{o}(G,B)) \cap_R ((F,A)\tilde{o}(H,C));$
2. $(F,A)\tilde{o}((G,B) \cap_{\varepsilon} (H,C)) \overset{\frown}{\subset} ((F,A)\tilde{o}(G,B)) \cap_{\varepsilon} ((F,A)\tilde{o}(H,C));$
3. $((F,A) \cap_R (G,B))\tilde{o}(H,C) \overset{\frown}{\subset} ((F,A)\tilde{o}(H,C)) \cap_R ((G,B)\tilde{o}(H,C));$
4. $((F,A) \cap_{\varepsilon} (G,B))\tilde{o}(H,C) \overset{\frown}{\subset} ((F,A)\tilde{o}(H,C)) \cap_{\varepsilon} ((G,B)\tilde{o}(H,C)).$

Proof: Let

$$(G,B) \cap_R (H,C) = (F_1, B \cap C)$$

and

$$(F,A)\tilde{o}((G,B) \cap_R (H,C)) = (F_2, A \cap (B \cap C))$$

Then

$$F_1(e) = G(e) \wedge H(e) \text{ for all } e \in B \cap C$$

and

$$\begin{aligned} F_2(e) &= F(e) \circ F_1(e) = F(e) \circ (G(e) \wedge H(e)) \subseteq (F(e) \circ G(e)) \wedge (F(e) \circ G(e)) \\ &= G_1(e) \text{ for all } e \in A \cap (B \cap C) = (A \cap B) \cap (A \cap C) \end{aligned}$$

where

$$((F,A)\tilde{o}(G,B)) \cap_R ((F,A)\tilde{o}(H,C)) = (G_1, ((A \cap B) \cap (A \cap C))$$

Thus

$$(F,A)\tilde{o}((G,B)\cap_R (H,C))\widehat{\subset}((F,A)\tilde{o}(G,B))\cap_R ((F,A)\tilde{o}(H,C))$$

Similarly, we can prove 2, 3 and 4.

Example 2: Let S be a semigroup of four elements {a,b,c,d} with the following multiplication table

.	a	b	c	d
a	a	a	a	a
b	a	a	a	a
c	a	a	b	a
d	a	a	b	b

Let E = {e_1, e_2, e_3, e_4} be a set of parameters.

Suppose that A = {e_1, e_2, e_3}, B = {e_2, e_3} and C = {e_3, e_4, e_5}. Let (F,A) and (G,B) and (H,C) be the fuzzy soft sets over S defined by the following tables:

F	e_1	e_2	e_3
a	0.1	0.4	0.2
b	0.7	0.5	0.5
c	0.3	0.6	0.7
d	0.4	0.9	0.8

G	e_2	e_3
a	0.1	0.6
b	0.3	0.7
c	0.4	0.8
d	0.5	0.2

H	e_2	e_3	e_4
a	0.1	0.3	0.2
b	0.2	0.4	0.9
c	0.4	0.5	0.7
d	0.2	0.7	0.3

Let

$$(F,A)\tilde{o}((G,B)\cap_R (H,C)) = (F_1, A\cap(B\cap C))$$

and

$$((F,A)\tilde{o}(G,B))\cap_R ((F,A)\tilde{o}(H,C)) = (G_1,(A\cup B)\cap(A\cup C))$$

Then

F_1	e_2	e_3
a	0.4	0.4
b	0.4	0.5
c	0	0
d	0	0

G_1	e_2	e_3
a	0.4	0.7
b	0.4	0.7
c	0	0
d	0	0

From above tables we see that $F_1(e_2) = G_1(e_2)$ and $F_1(e_3) \subseteq G_1(e_3)$. Thus

$$(F,A)\tilde{o}((G,B)\cap_R (H,C)) \neq ((F,A)\tilde{o}(G,B))\cap_R ((F,A)\tilde{o}(H,C))$$

Now let

$$(F,A)\tilde{o}((G,B)\cap_\varepsilon (H,C)) = (F_2, A\cap(B\cup C))$$

and

$$((F,A)\tilde{o}(G,B))\cap_\varepsilon ((F,A)\tilde{o}(H,C)) = (G_2,(A\cap B)\cup(A\cap C))$$

Then

F_2	e_2	e_4
a	0.4	0.4
b	0.4	0.5
c	0	0
d	0	0

G_2	e_2	e_3
a	0.4	0.7
b	0.4	0.7
c	0	0
d	0	0

From above tables we see that $F_2(e_2) = G_2(e_2)$ and $F_2(e_3) \subseteq G_2(e_3)$. Thus

$$(F,A)\tilde{o}((G,B) \cap_\varepsilon (H,C)) \neq ((F,A)\tilde{o}(G,B)) \cap_\varepsilon ((F,A)\tilde{o}(H,C))$$

Lemma 1: Let (F,A) and (G,B) and (H,C) be any fuzzy soft sets over S. If $(F,A)\widetilde{\subset}(G,B)$. Then

$$(H,C)\tilde{o}(F,A)\widetilde{\subset}(H,C)\tilde{o}(G,B);$$
$$(H,C)\tilde{o}_\varepsilon(F,A)\widetilde{\subset}(H,C)\tilde{o}_\varepsilon(G,B);$$
$$(F,A)\tilde{o}(H,C)\widetilde{\subset}(G,B)\tilde{o}(H,C);$$
$$(F,A)\tilde{o}_\varepsilon(H,C)\widetilde{\subset}(G,B)\tilde{o}_\varepsilon(H,C).$$

Proof: Let $(H,C)\tilde{o}(F,A) = (F_1, A \cap C)$. Then $F_1(e) = H(e) \circ F(e) \subseteq H(e) \circ G(e)$ for all $e \in A \cap C$ because $F(e) \subseteq G(e)$ for all $e \in A \subseteq B$. This implies that

$$(H,C)\tilde{o}(F,A)\widetilde{\subset}(H,C)\tilde{o}(G,B)$$

Let $(H,C)\tilde{o}_\varepsilon(F,A) = (F_1, A \cup C)$, then

$$F_1(e) = \begin{cases} H(e) & \text{if } e \in C - A \\ F(e) & \text{if } e \in A - C \\ H(e) \circ F(e) & \text{if } e \in C \cap A \end{cases}$$
$$\subseteq \begin{cases} H(e) & \text{if } e \in C - B \\ G(e) & \text{if } e \in B - C \\ H(e) \circ G(e) & \text{if } e \in C \cap B \end{cases}$$

because $F(e) \subseteq G(e)$ for all $e \in A \subseteq B$. This implies that $(H,C)\tilde{o}_\varepsilon(F,A)\widetilde{\subset}(H,C))\tilde{o}_\varepsilon(G,B)$. Similarly we can prove (3) and (4).

Definition 27: Let E, E' be the sets of parameters and A, B be the non-empty subsets of E and E' respectively. The External product of two fuzzy soft sets (F,A) and (G,B) over the semigroups S and T respectively, is the fuzzy soft set (H,C) over S×T denoted by (F,A)×(G,B) and defined as $(H,C) = (F,A) \times (G,B)$ where C = A×B and

$$H((a,b)) = F(a)\widehat{\otimes}G(b) \text{ for all } (a,b) \in A \times B.$$

Here ' $\widehat{\otimes}$ ' is defined as follows:

$$F(a)\widehat{\otimes}G(b)(e,e') = \min\{(F(a))(e), (G(b))(e')\}$$

Definition 28: The Internal product of two fuzzy soft sets (F,A) and (G,B) over the semigroup S is the fuzzy soft set (H,C) over S, denoted by $(F,A)\tilde{o}_\times(G,B)$ and defined as $(H,C) = (F,A)\tilde{o}_\times(G,B)$ where C = A×B and

$$H((a,b)) = F(a) \circ G(b) \text{ for all } (a,b) \in A \times B$$

The Cartesian product is not associative because $(A\times B)\times C \neq A\times(B\times C)$ but if we take

$$(A\times B)\times C = A\times B\times C = A\times(B\times C)$$

Then

$$((F,A)\tilde{o}_{\times}(G,B))\tilde{o}_{\times}(H,C) = (F,A)\tilde{o}_{\times}((G,B)\tilde{o}_{\times}(H,C))$$

Remark: We observe, from above definitions, that the internal product is defined for semigroup structure only, while the external product is defined for fuzzy soft sets in general and it has nothing to do with semigroup structure. Moreover, these notions are new and different from the notions of AND and OR operations of fuzzy soft sets given by Maji *et al.*

Definition 29: A fuzzy soft set (F,A) over a semigroup S is called a fuzzy soft semigroup ove S, if F(e) is a fuzzy subsemigroup of S for all $e\in A$.

Proposition 4: A non-empty fuzzy soft set (F,A) over S is a fuzzy soft semigroup over S if and only if $(F,A)\tilde{o}(F,A)\widetilde{\subset}(F,A)$.

Proof: Let (F,A) be a fuzzy soft semigroup over S and let $(F,A)\tilde{o}(F,A)=(H,A)$. Then $H(e)=F(e)\circ F(e)\subseteq F(e)$ for all $e\in A$, because F(e) is a fuzzy subsemigroup of S. Thus $(F,A)\tilde{o}(F,A)\widetilde{\subset}(F,A)$. Conversely assume that $(F,A)\tilde{o}(F,A)\widetilde{\subset}(F,A)$. Then $F(e)\circ F(e)\subseteq F(e)$ for all $e\in A$. This implies that F(e) is a fuzzy subsemigroup of S. Thus (F,A) is a fuzzy soft semigroup over S.

Definition 30: Let (F,A) and (G,B) be two fuzzy soft semigroups over S. Then (G,B) is said to be a fuzzy soft subsemigroup of (F,A) over S, if (G,B) is a fuzzy soft subset of (F,A) over S.

Remark 2: Note that, if (F,A) is a fuzzy soft semigroup over S then For every non-empty subset B of A, (F,B) is a fuzzy soft subsemigroup of (F,A) over S.

If (G,B) is a fuzzy soft subsemigroup of (F,A) over S and (H,C) is a fuzzy soft subsemigroup of (G,B) over S then (H,C) is a fuzzy soft subsemigroup of (F,A) over S.

Let (G,A) be a fuzzy soft semigroup over S such that $G(e)\subseteq F(e)$ for all $e\in A$, then (G,B) is a fuzzy soft subsemigroup of (F,A) over S.

(F,A) is a fuzzy soft subsemigroup of the absolute fuzzy soft semigroup (S,E) over S.

Proposition 5: Let $\{S_i: i\in I\}$ be a non-empty family of semigroups and $\{(F_i,A_i): i\in I\}$ be a non-empty family of fuzzy soft sets, such that (F_iA_i) is a fuzzy soft semigroup over S_i for each $i\in I$. Then Direct product (External) $\prod_{i\in I}(F_i,A_i)$ is a fuzzy soft semigroup over $\prod_{i\in I}S_i$.

Proof: By definition, for all

$$(e_i)\in\prod_{i\in I}A_i,\ (\widehat{\prod_{i\in I}}F_i(e_i))(x) = \wedge_{i\in I}((F_i(e_i))(x_i)) \text{ where } x=(x_i)\in\prod_{i\in I}S_i$$

and $F_i(e_i)$ is a fuzzy subsemigroup of S_i for all $i\in I$. Let $\prod_{i\in I}(F_i,A_i)=(H,\prod_{i\in I}A_i)$ where, $H((e_i))=\widehat{\prod_{i\in I}}F_i(e_i)$ for all $(e_i)\in\prod_{i\in I}A_i$. Then for any (x_i), $(y_i)\in\prod_{i\in I}S_i$

$$\begin{aligned}\{H((e_i))\}((x_i)(y_i)) &= (\widehat{\prod_{i\in I}}F_i(e_i))((x_iy_i)) = \wedge_{i\in I}((F_i(e_i))(x_iy_i)) \\ &\geq \wedge_{i\in I}\{(F_i(e_i))(x_i)\wedge(F_i(e_i))(y_i)\} \because F_i(e_i) \text{ is a fuzzy subsemigroup of } S_i \\ &= [\wedge_{i\in I}\{(F_i(e_i))(x_i)\}]\wedge[\wedge_{i\in I}\{(F_i(e_i))(y_i)\}] = (\widehat{\prod_{i\in I}}F_i(e_i))((x_i))\wedge(\widehat{\prod_{i\in I}}F_i(e_i))((y_i)) \\ &= \{H((e_i))\}((x_i))\wedge\{H((e_i))\}((y_i))\end{aligned}$$

This implies that H((e$_i$)) is a fuzzy subsemigroup of $\prod_{i\in I}S_i$ for all $(e_i)\in\prod_{i\in I}A_i$. Thus $\prod_{i\in I}(F_i,A_i)$ is a fuzzy soft semigroup over $\prod_{i\in I}S_i$.

Corollary 1: Let $\{(F_i,A_i):i\in I\}$ be a non-empty family of fuzzy soft subsemigroups of fuzzy soft semigroup (F,A) over S. Then Direct product (External) $\prod_{i\in I}(F_i,A_i)$ is a fuzzy soft subsemigroup of $\prod_{i\in I}(F,A)$ over $\prod_{i\in I}S_i$.

Proof: Follows directly from Proposition 7.

Theorem 2: Let $\{(F_i,A_i):i\in I\}$ be a non-empty family of fuzzy soft semigroups over S. Then $\widetilde{\cup}_{i\in I}(F_i,A_i)$ is a fuzzy soft semigroup over S, provided $A_i\cap A_j=\varnothing$ for all i, j$\in$I and i$\neq$j, $\cap_{\varepsilon\, i\in I}(F_i,A_i)$ is a fuzzy soft semigroup over S, $\cap_{R\, i\in I}(F_i,A_i)$ is a fuzzy soft semigroup over S provided $\cap_{i\in I}A_i\neq\varnothing$, $\wedge_{i\in I}(F_i,A_i)$ is a fuzzy soft semigroup over S.

Proof: Since A_i 's are non-empty, so $\cup_{i\in I}A_i$ is non-empty. Let $\widetilde{\cup}_{i\in I}(F_i,A_i)=(H,\cup_{i\in I}A_i)$. Then for any $e\in\cup_{i\in I}A_i\Rightarrow e\in A_j$ for some j$\in$I and H(e) = F$_j$(e). This implies that H(e) is a fuzzy subsemigroup of S for all $e\in\cup_{i\in I}A_i$. Thus $\widetilde{\cup}_{i\in I}(F_i,A_i)$ is a fuzzy soft semigroup over S.

Let $\cap_{\varepsilon\, i\in I}(F_i,A_i)=(H,\cup_{i\in I}A_i)$ where $H(e)=\wedge_{j\in\Lambda(e)}F_j(e)$, for all $e\in\cup_{i\in I}A_i$. We define the set $\Lambda(e)=\{j:e\in A_j\}$. Then for x, y$\in$S

$$\begin{aligned}\{H(e)\}(xy) &= \left\{\wedge_{j\in\Lambda(e)}F_j(e)\right\}(xy) = \wedge_{j\in\Lambda(e)}\{(F_j(e))(xy)\} \\ &\geq \wedge_{j\in\Lambda(e)}\{(F_j(e))(x)\wedge(F_j(e))(y)\} \because F_j(e) \text{ is a fuzzy subsemigroup of S} \\ &= \left(\wedge_{j\in\Lambda(e)}\{(F_j(e))(x)\}\right)\wedge\left(\wedge_{j\in\Lambda(e)}\{(F_j(e))(y)\}\right) = \left(\{\wedge_{j\in\Lambda(e)}(F_j(e))\}(x)\right)\wedge\left(\{\wedge_{j\in\Lambda(e)}(F_j(e))\}(y)\right) \\ &= \{(H(e))(x)\}\wedge\{(H(e))(y)\}\end{aligned}$$

This implies that H(e) is a fuzzy subsemigroup of S for all . Thus $\cap_{\varepsilon\, i\in I}(F_i,A_i)$ is a fuzzy soft subsemigroup over S.

Let $\cap_{R\, i\in I}(F_i,A_i)=(H,\cap_{i\in I}A_i)$ where $H(e)=\wedge_{i\in I}F_i(e)$ for all $e\in\cap_{i\in I}A_i$. Then for any x, y$\in$S

$$\{H(e)\}(xy)=\left\{\underset{i\in I}{\wedge}F_i(e)\right\}(xy)=\underset{i\in I}{\wedge}\{(F_i(e))(xy)\}\geq\underset{i\in I}{\wedge}\{(F_i(e))(x)\wedge(F_i(e))(y)\}\ \because F_i(e) \text{ is a fuzzy subsemigroup of S}$$
$$=\left(\underset{i\in I}{\wedge}\{(F_i(e))(x)\}\right)\wedge\left(\underset{i\in I}{\wedge}\{(F_i(e))(y)\}\right)=\left(\{\underset{i\in I}{\wedge}(F_i(e))\}(x)\right)\wedge\left(\{\underset{i\in I}{\wedge}(F_i(e))\}(y)\right)=\{(H(e))(x)\}\wedge\{(H(e))(y)\}$$

Hence H(e) is a fuzzy subsemigroup of S for all $e\in\underset{i\in I}{\cap}A_i$. Thus $\underset{i\in I}{\cap_R}(F_i,A_i)$ is a fuzzy soft semigroup over S.

Let $\underset{i\in I}{\wedge}(F_i,A_i)=(H,\prod_{i\in I}A_i)$ where, $H((e_i))=\underset{i\in I}{\wedge}F_i(e_i)$ for all $(e_i)\in\prod_{i\in I}A_i$. Then for any x, y∈ S

$$\{H((e_i))\}(xy)=\left\{\underset{i\in I}{\wedge}F_i(e_i)\right\}(xy)=\underset{i\in I}{\wedge}\{(F_i(e_i))(xy)\}\geq\underset{i\in I}{\wedge}[\{(F_i(e_i))(x)\}\wedge\{(F_i(e_i))(y)\}]\because F_i(e) \text{ is a fuzzy subsemigroup of S}$$
$$=[\underset{i\in I}{\wedge}\{(F_i(e_i))(x)\}]\wedge[\underset{i\in I}{\wedge}\{(F_i(e_i))(y)\}]=\{\underset{i\in I}{\wedge}(F_i(e_i))\}(x)\wedge\{\underset{i\in I}{\wedge}(F_i(e_i))\}(y)=\{H((e_i))(x)\}\wedge\{H((e_i))(y)\}$$

Which implies that H((e$_i$)) is a fuzzy subsemigroup of S for all $(e_i)\in\prod_{i\in I}A_i$. Thus $\underset{i\in I}{\wedge}(F_i,A_i)$ is a fuzzy soft semigroup over S.

Theorem 3: Let $\{(F_i,A_i):i\in I\}$ be a non-empty family of fuzzy soft subsemigroups of the fuzzy soft semigroup (F,A) over S. Then $\underset{i\in I}{\tilde{\cup}}(F_i,A_i)$ is a fuzzy soft subsemigroup of (F,A) over S, provided $A_i\cap A_j=\varnothing$ for all i, j∈ I, where i≠j $\underset{i\in I}{\cap_\varepsilon}(F_i,A_i)$ is a fuzzy soft subsemigroup of (F,A) over S $\underset{i\in I}{\cap_R}(F_i,A_i)$ is a fuzzy soft subsemigroup of (F,A) over S, provided $\underset{i\in I}{\cap}A_i\neq\varnothing$, $\underset{i\in I}{\wedge}(F_i,A_i)$ is a fuzzy soft subsemigroup of $\underset{i\in I}{\wedge}(F,A)$ over S.

Proof: Straightforward.

Remark 3: For any two fuzzy soft semigroups (F,A) and (G,B) over S, $(F,A)\cup_R(G,B)$ and $(F,A)\vee(G,B)$ may not be fuzzy soft semigroups over S.

Example 3: Let S be a semigroup of four elements {a,b,c,d} with the following multiplication table

.	a	b	c	d
a	a	b	c	d
b	b	b	b	b
c	c	c	c	c
d	d	c	b	d

Let E = {e_1, e_2, e_3, e_4} be a set of parameters.

Suppose that A = {e_1, e_2, e_3} and B = {e_2, e_3, e_4}. Let (F,A) and (G,B) be the fuzzy soft semigroups over S defined by the following tables:

F	e_1	e_2	e_3
a	1	1	1
b	1	0.1	0.2
c	0	0.2	1
d	0	0.1	0.2

G	e_2	e_3	e_4
a	1	1	1
b	0.1	0.2	1
c	0.1	1	0
d	1	0.2	0

Let

$$(F,A)\cup_R (G,B)=(F_1,A\cap B),(F,A)\,\widetilde{\cup}\,(G,B)=(F_2,A\cup B) \text{ and } (F,A)\vee(G,B)=(F_3,A\times B)$$

Then

F_1	e_2	e_3
a	1	1
b	0.1	0.2
c	0.2	1
d	1	0.2

F_2	e_1	e_2	e_3	e_4
a	1	1	1	1
b	1	0.1	0.2	1
c	0	0.2	1	0
d	0	1	0.2	0

F_3	(e_1,e_2)	(e_1,e_3)	(e_1,e_4)	(e_2,e_2)	(e_2,e_3)	(e_2,e_4)	(e_3,e_2)	(e_3,e_3)	(e_3,e_4)
a	1	1	1	1	1	1	1	1	1
b	1	1	1	0.1	0.2	1	0.2	0.2	1
c	0.1	1	0	0.2	1	0.2	1	1	1
d	1	0.2	0	1	0.2	0.1	1	0.2	0.2

There is one equation which is not
We see that

$$(F_1(e_2))(c)=0.2,\ (F_1(e_2))(d)=1 \text{ and } (F_1(e_2))(dc)=(F_1(e_2))(b)=0.1$$

so

$$(F_1(e_2))(dc)<(F_1(e_2))(c)\wedge(F_1(e_2))(d)$$

Thus, $(F,A)\cup_R (G,B)=(F_1,A\cap B)$ is not a fuzzy soft semigroup over S. Since $(F_2(e_2))(c)=0.2$, $(F_2(e_2))(d)=1$ and $(F_2(e_2))(dc)=(F_2(e_2))(b)=0.1$ so $(F_2(e_2))(dc)<(F_2(e_2))(c)\wedge(F_2(e_2))(d)$ and hence $(F,A)\,\widetilde{\cup}\,(G,B)=(F_2,A\cup B)$ is not a fuzzy soft semigroup over S. Also $\{F_3((e_1,e_2))\}(b)=1$, $\{F_3((e_1,e_2))\}(d)=1$ and

$$\{F_3((e_1,e_2))\}(db)=\{F_3((e_1,e_2))\}(c)=0.1$$

so

$$\{F_3((e_1,e_2))\}(db)<\{F_3((e_1,e_2))\}(d)\wedge\{F_3((e_1,e_2))\}(b)$$

Thus $(F,A)\vee(G,B)=(F_3,A\times B)$ is not a fuzzy soft semigroup over S.

Definition 31: A fuzzy soft set (F,A) over a semigroup S is said to be a fuzzy soft left (right) ideal over S if F(e) is a fuzzy left (right) ideal of S for all $e\in A$.

A fuzzy soft set (F,A) over S is said to be a fuzzy soft two-sided ideal or simply a fuzzy soft ideal over S, if it is both a fuzzy soft left and a fuzzy soft right ideal over S.

Proposition 6: A fuzzy soft set (F,A) over S is a fuzzy soft left (right) ideal over S if and only if

$$(\mathbf{S},E)\tilde{\circ}(F,A)\,\widetilde{\subset}\,(F,A)\big((F,A)\tilde{\circ}(\mathbf{S},E)\,\widetilde{\subset}\,(F,A)\big)$$

Proof: Suppose (F,A) is a fuzzy soft left ideal over S. Let $(\mathbf{S},E)\tilde{\circ}(F,A)=(H,A)$. Then H(e) = S $(e)\circ F(e)=\mathbf{S}\circ F(e)\subseteq F(e)$ for all $e\in A$, because F(e) is a fuzzy left ideal of S. This implies that $(\mathbf{S},E)\tilde{\circ}(F,A)\,\widetilde{\subset}\,(F,A)$. Conversely, if $(\mathbf{S},E)\tilde{\circ}(F,A)\,\widetilde{\subset}\,(F,A)$ then $\mathbf{S}(e)\circ F(e)\subseteq F(e)$ for all $e\in A$. So $\mathbf{S}\circ F(e)\subseteq F(e)$ for all $e\in A$, implies that F(e) is a fuzzy left ideal of S.
Similarly we can prove for the fuzzy soft right ideals.

Theorem 4: Let $\{(F_i, A_i): i \in I\}$ be a non-empty family of fuzzy soft left (right, two-sided) ideals over S. Then

1. $\bigcap_{R\, i\in I} (F_i, A_i)$ is a fuzzy soft left (right, two-sided) ideal over S, provided $\bigcap_{i\in I} A_i \neq \varnothing$;
2. $\bigcap_{\varepsilon\, i\in I} (F_i, A_i)$ is a fuzzy soft left (right, two-sided) ideal over S;
3. $\bigcup_{R\, i\in I} (F_i, A_i)$ is a fuzzy soft left (right, two-sided) ideal over S, provided $\bigcap_{i\in I} A_i \neq \varnothing$;
4. $\tilde{\bigcup}_{i\in I} (F_i, A_i)$ is a fuzzy soft left (right, two-sided) ideal over S;
5. $\wedge_{i\in I} (F_i, A_i)$ is a fuzzy soft left (right, two-sided) ideal over S;
6. $\vee_{i\in I} (F_i, A_i)$ is a fuzzy soft left (right, two-sided) ideal over S.

Proof: Let $\bigcap_{R\, i\in I} (F_i, A_i) = (H, \bigcap_{i\in I} A_i)$ where $H(e) = \wedge_{i\in I} F_i(e)$ for all $e \in \bigcap_{i\in I} A_i$. Then for any x, y$\in$ S

$$\{H(e)\}(xy) = \left\{\wedge_{i\in I} F_i(e)\right\}(xy) = \wedge_{i\in I}\{(F_i(e))(xy)\} \geq \wedge_{i\in I}\{(F_i(e))(y)\} \because F_i(e) \text{ is a fuzzy left ideal of S}$$
$$= \{\wedge_{i\in I}(F_i(e))\}(y) = \{H(e)\}(y)$$

Hence H(e) is a fuzzy left ideal of S for all $e \in \bigcap_{i\in I} A_i$. Thus $\bigcap_{R\, i\in I} (F_i, A_i)$ is a fuzzy soft left ideal over S.

Let $\bigcap_{\varepsilon\, i\in I} (F_i, A_i) = (H, \bigcup_{i\in I} A_i)$ where $H(e) = \wedge_{j\in\Lambda(e)} F_j(e)$, for all $e \in \bigcup_{i\in I} A_i$. We define the set $\Lambda(e) = \{ j : e \in A_j\}$. Then for x, y$\in$ S

$$\{H(e)\}(xy) = \left\{\wedge_{i\in\Lambda(e)} F_i(e)\right\}(xy)$$
$$= \wedge_{i\in\Lambda(e)}\{(F_i(e))(xy)\}$$
$$\geq \wedge_{i\in\Lambda(e)}\{(F_i(e))(y)\} \qquad \because F_i(e) \text{ is a fuzzy left ideal of S}$$
$$= \left(\{\wedge_{i\in\Lambda(e)}(F_i(e))\}(y)\right) = (H(e))(y)$$

implies that H(e) is a fuzzy left ideal of S for all $e \in \bigcup_{i\in I} A_i$. Thus $\bigcap_{\varepsilon\, i\in I} (F_i, A_i)$ is a fuzzy soft left ideal over S.

Let $\bigcup_{R\, i\in I} (F_i, A_i) = (G, \bigcap_{i\in I} A_i)$ where $G(e) = \vee_{i\in I} F_i(e)$ for all $e \in \bigcap_{i\in I} A_i$. Then for any x, y$\in$ S

$$\{G(e)\}(xy) = \left\{\vee_{i\in I} F_i(e)\right\}(xy)$$
$$= \vee_{i\in I}\{(F_i(e))(xy)\}$$
$$\geq \vee_{i\in I}\{(F_i(e))(y)\} \because F_i(e) \text{ is a fuzzy left ideal of S}$$
$$= \{\vee_{i\in I}(F_i(e))\}(y) = \{G(e)\}(y)$$

Hence G(e) is a fuzzy left ideal of S for all $e \in \bigcap_{i\in I} A_i$. Thus $\bigcup_{R\, i\in I} (F_i, A_i)$ is a fuzzy soft left ideal over S.

Let $\tilde{\bigcup}_{i\in I} (F_i, A_i) = (G, \bigcup_{i\in I} A_i)$ where $G(e) = \vee_{j\in\Lambda(e)} F_j(e)$, for all $e \in \bigcup_{i\in I} A_i$. We define the set $\Lambda(e) = \{j : e \in A_j\}$. Then for x, y$\in$ S

$$\{G(e)\}(xy) = \left\{ \underset{j\in\Lambda(e)}{\vee} F_j(e) \right\}(xy)$$
$$= \underset{j\in\Lambda(e)}{\vee} \{(F_j(e))(xy)\}$$
$$\geq \underset{j\in\Lambda(e)}{\vee} \{(F_j(e))(y)\} \because F_j(e) \text{ is a fuzzy left ideal of S}$$
$$= \{\underset{j\in\Lambda(e)}{\vee} (F_j(e))\}(y) = \{G(e)\}(y)$$

implies that G(e) is a fuzzy left ideal of S for all $e \in \underset{i\in I}{\cup} A_i$. Thus $\underset{i\in I}{\tilde{\cup}}(F_i, A_i)$ is a fuzzy soft left ideal over S.

Let $\underset{i\in I}{\wedge}(F_i, A_i) = (I, \prod_{i\in I} A_i)$ where $I((e_i)) = \underset{i\in I}{\wedge} F_i(e_i)$ for all $(e_i) \in \prod_{i\in I} A_i$. Then for any x, y∈ S

$$\{I((e_i))\}(xy) = \left\{ \underset{i\in I}{\wedge} F_i(e_i) \right\}(xy) = \underset{i\in I}{\wedge} \{(F_i(e_i))(xy)\} \geq \underset{i\in I}{\wedge} \{(F_i(e_i))(y)\} \because F_i(e) \text{ is a fuzzy left ideal of S}$$
$$= \{\underset{i\in I}{\wedge} (F_i(e_i))\}(y) = \{I((e_i))\}(y)$$

implies that $I((e_i))$ is a fuzzy left ideal of S for all $(e_i) \in \prod_{i\in I} A_i$. Thus $\underset{i\in I}{\wedge}(F_i, A_i)$ is a fuzzy soft left ideal over S.

Let $\underset{i\in I}{\vee}(F_i, A_i) = (I, \prod_{i\in I} A_i)$ where $I((e_i)) = \underset{i\in I}{\vee} F_i(e_i)$, for all $(e_i) \in \prod_{i\in I} A_i$. Then for any x, y∈ S

$$\{I((e_i))\}(xy) = \left\{ \underset{i\in I}{\vee} F_i(e_i) \right\}(xy) = \underset{i\in I}{\vee} \{(F_i(e_i))(xy)\} \geq \underset{i\in I}{\vee} \{(F_i(e_i))(y)\} \because F_i(e) \text{ is a fuzzy left ideal of S}$$
$$= \{\underset{i\in I}{\vee} (F_i(e_i))\}(y) = \{I((e_i))\}(y)$$

implies that $I((e_i))$ is a fuzzy left ideal of S for all $(e_i) \in \prod_{i\in I} A_i$. Thus $\underset{i\in I}{\vee}(F_i, A_i)$ is a fuzzy soft left ideal over S. Similarly we can prove for fuzzy soft right ideals and hence for fuzzy soft two-sided ideals over S.

Proposition 7: Let (F,A) and (G,B) be two fuzzy soft left (right, two-sided) ideals over S and T respectively. Then their Cartesian product (F,A)×(G,B) is a fuzzy soft left (right, two-sided) ideal over S×T.

Proof: Let $(F,A)\times(G,B) = (H, A\times B)$ where $F(a)\hat{\otimes}G(b) = H((a,b))$ for all $(a,b)\in A\times B$.

Then for any (x,y), (z,t)∈ S×T

$$\{H((a,b))\}((x,y)(z,t)) = \{H((a,b))\}((xz, yt)) = \{F(a)\hat{\otimes}G(b)\}((xz, yt)) = \{(F(a))(xz)\} \wedge \{(G(b))(yt)\}$$
$$\geq \{(F(a))(z)\} \wedge \{(G(b))(t)\} = \{F(a)\hat{\otimes}G(b)\}((z, y)) = \{H((a,b))\}((z,t))$$

implies that H((a,b)) is a fuzzy left ideal of S for all (a,b)∈ A×B. Thus (F,A)×(G,B) is a fuzzy soft left ideal over S×T.

Similarly we can prove for fuzzy soft right ideals and hence for fuzzy soft two-sided ideals over S×T.

Proposition 8: Let $\{S_i : i\in I\}$ be a non-empty family of semigroups and $\{(F_i, A_i) : i\in I\}$ be a non-empty family of fuzzy soft sets, such that (F_i, A_i) is a fuzzy soft left (right, two-sided) ideal over S_i for each i∈ I. Then $\prod_{i\in I}(F_i, A_i)$ is a fuzzy soft left (right, two-sided) ideal over $\prod_{i\in I} S_i$.

Proof: For all i∈ I, $F_i(e_i)$ is a fuzzy left ideal of S_i. By definition, for all $(e_i)\in \prod_{i\in I} A_i$,

$$(\widehat{\prod_{i\in I}} F_i(e_i))(x) = \wedge_{i\in I}((F_i(e_i))(x_i)) \text{ where } x = (x_i) \in \prod_{i\in I} S_i$$

Let $\prod_{i\in I}(F_i, A_i) = (H, \prod_{i\in I} A_i)$ where, $H((e_i)) = \widehat{\prod_{i\in I}} F_i(e_i)$ for all $(e_i)\in \prod_{i\in I} A_i$. Then for any (x_i), $(y_i)\in \prod_{i\in I} S_i$

$$\{H((e_i))\}((x_i)(y_i)) = (\widehat{\prod_{i\in I}} F_i(e_i))((x_i y_i)) = \wedge_{i\in I}((F_i(e_i))(x_i y_i)) \geq \wedge_{i\in I}((F_i(e_i))(y_i)) \because F_i(e_i) \text{ is a fuzzy left ideal of } S_i$$
$$= (\widehat{\prod_{i\in I}} F_i(e_i))((y_i)) = \{H((e_i))\}((y_i))$$

implies that $H((e_i))$ is a fuzzy left ideal of the product semigroup $\prod_{i\in I} S_i$ for all $(e_i)\in \prod_{i\in I} A_i$. Thus $\prod_{i\in I}(F_i, A_i)$ is a fuzzy soft left ideal over $\prod_{i\in I} S_i$.
Similarly we can prove for fuzzy soft right ideals and hence for fuzzy soft two-sided ideals.

Lemma 2: If (F,A) is a non-empty fuzzy soft set over S, then $(\mathbf{S}, E)\tilde{o}(F,A)\left((F,A)\tilde{o}(\mathbf{S},E)\right)$ is a fuzzy soft left (right) ideal over S.

Proof: Straightforward.

Lemma 3: If (F,A) is a fuzzy soft right (left) ideal over S, then $(F,A)\widehat{\cup}((\mathbf{S},E)\tilde{o}(F,A))$ $((F,A)\widehat{\cup}((F,A)\tilde{o}(\mathbf{S},E)))$ is a fuzzy soft ideal over S.

Proof: Suppose (F,A) is a fuzzy soft right ideal over S. Then

$$(\mathbf{S},E)\tilde{o}((F,A)\widehat{\cup}((\mathbf{S},E)\tilde{o}(F,A))) = ((\mathbf{S},E)\tilde{o}(F,A))\widehat{\cup}((\mathbf{S},E)\tilde{o}(F,A))\widehat{\cup}((\mathbf{S},E)\tilde{o}(F,A))$$
$$= ((\mathbf{S},E)\tilde{o}(F,A))\widehat{\cup}(((\mathbf{S},E)\tilde{o}(F,A))\widehat{\cup}((\mathbf{S},E))\tilde{o}(F,A))\widehat{\subset}(\mathbf{S},E)\tilde{o}(F,A))\widehat{\cup}(\mathbf{S},E)\tilde{o}(F,A))$$
$$= (\mathbf{S},E)\tilde{o}(F,A)\widehat{\subset}(F,A)\widehat{\cup}((\mathbf{S},E)\tilde{o}(F,A))$$

So, $(F,A)\widehat{\cup}((\mathbf{S},E)\tilde{o}(F,A))$ is a fuzzy soft left ideal over S. Now,

$$((F,A)\widehat{\cup}((\mathbf{S},E)\tilde{o}(F,A)))\tilde{o}(\mathbf{S},E) = ((F,A)\tilde{o}(\mathbf{S},E))\widehat{\cup}(((\mathbf{S},E)\tilde{o}(F,A))\tilde{o}(\mathbf{S},E))$$
$$\widehat{\subset}(F,A)\widehat{\cup}((\mathbf{S},E)\tilde{o}((F,A)\tilde{o}(\mathbf{S},E)))\widehat{\subset}(F,A)\widehat{\cup}((\mathbf{S},E)\tilde{o}(F,A)) \quad \text{because is a fuzzy soft right ideal over S}$$

Thus $(F,A)\widehat{\cup}((\mathbf{S},E)\tilde{o}(F,A))$ is a fuzzy soft ideal over S.

Definition 32: A fuzzy soft set (F,A) over S is said to be a fuzzy soft quasi-ideal over S if F(e) is a fuzzy quasi-ideal of S, for all $e\in A$.

Proposition 9: A non-empty fuzzy soft set (F,A) over S is a fuzzy soft quasi-ideal over S if and only if

$$((F,A)\tilde{o}(\mathbf{S},E)) \cap_R ((\mathbf{S},E)\tilde{o}(F,A))\widehat{\subset}(F,A)$$

Proof. Suppose that $((F,A)\tilde{o}(\mathbf{S},E)) \cap_R ((\mathbf{S},E)\tilde{o}(F,A))\widehat{\subset}(F,A)$
Then $(F(e)\circ \mathbf{S}(e)) \wedge (\mathbf{S}(e)\circ F(e)) \subseteq F(e) \Rightarrow (F(e)\circ \mathbf{S}) \wedge (\mathbf{S}\circ F(e)) \subseteq F(e)$

for all $e \in A$. This shows that F(e) is a fuzzy quasi-ideal of S. Conversely assume that (F,A) is a fuzzy soft quasi-ideal over S. Let $((F,A)\tilde{o}(\mathbf{S},E)) \cap_R ((\mathbf{S},E)\tilde{o}(F,A)) = (G,A)$, then

$$G(e) = (F(e) \circ \mathbf{S}(e)) \wedge (\mathbf{S}(e) \circ F(e)) = (F(e) \circ \mathbf{S}) \wedge (\mathbf{S} \circ F(e)) \subseteq F(e) \text{ for all } e \in A$$

Thus

$$((F,A)\tilde{o}(\mathbf{S},E)) \cap_R ((\mathbf{S},E)\tilde{o}(F,A)) \widetilde{\subset} (F,A)$$

Remark 4: A fuzzy soft left or right ideal over S is a fuzzy soft quasi-ideal over S, but the converse is not true in general.

Example 4: Let S be a semigroup of four elements {0,a,b,c} with the following multiplication table:

.	0	a	b	c
0	0	0	0	0
a	0	a	b	0
b	0	0	0	0
c	0	c	0	0

Suppose that $E = \{e_1, e_2, e_3, e_4\}$ be a set of parameters.
Let $A = \{e_1, e_2, e_3\}$ and (F,A) be a fuzzy soft set over S, defined by the following table

F	e_1	e_2	e_3
0	0.7	0.1	0.2
a	0.7	0.1	0.2
b	0	0	0
c	0	0	0.3

Then (F,A) is a fuzzy soft quasi-ideal over S, but not a fuzzy soft left (right, two-sided) ideal over S.

Proposition 10: Let $\{(F_i, A_i) : i \in I\}$ be a non-empty family of fuzzy soft quasi-ideals over S. Then

1. $\bigcap_{R\, i \in I} (F_i, A_i)$ is a fuzzy soft quasi-ideal over S, provided $\bigcap_{i \in I} A_i \neq \varnothing$;
2. $\bigcap_{\varepsilon\, i \in I} (F_i, A_i)$ is a fuzzy soft quasi-ideal over S;
3. $\bigwedge_{i \in I} (F_i, A_i)$ is a fuzzy soft quasi-ideal over S;
4. $\widetilde{\bigcup}_{i \in I} (F_i, A_i)$ is a fuzzy soft quasi-ideal over S, provided $A_i \cap A_j = \varnothing$ for all $i, j \in I$ and $i \neq j$.

Proof: Let $\bigcap_{\varepsilon\, i \in I} (F_i, A_i) = (H, \bigcup_{i \in I} A_i)$ where $H(e) = \bigwedge_{j \in \Lambda(e)} F_j(e)$, for all $e \in \bigcup_{i \in I} A_i$. We define the set $\Lambda(e) = \{ j : e \in A_j \}$. Now H(e) is a fuzzy quasi-ideal of S being the intersection of fuzzy quasi-ideals of S. Thus $\bigcap_{\varepsilon\, i \in I} (F_i, A_i)$ is a fuzzy soft quasi-ideal over S.
Similarly we can prove 2, 3 and 4.

Definition 33: A fuzzy soft semigroup (F,A) over a semigroup S is said to be a fuzzy soft bi-ideal over S, if F(e) is a fuzzy bi-ideal of S for every $e \in A$.

Proposition 11: A fuzzy soft semigroup (F,A) over S is a fuzzy soft bi-ideal over S if and only if $(F,A)\tilde{o}(\mathbf{S},E)\tilde{o}(F,A)\widehat{\subset}(F,A)$.

Proof: Suppose that (F,A) is a fuzzy soft bi-ideal over S.
Let $(F,A)\tilde{o}(\mathbf{S},E)\tilde{o}(F,A)=(H,A)$. Then

$$H(e)=F(e)\circ\mathbf{S}(e)\circ F(e)=F(e)\circ\mathbf{S}\circ F(e)\subseteq F(e) \text{ for all } e\in A$$

Thus $(F,A)\tilde{o}(\mathbf{S},E)\tilde{o}(F,A)\widehat{\subset}(F,A)$. Conversely, suppose that $(F,A)\tilde{o}(\mathbf{S},E)\tilde{o}(F,A)\widehat{\subset}(F,A)$. Then

$$F(e)\circ\mathbf{S}(e)\circ F(e)\subseteq F(e)\Rightarrow F(e)\circ\mathbf{S}\circ F(e)\subseteq F(e)$$

Thus F(e) is a fuzzy bi-ideal of S for all $e\in A$. Hence (F,A) is a fuzzy soft bi-ideal over S.

Proposition 12: Let (F,A) be a non-empty fuzzy soft set over S and (G,B) be a fuzzy soft bi-ideal over S. Then the products $(F,A)\tilde{o}(G,B)$ and $(G,B)\tilde{o}(F,A)$ are fuzzy soft bi-ideals over S, provided $A\cap B\neq\varnothing$.

Proof: First we note that if $(F,A)\tilde{o}(G,B)=(H,A\cap B)$ where $H(e)=F(e)\circ G(e)$, for all $e\in A\cap B$ then

$$\begin{aligned} H(e)\circ H(e) &= (F(e)\circ G(e))\circ(F(e)\circ G(e)) \\ &= F(e)\circ(G(e)\circ F(e)\circ G(e)) \\ &\subseteq F(e)\circ(G(e)\circ\mathbf{S}\circ G(e)) \\ &\subseteq F(e)\circ G(e) \qquad \because G(e) \text{ is a fuzzy bi-ideal of } S \\ &= H(e) \end{aligned}$$

So

$$(H,A\cap B)\tilde{o}(H,A\cap B)\widehat{\subset}(H,A\cap B)$$

Thus $(F,A)\tilde{o}(G,B)$ is a fuzzy soft semigroup over S. Now we suppose that

$$(H,A\cap B)\tilde{o}(\mathbf{S},E)\tilde{o}(H,A\cap B)=(I,A\cap B) \text{ where } I(e)=H(e)\circ\mathbf{S}(e)\circ H(e)$$

Then

$$\begin{aligned} I(e) &= H(e)\circ\mathbf{S}(e)\circ H(e) \\ &= (F(e)\circ G(e))\circ\mathbf{S}\circ(F(e)\circ G(e)) \\ &= F(e)\circ(G(e)\circ\mathbf{S}\circ F(e)\circ G(e) \\ &\subseteq F(e)\circ(G(e)\circ\mathbf{S}\circ G(e)) \\ &\subseteq F(e)\circ G(e) \\ &= H(e) \text{ for all } e\in A\cap B. \end{aligned}$$

This implies that $(H,A\cap B)\tilde{o}(\mathbf{S},E)\tilde{o}(H,A\cap B)\widehat{\subset}(H,A\cap B)$. Thus $(F,A)\tilde{o}(G,B)$ is a fuzzy soft bi-ideal over S. Similarly $(G,B)\tilde{o}(F,A)$ is a fuzzy soft bi-ideal over S.

Proposition 1: Let $\{(F_i,A_i):i\in I\}$ be a non-empty family of fuzzy soft bi-ideals over S. Then

1. $\bigcap_{R\, i\in I}(F_i, A_i)$ is a fuzzy soft bi-ideal over S, provided $\underset{i\in I}{\cap} A_i \neq \varnothing$;
2. $\bigcap_{\varepsilon\, i\in I}(F_i, A_i)$ is a fuzzy soft bi-ideal over S;
3. $\underset{i\in I}{\wedge}(F_i, A_i)$ is a fuzzy soft bi-ideal over S;
4. $\underset{i\in I}{\tilde{\cup}}(F_i, A_i)$ is a fuzzy soft bi-ideal over S, provided $A_i \cap A_j = \varnothing$ for all i, j∈ I and i≠j.

Proof: We have already proved in Theorem 4, that the intersections of such family of fuzzy soft semigroups is a fuzzy soft semigroup over S, so we only need to show the following:
Let $\bigcap_{R\, i\in I}(F_i, A_i) = (H, \underset{i\in I}{\cap} A_i)$ where $H(e) = \underset{i\in I}{\wedge} F_i(e)$ for all $e \in \underset{i\in I}{\cap} A_i$. Then for any x, y, z∈ S

$$\{H(e)\}(xyz) = \left\{\underset{i\in I}{\wedge} F_i(e)\right\}(xyz) = \underset{i\in I}{\wedge}\{(F_i(e))(xyz)\} \geq \underset{i\in I}{\wedge}\{(F_i(e))(x) \wedge (F_i(e))(z)\} \because F_i(e) \text{ is a fuzzy bi-ideal of S}$$

$$= \left(\underset{i\in I}{\wedge}\{(F_i(e))(x)\}\right) \wedge \left(\underset{i\in I}{\wedge}\{(F_i(e))(z)\}\right) = \left(\{\underset{i\in I}{\wedge}(F_i(e))\}(x)\right) \wedge \left(\{\underset{i\in I}{\wedge}(F_i(e))\}(z)\right) = \{(H(e))(x)\} \wedge \{(H(e))(z)\}$$

Hence H(e) is a fuzzy bi-ideal of S for all $e \in \underset{i\in I}{\cap} A_i$. Thus $\bigcap_{R\, i\in I}(F_i, A_i)$ is a fuzzy soft bi-ideal over S.

Similarly we can prove 2, 3 and 4.

Remark 5: A fuzzy soft quasi-ideal over S is a fuzzy soft bi-ideal over S, but the converse of this statement is not true. This is shown in next example:

Example 5: Let S be a semigroup of four elements {0,a,b,c} with the following multiplication table:

.	0	a	b	c
0	0	0	0	0
a	0	0	0	0
b	0	0	0	a
c	0	0	a	b

Suppose that E = {e_1, e_2, e_3} be a set of parameters.
Let A = {e_1, e_3} and (F,A) be a fuzzy soft set over S, defined by the following table

F	e_1	e_3
0	0.7	0.5
a	0	0.1
b	0.7	0.5
c	0	0.1

Then (F,A) is a fuzzy soft bi-ideal over S, but not a fuzzy soft quasi-ideal over S.

Definition 34: A fuzzy soft set (F,A) over a semigroup S is said to be a fuzzy soft generalized bi-ideal over S, if F(e) is a fuzzy generalized bi-ideal of S for every e∈ A.

Proposition 14: A fuzzy soft set (F,A) over S is a fuzzy soft generalized bi-ideal over S if and only if $(F,A)\tilde{o}(\mathbf{S},E)\tilde{o}(F,A)\tilde{\subset}(F,A)$.

Proof: Straightforward.

Proposition 15: Let (F,A) and (G,B) be two fuzzy soft generalized bi-ideals over S. Then their restricted product $(F,A)\tilde{o}(G,B)$ is a fuzzy soft generalized bi-ideal over S.

Proof: Let $(F,A)\tilde{o}(G,B))=(H,A\cap B)$ where $H(e)=F(e)\circ G(e)$ for all $e\in A\cap B$. Since F(e) and G(e) are fuzzy generalized bi-ideals of S, so their product $F(e)\circ G(e)$ is also a fuzzy generalized bi-ideal of S for all $e\in A\cap B$. Thus $(F,A)\tilde{o}(G,B)$ is a fuzzy soft generalized bi-ideal over S.

Proposition 16: Let (F,A) and (G,B) be two fuzzy soft generalized bi-ideals over S. Then their extended product $(F,A)\tilde{o}_{\varepsilon}(G,B)$ is a fuzzy soft generalized bi-ideal over S.

Proof: Let $(F,A)\tilde{o}_{\varepsilon}(G,B)=(H,A\cup B)$ where

$$H(e)=\begin{cases} F(e) & \text{if } e\in A-B \\ G(e) & \text{if } e\in B-A \\ F(e)\circ G(e) & \text{if } e\in A\cap B \end{cases}$$

for all $e\in A\cup B$. Since F(e) and G(e) are fuzzy generalized bi-ideals of S, so their product $F(e)\circ G(e)$ is also a fuzzy generalized bi-ideal of S for all $e\in A\cap B$. Therefore in any of the above three cases, H(e) is a fuzzy generalized bi-ideal of S for all $e\in A\cup B$. Thus $(F,A)\tilde{o}_{\varepsilon}(G,B)$ is a fuzzy soft generalized bi-ideal over S.

Proposition 17: Let (F,A) and (G,B) be two fuzzy soft generalized bi-ideals over S. Then their Cartesian product (internal) i.e. $(F,A)\tilde{o}_{\times}(G,B)$ is a fuzzy soft generalized bi-ideal over S.

Proof: Let $(F,A)\tilde{o}_{\times}(G,B)=(H,A\times B)$ where $H((a,b))=F(a)\circ G(b)$ for all $(a,b)\in A\times B.$

Since F(a) and G(b) are fuzzy generalized bi-ideals of S, so their product $F(a)\circ G(b)$ is also a fuzzy generalized bi-ideal of S for all $(a,b)\in A\times B$. Thus $(F,A)\tilde{o}_{\times}(G,B)$ is a fuzzy soft generalized bi-ideal over S.

A fuzzy soft bi-ideal over S is a fuzzy soft generalized bi-ideal over S, but the converse is not true. This is shown in following example:

Example 6: Let S be a semigroup of four elements {a,b,c,d} with the following multiplication table:

.	a	b	c	d
a	a	a	a	a
b	a	a	a	a
c	a	a	b	a
d	a	a	b	b

Suppose that E = {e_1, e_2, e_3, e_4} be a set of parameters.
Let A = {e_1, e_2, e_3} and (F,A) be a fuzzy soft set over S, defined by the following table

F	e_1	e_2	e_3
a	0.5	0.7	1
b	0	0	1
c	0.2	0.7	1
d	0	0	1

Then (F,A) is a fuzzy soft generalized bi-ideal over S, but not a fuzzy soft bi-ideal over S.

Definition 35: Let (F,A) be a fuzzy soft set over S then it is said to be a fuzzy soft interior ideal over S if F(e) is a fuzzy interior ideal of S, for all $e \in A$.

Proposition 18: A fuzzy soft set (F,A) over S is a fuzzy soft interior ideal over S if and only if $(\mathbf{S},E)\tilde{o}(F,A)\tilde{o}(\mathbf{S},E)\widehat{\subset}(F,A)$.

Proof: Let (F,A) be a fuzzy soft interior ideal over S and $(\mathbf{S},E)\tilde{o}(F,A)\tilde{o}(\mathbf{S},E)=(H,A)$ where $H(e)=\mathbf{S}(e)\circ F(e)\circ \mathbf{S}(e)$, for all $e\in A$. Then $H(e)=\mathbf{S}(e)\circ F(e)\circ \mathbf{S}(e)=\mathbf{S}\circ F(e)\circ \mathbf{S}\subseteq F(e)$ because F(e) is a fuzzy interior ideal of S. Thus $(\mathbf{S},E)\tilde{o}(F,A)\tilde{o}(\mathbf{S},E)\widehat{\subset}(F,A)$.
Conversely, assume that $(\mathbf{S},E)\tilde{o}(F,A)\tilde{o}(\mathbf{S},E)\widehat{\subset}(F,A)$. Then $\mathbf{S}(e)\circ F(e)\circ \mathbf{S}(e)\subseteq F(e)\Rightarrow \mathbf{S}\circ F(e)\circ \mathbf{S}\subseteq F(e)$ for all $e\in A$. So F(e) is a fuzzy interior ideal of S. Thus (F,A) is a fuzzy soft interior ideal over S.

Proposition 19: Let $\{(F_i, A_i):i\in I\}$ be a non-empty family of fuzzy soft interior ideals over S. Then

1. $\bigcap_{R\, i\in I}(F_i,A_i)$ is a fuzzy soft interior ideal over S, provided $\bigcap_{i\in I}A_i\neq\varnothing$;
2. $\bigcap_{\varepsilon\, i\in I}(F_i,A_i)$ is a fuzzy soft interior ideal over S;
3. $\bigwedge_{i\in I}(F_i,A_i)$ is a fuzzy soft interior ideal over S;
4. $\widetilde{\bigcup}_{i\in I}(F_i,A_i)$ is a fuzzy soft interior ideal over S, provided $A_i\cap A_j=\varnothing$ for all i, $j\in I$ and $i\neq j$.

Proof: Let $\bigcap_{R\, i\in I}(F_i,A_i)=(H,\bigcap_{i\in I}A_i)$ where $H(e)=\bigwedge_{i\in I}F_i(e)$ for all $e\in\bigcap_{i\in I}A_i$. Then for any x, y, $a\in S$

$$(H(e))(xay)=\left\{\bigwedge_{i\in I}F_i(e)\right\}(xay)=\bigwedge_{i\in I}\{(F_i(e))(xay)\}\geq\bigwedge_{i\in I}\{(F_i(e))(a)\}\ \because F_i(e)\text{ is a fuzzy interior ideal of S}$$
$$=\{\bigwedge_{i\in I}(F_i(e))\}(a)=(H(e))(a)$$

Hence H(e) is a fuzzy interior ideal of S for all $e\in\bigcap_{i\in I}A_i$. Thus $\bigcap_{R\, i\in I}(F_i,A_i)$ is a fuzzy soft interior ideal over S.
Similarly we can prove 2, 3 and 4.

Remark 6: Note that every fuzzy soft ideal is a fuzzy soft interior ideal over S, but the converse is not true.

Example 7: Let S be a semigroup of four elements {a,b,c,d} with the following multiplication table

.	a	b	c	d
a	a	a	a	a
b	a	a	a	a
c	a	a	b	a
d	a	a	b	b

Let E = {e_1, e_2, e_3, e_4, e_5} be a set of parameters and A = {e_1, e_2}. Let (F,A) be a fuzzy soft set over S, given by the following multiplication table

F	e_2	e_3
a	0.7	0.6
b	0	0
c	0.3	0.2
d	0	0

Then (F,A) is a fuzzy soft interior ideal over S which is not a fuzzy soft two-sided ideal.

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