A method for multiple attribute decision making with incomplete weight information under uncertain linguistic environment

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Abstract

The multi-attribute decision making problems are studied, in which the information about the attribute values take the form of uncertain linguistic variables. The concept of deviation degree between uncertain linguistic variables is defined, and ideal point of uncertain linguistic decision making matrix is also defined. A formula of possibility degree for the comparison between uncertain linguistic variables is proposed. Based on the deviation degree and ideal point of uncertain linguistic variables, an optimization model is established, by solving the model, a simple and exact formula is derived to determine the attribute weights where the information about the attribute weights is completely unknown. For the information about the attribute weights is partly known, another optimization model is established to determine the weights, and then to aggregate the given uncertain linguistic decision information, respectively. A method based on possibility degree is given to rank the alternatives. Finally, an illustrative example is also given.

1. Introduction

Multiple attribute decision making is a prominent area of research in normative decision theory. This topic has been widely studied [1,4,6,12,18,24]. It generally involves the following three phases:

1. It needs collecting the information about attribute weights and attribute values.
2. It involves weighted aggregation of the attribute values across all attributes for each alternative to obtain an overall value.
3. It orders the overall values to obtain the best alternative(s).

In the real world, the decision maker (DM) may have vague knowledge about the preferences degree of one alternative over another. Furthermore, it is too complex or too ill-defined to be amenable for description in conventional quantitative expressions. It is more suitable to provide their preferences by means of linguistic variables rather than numerical one, the information about attribute weight is completely unknown or partly known. The above approaches will fail in dealing with the situations in which the decision information takes the form of uncertain linguistic variables. Therefore, it is necessary to pay attention to this issue.

Technique for order performance by similarity to ideal solution (TOPSIS), one of the known classical MCDM method, was first developed by Hwang and Yoon [12] for solving a MCDM problem. It based upon the concept that the chosen alternative should have the shortest distance from the ideal solution (IS). In this paper, we extend the TOPSIS method to uncertain linguistic environment [22,23], to overcome the above limitation, and we finally get the attribute weights. In order to do this, this paper is structured as follows. Section 2 gives the concept of the uncertain linguistic variables and some operational laws of uncertain linguistic variables. In order to compare uncertain linguistic variables, we present a formula of possibility degree of uncertain linguistic variables and propose the properties of the possibility degree. In Section 3 we define some useful concepts. In this section, we extend the concepts of TOPSIS to the uncertain linguistic environment and define the concept of the distance of two uncertain linguistic variables.
Section 4, based on the deviation degree and ideal point of uncertain linguistic variables, an optimization model is established, by solving the model, a simple and exact formula is derived to determine the attribute weights where the information about the attribute weights is completely unknown. For the information about the attribute weights is partly known, another optimization model is established to determine the weights, then we utilize the uncertain linguistic weighted average (ULWA) operator to aggregate the uncertain linguistic variables corresponding to each alternative, respectively. A method based on the possibility degree of uncertain linguistic variables to rank the alternatives. Section 5, a practical application of the developed method to evaluate the technological innovation capability of enterprises has also been given to show the effectiveness of the proposed method. Section 6, we conclude the paper.

2. Preliminaries

The linguistic approach is an approximate technique which represents qualitative aspects as linguistic variables [10,11,20,22,25]. Suppose that $S = \{s| i = -t, \ldots, t\}$ is a finite and totally ordered discrete term set, where $s_i$ represents a possible value for a linguistic variable. For example, a set of nine terms $S$ could be

$$S = \{s_{-4} = \text{extremely poor}, s_{-3} = \text{very poor}, s_{-2} = \text{poor},$$

$$s_{-1} = \text{slightly poor}, s_0 = \text{fair}, s_1 = \text{slightly good}, s_2 = \text{good},$$

$$s_3 = \text{very good}, s_4 = \text{extremely good}\}.$$  

Obviously, the mids linguistic label $s_0$ represents an assessment of “indifference”, and with the rest of the linguistic labels being placed symmetrically around it.

In these cases, it is usually required that there exist the following [8,20,22,23]:

1. The set is ordered: $s_i \succ s_j$ if $i > j$;
2. There is the negation operator: $\neg(s_i) = s_{-i}$ such that $i + j = 0$;
3. Max operator: $\max(s_i, s_j) = s_i$ if $s_i \succ s_j$;
4. Min operator: $\min(s_i, s_j) = s_j$ if $s_j \prec s_i$.

To preserve all the given information, we extend the discrete term set $S$ to a continuous term set [21,22] $S = \{s|x \in [-t, t]\}$. If $s_x \in S$, then we call $s_x$ an original linguistic term, otherwise, we call $s_x$ a virtual linguistic term. In general, the decision maker used the original linguistic terms to evaluate alternatives, and the virtual linguistic terms can only appear in operation.

**Definition 1.** [23] Let $\hat{s} = [s_x, s_y]$, where $s_x, s_y \in \bar{S}$, $s_x$ and $s_y$ are the lower and the upper limits, respectively, then we call $\hat{s}$ the uncertain linguistic variable.

Let $\tilde{S}$ be the set of all uncertain linguistic variables. Consider any three uncertain linguistic variables $\tilde{s} = [s_x, s_y], \tilde{s}_1 = [s_x, s_y]_1$, and $\tilde{s}_2 = [s_x, s_y]_2$, then their operational laws are defined as:

1. $\tilde{s}_1 \oplus \tilde{s}_2 = [s_x, s_y]_1 \oplus [s_x, s_y]_2 = [s_x, s_y] = [s_{x-1}, s_{y+1}]$;
2. $\tilde{x} \tilde{s} = \tilde{s}_x \tilde{s}_y = [\tilde{s}_x, \tilde{s}_y] = [s_{x}, s_{y}]$, where $x \in [0, 1]$;
3. $\tilde{s}_1 \oplus \tilde{s}_2 = \tilde{s}_2 \oplus \tilde{s}_1$;
4. $\tilde{x}(\tilde{s}_1 \oplus \tilde{s}_2) = \tilde{s}_x \tilde{s}_y \oplus \tilde{s}_y \tilde{s}_2$, where $x \in [0, 1]$;
5. $\tilde{x}(\tilde{s}_1 + \tilde{s}_2) = \tilde{s}_x \tilde{s}_y + \tilde{s}_y \tilde{s}_2$, where $x_1, x_2 \in [0, 1]$.  

In order to compare uncertain linguistic variables, we give the following definition:

**Definition 2.** Let $\tilde{s}_1 = [s_x, s_y], \tilde{s}_2 = [s_x, s_y] \in \hat{S}$ be two uncertain linguistic variables, and let $\text{len}(\tilde{s}_1) = \beta_1 - \eta_1$, $\text{len}(\tilde{s}_2) = \beta_2 - \eta_2$, $\text{win}(\tilde{s}_1) = \beta_1 + \eta_1$, $\text{win}(\tilde{s}_2) = \beta_2 + \eta_2$, then the degree of possibility of $\tilde{s}_1 \succ \tilde{s}_2$ is defined as

$$p(\tilde{s}_1 \succ \tilde{s}_2) = \min \left\{ \frac{\text{win}(\tilde{s}_1) - \text{win}(\tilde{s}_2)}{\text{len}(\tilde{s}_1) + \text{len}(\tilde{s}_2)} + 1 \right\} [0, 1]$$

From Definition 2, we can easily get the following properties of the degree of possibility.

**Theorem 1.** Let $\tilde{s}_1 = [s_x, s_y], \tilde{s}_2 = [s_x, s_y]$ and $\tilde{s}_3 = [s_x, s_y] \in \tilde{S}$ be three uncertain linguistic variables, then

1. $0 \leq p(\tilde{s}_1 \succ \tilde{s}_2) \leq 1$;
2. $p(\tilde{s}_1 \succ \tilde{s}_2) = 1$, if and only if $\beta_2 \leq \eta_1$;
3. $p(\tilde{s}_1 \succ \tilde{s}_2) = 0$, if and only if $\beta_1 \leq \eta_2$;
4. $p(\tilde{s}_1 \succ \tilde{s}_2) + p(\tilde{s}_2 \succ \tilde{s}_1) = 1$. Especially, $p(\tilde{s}_1 \succ \tilde{s}_1) = 1/2$;
5. $p(\tilde{s}_1 \succ \tilde{s}_2) \geq 1/2$, if and only if $\beta_2 + \beta_1 \geq \eta_1 + \eta_2$. Especially, $p(\tilde{s}_1 \succ \tilde{s}_2) = 1/2$, if and only if $\beta_1 = \eta_2$.
6. $p(\tilde{s}_1 \succ \tilde{s}_2) \geq 1/2$ and $p(\tilde{s}_2 \succ \tilde{s}_3) \geq 1/2$, then $p(\tilde{s}_1 \succ \tilde{s}_3) \geq 1/2$.

3. Uncertain linguistic weighted averaging operator

For an uncertain multiple attribute decision making problem, let $X = \{x_1, x_2, \ldots, x_m\}$ be a discrete set of alternatives, $U = \{u_1, u_2, \ldots, u_n\}$ be a set of attributes. For each alternative $x_i \in X$, the decision maker gives his/her preference value $\bar{a}_i$ with respect to attribute $u_j \in U$, where $\bar{a}_i$ takes the form of uncertain linguistic variable, that is $\bar{a}_i = \bar{a}_i = [a_{i1}, a_{i2}, \ldots, a_{il}]$, and $a_{ij} \in [0, 1], j = 1, 2, \ldots, n$. Then, the overall preference values of the alternatives consists the decision matrix $A = (\bar{a}_i)_{m \times n}$.

**Definition 3.** [23] Let ULWA: $\hat{S}^n \rightarrow \hat{S}$, if

$$\text{ULWA}_w(\bar{s}_1, \bar{s}_2, \ldots, \bar{s}_n) = \frac{w_1 \bar{s}_1 + w_2 \bar{s}_2 + \cdots + w_n \bar{s}_n} {w_1 + w_2 + \cdots + w_n}$$

where $w = (w_1, w_2, \ldots, w_n)^T$ is the weighting vector of uncertain linguistic variables $\bar{s}_i (i = 1, 2, \ldots, n)$, and $w_i \in [0, 1], i = 1, 2, \ldots, n$. Then, ULWA is called the uncertain linguistic weighted averaging (ULWA) operator. Especially, if $w = (1/n, 1/n, \ldots, 1/n)^T$, then the ULWA operator is reduced to the ULA operator.

**Definition 4.** Let $\hat{A} = (\bar{a}_i)_{m \times n}$ be the uncertain linguistic decision matrix, $\tilde{a}_i = (\bar{a}_{i1}, \bar{a}_{i2}, \ldots, \bar{a}_{il})$ be the vector of attribute values corresponding to the alternative $x_i, i = 1, 2, \ldots, m$, then we call

$$\tilde{z}(w) = \text{ULWA}_w(\tilde{a}_1, \tilde{a}_2, \ldots, \tilde{a}_m) = w_1 \tilde{a}_1 \oplus w_2 \tilde{a}_2 \oplus \cdots \oplus w_n \tilde{a}_n$$

the overall value of the alternative $x$, where $w = (w_1, w_2, \ldots, w_n)^T$ is the weighting vector of attributes.

The uncertain linguistic multiple attribute decision making problems generally consist of finding the most desirable alternative(s) from a given alternative set. If the decision matrix $A = (\bar{a}_i)_{m \times n}$ and the corresponding weight value of attributes are known, we can get the overall value of the alternative $x_i$ by Eq. (3). TOPSIS (technique for order performance by similarity to ideal solution) is a useful technique in dealing with multi-attribute or multi-criteria decision making problems in the real world [12]. It originates from the concept of a displaced ideal point from which the compromise solution has the shortest distance [22,26]. Hwang and Yoon [12] further propose that the ranking of alternatives will be based on the shortest distance from the (positive) ideal solution (PIS). Therefore, we can know that the TOPSIS technique is a sound logic that represents the rationale of human choice. In the following, we extend the TOPSIS to the uncertain linguistic variables.

**Definition 5.** Let $\tilde{A} = (\bar{a}_i)_{m \times n}$ be the uncertain linguistic decision matrix, then the ideal solution of attribute values are defined as following:

$$\bar{a}^* = \{\bar{a}_1, \bar{a}_2, \ldots, \bar{a}_l\}$$
where \( \tilde{a}_j = [\tilde{a}_{ij}^-, \tilde{a}_{ij}^+] = [\max_{i} a_{ij}^-, \max_{i} a_{ij}^+] \), \( j = 1, 2, \ldots, n \) and \( \tilde{a}_{ij}^-, \tilde{a}_{ij}^+ \) are the lower bound and upper bound of \( \tilde{a}_{ij} \).

**Definition 6.** Let \( s_1 = [s_{11}, s_{12}] \) and \( s_2 = [s_{21}, s_{22}] \in \tilde{S} \) be two uncertain linguistic variables, then we call
\[
D(s_1, s_2) = |x_2 - x_1| + |\beta_2 - \beta_1|
\]
the distance between \( s_1 \) and \( s_2 \).

**Definition 7.** Let \( \tilde{A} = (\tilde{a}_{ij})_{m \times n} \) be the uncertain linguistic decision matrix, \( \tilde{a}^* = (\tilde{a}_{1}^*, \tilde{a}_{2}^*, \ldots, \tilde{a}_{n}^*) \) be the ideal point of attribute values, where defined as **Definition 5**, we then call
\[
\tilde{z}(w) = ULM\tilde{W}(\tilde{a}_1^*, \tilde{a}_2^*, \ldots, \tilde{a}_n^*) = w_1 \tilde{a}_1^* \oplus w_2 \tilde{a}_2^* \oplus \ldots \oplus w_n \tilde{a}_n^*
\]
the overall value of ideal point \( \tilde{a}^* \).

### 4. Models and methods

In the real life, there always exist some differences between the ideal point of attribute values and the vector of attribute values corresponding to the alternative \( x_i (i = 1, 2, \ldots, m) \). By **Definitions 3–6**, in what follows we define the distance \( d_i \) between the overall value \( \tilde{z}(w) \) of ideal point and the overall value \( \tilde{z}_i(w) \) of the alternative \( x_i \):
\[
d_i = \sum_{j=1}^{m} (D(\tilde{a}_{ij}^-, \tilde{a}_{ij}^+)w_{j})^2, \quad i = 1, 2, \ldots, m
\]
Obviously, the smaller \( d_i \) is, the better the alternative \( x_i \) will be. Thus, a reasonable weight vector \( w^* = (w_1, w_2, \ldots, w_{n})^T \) should be determined so as to make all the distances \( d_i (i = 1, 2, \ldots, m) \) as smaller as possible, which means to minimize the following distance vector:
\[
d(w) = (d_1, d_2, \ldots, d_m)
\]
In order to do that, we establish the following multiple objective optimization model:
\[
\begin{align*}
(M-1) \min & F(w) = (d_1, d_2, \ldots, d_m) \\
\text{s.t.} & \sum_{j=1}^{m} w_{j} = 1, \quad w_{j} \geq 0 \quad j = 1, 2, \ldots, n
\end{align*}
\]
Generally, all the objectives are fairly competitive and there is no preference relationship among them, therefore the above model can be transformed into the following goal programming problem:
\[
\begin{align*}
(M-2) \min & F(w) = \sum_{j=1}^{m} n \left( D(\tilde{a}_{ij}^-, \tilde{a}_{ij}^+)w_{j} \right)^2 \\
\text{s.t.} & w_{j} \geq 0, \quad \sum_{j=1}^{m} w_{j} = 1
\end{align*}
\]
To solve this model, we construct the Lagrange function:
\[
F(w, \lambda) = \sum_{j=1}^{m} n \left( D(\tilde{a}_{ij}^-, \tilde{a}_{ij}^+)w_{j} \right)^2 + 2\lambda \left( \sum_{j=1}^{m} w_{j} - 1 \right)
\]
where \( \lambda \) is the Lagrange multiplier.

Differentiating Eq. (8) with respect to \( w_{j} (j = 1, 2, \ldots, n) \) and \( \lambda \), and setting these partial derivatives equal to zero, the following set of equations is obtained:
\[
\begin{align*}
\frac{\partial F}{\partial w_{j}} &= 2 \sum_{j=1}^{m} D(\tilde{a}_{ij}^-, \tilde{a}_{ij}^+)w_{j} + 2\lambda = 0, \quad j = 1, 2, \ldots, n \\
\frac{\partial F}{\partial \lambda} &= \sum_{j=1}^{m} w_{j} - 1 = 0
\end{align*}
\]
By solving Eq. (9), then we can get:
\[
w_{j} = -\frac{2 \sum_{j=1}^{m} D(\tilde{a}_{ij}^-, \tilde{a}_{ij}^+)}{\sum_{j=1}^{m} D(\tilde{a}_{ij}^-, \tilde{a}_{ij}^+)}, \quad j = 1, 2, \ldots, n
\]
\[
\sum_{j=1}^{m} w_{j} = 1
\]
By solving Eqs. (10) and (12), we get:
\[
\lambda = -\frac{1}{\sum_{j=1}^{m} D(\tilde{a}_{ij}^-, \tilde{a}_{ij}^+)}
\]
By solving Eqs. (10) and (11), we can get:
\[
w_{j} = \frac{1}{\sum_{j=1}^{m} D(\tilde{a}_{ij}^-, \tilde{a}_{ij}^+)}, \quad j = 1, 2, \ldots, n
\]
which can be used as the weight vector of attributes. Obviously, \( w_{j} \geq 0 \), for all \( j \).

In the real world, the information about attribute weights is incompletely known. Let \( w = (w_1, w_2, \ldots, w_{n})^T \in H \) be the weight vector of attributes, where \( w_{j} \in [0, 1], j = 1, 2, \ldots, n, \sum_{j=1}^{m} w_{j} = 1 \). \( H \) is a set of the known weight information, which can be constructed by the following forms \([2,14–17,21,26]\), for \( i \neq j \):

**Form 1.** A weak ranking: \( [w_{j} \geq w_{i}] \);
**Form 2.** A strict ranking: \( [w_{j} - w_{i} \geq z_i (>0)] \);
**Form 3.** A ranking of differences: \( [w_{j} - w_{i} \geq w_{k} - w_{i}] \), for \( j \neq k \neq i; \)
**Form 4.** A ranking with multiples: \( [w_{i} \geq z_i w_{j}], 0 \leq z_i \leq 1 \);
**Form 5.** An interval form: \( z_i \leq w_i \leq z_i + e_i, 0 \leq z_i < z_i + e_i \leq 1 \).

Forms 1–2 and Forms 4–5 are well known types of imprecise information, and Form 3 is ranking of differences of adjacent parameters obtained by ranking between two parameters, which can be constructed based on Form 1.

If the information about attribute weights is partly known, and Eq. (7) is replaced with the following deviation function
\[
d_i = \sum_{j=1}^{m} D(\tilde{a}_{ij}^-, \tilde{a}_{ij}^+)w_{j}, \quad i = 1, 2, \ldots, m
\]
Then, we can establish the following multiple-objective programming model:
\[
\begin{align*}
(M-3) \min & F(w) = (d_1, d_2, \ldots, d_m) \\
\text{s.t.} & w \in H, \quad \sum_{j=1}^{m} w_{j} = 1, w_{j} \geq 0 \quad j = 1, 2, \ldots, n
\end{align*}
\]
Generally, all the objectives are fairly competitive and there is no preference relationship among them, therefore the above model can be transformed into the following linear goal programming problem:
\[
\begin{align*}
(M-4) \min & \sum_{i=1}^{m} \lambda_i \\
\text{s.t.} & \sum_{j=1}^{m} D(\tilde{a}_{ij}^-, \tilde{a}_{ij}^+)w_{j} \leq \lambda_i, \quad i = 1, 2, \ldots, m, \quad w \in H, \quad \sum_{j=1}^{m} w_{j} = 1, \quad w_{j} \geq 0 \quad j = 1, 2, \ldots, n.
\end{align*}
\]
From the above analysis, we know that both the models (M-2) and (M-4) can be used to determine the attribute weights in a multiple attribute decision making problem with incomplete weight information under linguistic environment. Then we can utilize the Eq. (3) to get the overall value \( \tilde{z}(w)(i = 1, 2, \ldots, m) \) of each alternative. As \( \tilde{z}(w) \) is still the uncertain linguistic variable, it is difficult to rank them directly. Therefore, we can utilize the formula (13) to compare each \( \tilde{z}(w) \) with all \( \tilde{z}(w)(i = 1, 2, \ldots, m) \). For simplicity, we let \( p_{ij} = p_0(\tilde{z}(w) \geq \tilde{z}(w)) \), then we develop a complementary matrix as \( P = (p_{ij})_{m \times m} \), where
\[
p_{ij} \geq 0 \quad p_{ij} + p_{ji} = 1, \quad p_{ii} = \frac{1}{2}, \quad i, j = 1, 2, \ldots, m
\]
Summing all elements in each line of matrix $P$, we have

$$p_i = \sum_{j=1}^{m} p_{ij}, \quad i = 1, 2, \ldots, m \tag{15}$$

Then we rank $2(w)$ in descending order in accordance with the values of $p_i (i = 1, 2, \ldots, m)$.

Based on the above model, we develop a practical method for solving the uncertain linguistic multiple attribute decision making problems, in which the information about attribute weights is incompletely known, and the attribute values take the form of uncertain linguistic variables. The method involves the following steps:

**Step 1:** Let $X = \{x_1, x_2, \ldots, x_m\}$ be a discrete set of alternatives, $U = \{u_1, u_2, \ldots, u_n\}$ be a set of attributes, and $w = \{w_1, w_2, \ldots, w_n\} \in H$ be the weight vector of attributes, where $w_j \in [0, 1], j = 1, 2, \ldots, n$. $\sum_{j=1}^{n} w_j = 1$ is a set of the known weight information, which can be constructed by the Formulas 1–5. For each alternative $x_i \in X$, the decision maker gives his/her preference value $\tilde{a}_{ij}$ with respect to attribute $u_j \in U$, where $\tilde{a}_{ij}$ takes the form of an uncertain linguistic variable, that is $\tilde{a}_{ij} \in \tilde{S} \left( \tilde{a}_{ij} \in \tilde{S} \left( \tilde{a}_{ij} \in S, \tilde{a}_{ij} \in S \right) \right)$, then all the preference values of the alternatives consists the decision matrix $\tilde{A} = (\tilde{a}_{ij})_{m \times n}$ and by Eq. (4) we can get the ideal point $\tilde{a}^+ = (\tilde{a}_{ij}^+, \tilde{a}_{ij}^+, \tilde{a}_{ij}^+)$ of attribute values.

**Step 2:** If the information about the attribute weights is completely unknown, we solve the model (M-2) to obtain the optimal weight vector $w^* = \{w_1^*, w_2^*, \ldots, w_n^*\}$. If the information about the weights is partly known, then we solve the (M-4) to determine the attribute weights, and then by Eq. (3), we obtain the overall values $\tilde{z}_i(w^*) (i = 1, 2, \ldots, m)$ of the alternatives $\tilde{x}_i (i = 1, 2, \ldots, m)$.

**Step 3:** Utilize the formula (1) to compare each $\tilde{z}_i(w^*)$ with all $\tilde{z}_i(w) (i = 1, 2, \ldots, m)$, we get the possibility degree $p_{ij} = P(\tilde{z}_i(w^*) \geq \tilde{z}_j(w^*))$, and then construct a complementary matrix as $P = (p_{ij})_{m \times m}$, where $p_{ij} \geq 0, p_{ij} = 1$. $p_{ij} \geq 1, j = 1, 2, \ldots, n$.

**Step 4:** Utilize the Eq. (15) to get the sum in each line of the matrix $P = (p_{ij})_{m \times m}$. Then we rank the $\tilde{z}_i(w^*) (i = 1, 2, \ldots, m)$ in descending order in accordance with the values of $p_i (i = 1, 2, \ldots, m)$.

**Step 5:** Rank all the alternatives $x_i (i = 1, 2, \ldots, m)$ and select the best one(s) in accordance with the $\tilde{z}_i(w^*) (i = 1, 2, \ldots, m)$.

**Step 6:** End.

5. An illustrative example

Let us suppose to evaluate the technological innovation capability of enterprises, there are four enterprises denoted as $x_1, x_2, x_3, x_4$ to be evaluated. These attributes, which are critical for the selection of the best enterprise, are following (adapted from [7]): (1) $u_1$: innovation input capacity (2) $u_2$: innovation management capacity (3) $u_3$: innovation inclined. (4) $u_4$: research and development capabilities (5) $u_5$: manufacturing capacity. (6) $u_6$: marketing ability.

The four possible alternatives are to be evaluated using the linguistic term set $\tilde{S} = \{s_{-4} = \text{extremely poor}, s_{-3} = \text{very poor}, s_{-2} = \text{poor}, s_{-1} = \text{slightly poor}, s_0 = \text{fair}, s_1 = \text{slightly good}, s_2 = \text{good}, s_3 = \text{very good}, s_4 = \text{extremely good}\}$

and the decision maker give the uncertain linguistic decision matrix as listed in Table 1.

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**Table 1 Uncertain linguistic decision matrix $\tilde{A}$**

**Step 1:** From Table 1, we can get the ideal solution of the attribute values as follows:

$$\tilde{a}^+ = (\tilde{a}_{ij}^{1}, \tilde{a}_{ij}^{1}, \tilde{a}_{ij}^{1}, \tilde{a}_{ij}^{1}, \tilde{a}_{ij}^{1}, \tilde{a}_{ij}^{1})$$

**Step 2:** We solve the formula Eq. (13), then we have $w^* = 0.0687, 0.1458, 0.2292, 0.3437, 0.1266, 0.0859$. By Eq. (3), we obtain the overall values $\tilde{z}_i(w^*) (i = 1, 2, 3, 4)$ of the alternatives $x_i (i = 1, 2, 3, 4)$.

**Step 3:** To rank these collective overall preference values $\tilde{z}_i(w^*) (i = 1, 2, 3, 4)$, we first compare each $\tilde{z}_i(w^*)$ with all $\tilde{z}_i(w^*) (i = 1, 2, 3, 4)$ by using formula (1), and develop a complementary matrix.

**Step 4:** Summing all elements in each line of the matrix $P$, we have $p_1 = 1.8545, p_2 = 1.7145, p_3 = 2.4513, p_4 = 1.9788$

Then we rank the collective overall preference values $\tilde{z}_i(w^*) (i = 1, 2, 3, 4)$ in descending order in accordance with the values of $p_i (i = 1, 2, 3, 4)$.

**Step 5:** Rank all the alternatives $x_i (i = 1, 2, 3, 4)$ and select the best one(s) in accordance with the $\tilde{z}_i(w^*) (i = 1, 2, 3, 4)$:

$$x_2 \succ x_3 \succ x_4 \succ x_1$$

the best alternative is $x_2$.

If the information about the attribute weights is partly unknown, and the attribute weights information are follows:

$$0.06 \leq w_1 \leq 0.1, 0.1 \leq w_2 \leq 0.2, 0.2 \leq w_3 \leq 0.3, 0.3 \leq w_4 \leq 2.0, w_5 \leq 0.1, w_6 \leq 0.25$$

**Step 1:** From Table 1, we can get the ideal solution of the attribute values as follows:

$$\tilde{a}^+ = (\tilde{a}_{ij}^{1}, \tilde{a}_{ij}^{1}, \tilde{a}_{ij}^{1}, \tilde{a}_{ij}^{1}, \tilde{a}_{ij}^{1}, \tilde{a}_{ij}^{1})$$

**Step 2:** We utilize the model (M-4) to establish the following single-objective programming model:

$$\min \lambda_1 + \lambda_2 + \lambda_3 + \lambda_4$$

$$3w_1 + 2w_2 + 4w_3 + w_4 + 3w_5 + 2w_6 \leq \lambda_1$$

$$5w_2 + 3w_4 + w_5 + 6w_6 \leq \lambda_2$$

$$6w_1 + 3w_3 + 5w_5 \leq \lambda_3$$

$$5w_1 + 2w_2 + 2w_3 + 5w_4 \leq \lambda_4$$

$$0.06 \leq w_1 \leq 0.1, 0.1 \leq w_2 \leq 0.2, 0.2 \leq w_3 \leq 0.3, 0.3 \leq w_4 \leq 2.0, w_5 \leq 0.1, w_6 \leq 0.25$$

$$w_6 \geq 0.6w_5$$

The best alternative(s), the following steps are involved:
\[ w_j \geq 0, \quad j = 1, \ldots, 6, \sum_{j=1}^{6} w_j = 1 \]

Solving this model, we get the weight vector of attributes:

\[ w^* = (0.06, 0.172, 0.272, 0.344, 0.094, 0.057)^T \]

and \( \lambda_1 = 2.35, \quad \lambda_2 = 2.42, \quad \lambda_3 = 1.1, \quad \lambda_4 = 2.1 \) By Eq. (3), we obtain the overall values \( z_i(w^*) (i = 1, 2, 3, 4) \) of the alternatives \( x_i (i = 1, 2, 3, 4) \).

Step 2: To rank these collective overall preference values \( z_i(w^*) (i = 1, 2, 3, 4) \), we first compare each \( z_i(w^*) \) with all \( z_j(w^*) (i = 1, 2, 3, 4) \) by using formula (1), and develop a complementary matrix:

\[
P = \begin{bmatrix}
0.5 & 0.5122 & 0.3165 & 0.46 \\
0.4878 & 0.5 & 0.3138 & 0.4509 \\
0.6835 & 0.6862 & 0.5 & 0.6327 \\
0.54 & 0.5491 & 0.3673 & 0.5
\end{bmatrix}
\]

Step 4: Summing all elements in each line of the matrix \( P \), we have

\[ p_1 = 1.7887, \quad p_2 = 1.7525, \quad p_3 = 2.5024, \quad p_4 = 1.9564 \]

Then we rank the collective overall preference values \( z_i(w^*) (i = 1, 2, 3, 4) \) in descending order in accordance with the values of \( p_i (i = 1, 2, 3, 4) \):

\[ z_2(w^*) > z_4(w^*) > z_3(w^*) > z_1(w^*) \]

Step 5: Rank all the alternatives \( x_i (i = 1, 2, 3, 4) \) and select the best one(s) in accordance with the \( z_i(w^*) (i = 1, 2, 3, 4) \):

\[ x_3 > x_4 > x_1 > x_2 \]

thus the best alternative is \( x_3 \).

6. Concluding remarks

In the real world, sometimes the experts may estimate their preferences with uncertain linguistic variables and construct uncertain linguistic preference relations due to their vague knowledge, environment or time pressure, or his/her limited expertise about the problem domain, etc. In this paper, we have investigated the multiple attribute decision making problems, in which the attribute values take the form of uncertain linguistic variables, and the information about attribute weights is completely unknown. To determine the attribute weights, we have established a simple optimization model based on the concept of deviation degree of uncertain linguistic variables and the ideal point of uncertain linguistic multiple attribute values. Solving the model, we obtain a simple and exact formula for obtaining the attribute weights. The prominent characteristic of the developed method is that it can relieve the influence of subjectivity of the DMs and utilize the decision information sufficiently. For the situations where the information about the attribute weights is partly known, we establish an optimization model to determine the weights, and then we utilize the uncertain linguistic weighted average (ULWA) operator to aggregate the uncertain linguistic variables corresponding to each alternative. We utilize the formula of possibility degree of uncertain linguistic variables to compare the aggregated uncertain linguistic information, and then rank the alternatives. Finally, we give an example of practical application of the developed method to evaluate the technological innovation capability of enterprises.

References