Conditional entropy for the union of fuzzy and crisp partitions

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Abstract

In this paper we introduce the conditional entropy without a fuzzy measure for the union of fuzzy partitions. We recall its properties and we solve the system of functional equations which derives from these conditions, taking into account the *locality principle* and the *independence axiom*.

Keywords: Fuzzy partitions, Entropy, Functional equations.

1 Introduction

In [4] B.Forte has introduced the so called locality property in the crisp setting; later, in [2] Divari and Pandolfi have given some compositive laws for crisp partitions. Later, the authors have used the locality property in the fuzzy setting in [8].

The aim of this paper is to find a class of conditional entropy for the union of fuzzy partitions, by using the locality property and the independence axiom.

Finally, we present the crisp case: the system is analogous.

2 Preliminaires

From algebraic point of view, similar concepts have been introduced in the setting of MV-algebras by Mundici in [5].

Let X be an abstract space and $\mathcal{F}$ a family of fuzzy sets F [9] with membership function $F(x)$, $0 \leq F(x) \leq 1$ for all $x \in X$. Now we recall some definitions which we will use later.

Let F_1 and F_2 be two fuzzy sets, with membership functions $F_1(x)$ and $F_2(x)$. We recall that $(F_1 \cap^+ F_2)(x) = F_1(x) \wedge F_2(x), \forall x \in X$ is the membership function of the intersection set $F_1 \cap^+ F_2$ and $(F_1 \cup^+ F_2)(x) = F_1(x) \vee F_2(x), \forall x \in X$ is the membership function of the union set $F_1 \cup^+ F_2$.

Moreover, we say that two fuzzy sets F_1 and F_2 are disjoint if $F_1 \cap^+ F_2 = \emptyset$. A family $\{F_1, ..., F_n\}$ is a finite partition of a set F, called support, if $F_i \neq \emptyset, F_i$ are disjoint sets and $\Sigma_{i=1}^n F_i(x) = F(x) \ \forall \ x \in X$. We shall indicate with $\mathcal{P}(\mathrm{F})$ the fuzzy partition with support F, with $\{F\}$ the fuzzy set thought as a partition and with $\mathcal{K}$ the family of all partitions of X.

We consider the whole space X as a fuzzy partition, which shall be indicated by $\{X\}$.

In [1] we have recognized that for every $\mathcal{P}(\mathrm{F})$ an algebra $\mathcal{C}_{\mathcal{P}(\mathrm{F})}$ of crisp sets is associated, whose elements are the inverse images of Borel sets of $[0,1]^n$ through the map $F \to [0,1]^n$.

Given two partitions $\mathcal{P}(\mathrm{F})$ and $\mathcal{P}'(\mathrm{F})$ of the same set F :

$$\mathcal{P}(\mathrm{F}) = \{\mathrm{F}_1, ..., \mathrm{F}_i, ..., \mathrm{F}_n / \mathrm{F}_i \cap^+ \mathrm{F}_h = \emptyset, \mathrm{i} \neq \mathrm{h}, \qquad (1)$$

$$\Sigma_{i=1}^n F_i(x) = F(x) \forall \ x \in X\},$$

$$\mathcal{P}'(\mathrm{F}) = \{\mathrm{F}'_1, ..., \mathrm{F}'_j, ..., \mathrm{F}'_m / \mathrm{F}'_j \cap^+ \mathrm{F}'_k = \emptyset, \mathrm{j} \neq \mathrm{k},$$

$$\Sigma_{j=1}^m F'_j(x) = F(x) \forall \ x \in X\}$$

we say that $\mathcal{P}'(\mathrm{F})$ is a less fine than $\mathcal{P}(\mathrm{F})$ ($\mathcal{P}'(\mathcal{F}) \preceq \mathcal{P}(\mathrm{F})$) if $\mathcal{C}_{\mathcal{P}'(\mathrm{F})} \subset \mathcal{C}_{\mathcal{P}(\mathrm{F})}$.

Later, as in [3], we shall use the operation algebraic joint between two partitions of X. Now, we consider two partitions $\mathcal{P}(\mathrm{F})$ as in (1) and $\mathcal{P}(\mathrm{G})$ in $\mathcal{K}$:

$$\mathcal{P}(\mathrm{G}) = \{\mathrm{G}_1, ..., \mathrm{G}_j, ..., \mathrm{G}_m / \mathrm{G}_j \cap^+ \mathrm{G}_k = \emptyset, \mathrm{j} \neq \mathrm{k},$$

$$\Sigma_{j=1}^m G_j(x) = G(x) \forall \ x \in X\}.$$

The algebraic joint $\mathcal{P}(\mathcal{F}) \nabla \mathcal{P}(\mathrm{G})$ is

$$\mathcal{P}(\mathrm{F}) \nabla \mathcal{P}(\mathrm{G}) = \{\mathrm{F}_i \cdot \mathrm{G}_j / 1 \leq \mathrm{i} \leq \mathrm{n}, \mathrm{i} \leq \mathrm{j} \leq \mathrm{m}\}$$

with lexicographic order, where $(F_i \cdot G_j)(x) = F_i(x) \cdot G_j(x), \forall\ x \in X$. We say that two fuzzy partitions $\mathcal{P}$ and $\mathcal{P}(\mathrm{F})$ are algebraically independent if

$$(F_i \cdot G_j)(x) \neq 0, \forall\ x \in X. \tag{2}$$

Now, we recall the definition of the union of two partitions, given in [6, 8].

Given two partitions $\mathcal{P}(\mathrm{F})$ and $\mathcal{P}(\mathrm{G}) \in \mathcal{K}$, the union $\mathcal{P}(\mathrm{F}) \cup^+ \mathcal{P}(\mathrm{G})$ is defined by

$$\mathcal{P}(\mathrm{F}) \cup^+ \mathcal{P}(\mathrm{G}) = \{\mathrm{F_1}, ..., \mathrm{F_i}, ..., \mathrm{F_n}, \mathrm{G_1}, ..., \mathrm{G_j}, ..., \mathrm{G_m}/$$

$$\forall x \in X,\ F_i(x) \wedge G_j(x) = 0\ (i \neq j),$$

$$\Sigma_{i=1}^{n} F_i(x) + \Sigma_{j=1}^{m} G_j(x) \leq 1\}. \tag{3}$$

In this case, we call $\mathcal{P}(\mathrm{F})$ and $\mathcal{P}(\mathrm{G})$ compositive partitions. Let $\mathcal{H} \subset \mathcal{K}$ be the family of compositive partitions.

As in [7], we introduce the entropy without a fuzzy measure for a fuzzy partition $\mathcal{P}(\mathrm{F}) \in \mathcal{K}$ conditionated by $\mathcal{Q} \in \mathcal{K}$ with $H(\mathcal{Q}) \neq +\infty$ as a function

$$H^*_{\mathcal{Q}}(\cdot) : \mathcal{K} \to [0, +\infty]$$

with the following properties:

(I) $\mathcal{P}'(\mathrm{F}) \preceq \mathcal{P}(\mathrm{F}) \Rightarrow H^*_{\mathcal{Q}}(\mathcal{P}'(\mathrm{F})) \leq H^*_{\mathcal{Q}}(\mathcal{P}(\mathrm{F}))$,

$\mathcal{P}'(\mathrm{F}) \in \mathcal{K}$.

(II) $H^*_{\mathcal{Q}}(\mathcal{P}(\mathrm{F})) = H^*(\mathcal{P}(\mathrm{F}))$,

if $\mathcal{P}(\mathrm{F})$ is not conditioned by $\mathcal{Q}$.

It follows that : $H^*_{\mathcal{Q}}(\{X\}) = 0$ and $H^*_{\mathcal{Q}}(\{\emptyset\}) = +\infty$.

(III) $H^*_{\mathcal{Q}}(\mathcal{P}(\mathrm{F}) \nabla \mathcal{P}(\mathrm{G})) = H^*_{\mathcal{Q}}(\mathcal{P}(\mathrm{F})) + H^*_{\mathcal{Q}}(\mathcal{P}(\mathrm{G})$,

if $\mathcal{P}(\mathrm{F})$ and $\mathcal{P}(\mathrm{G})$ are algebraically independent and $\mathcal{P}(\mathrm{G}) \in \mathcal{K}$.

3 Conditional entropy for the union of the partitions

If $\mathcal{P}(\mathrm{F}), \mathcal{P}'(\mathrm{F}), \mathcal{P}(\mathrm{G}) \in \mathcal{H}, \mathcal{Q}, \mathcal{R} \in \mathcal{K}$ and $H^*(\mathcal{Q}) \neq +\infty$, the conditional entropy $H^*_{\mathcal{Q}}\Big(\mathcal{P}(\mathrm{F}) \cup^+ \mathcal{P}(\mathrm{G})\Big)$ for the union of the fuzzy partitions satisfies the following conditions:

(α) $0 \leq H^*_{\mathcal{Q}}\Big(\mathcal{P}(\mathrm{F}) \cup^+ \mathcal{P}(\mathrm{G})\Big) \leq +\infty$

as $\mathcal{P}(\mathrm{F}) \cup^+ \mathcal{P}(\mathrm{G}) = \{\mathrm{X}\} \to \mathrm{H}^*_{\mathcal{Q}}\Big(\mathcal{P}(\mathrm{F}) \cup^+ \mathcal{P}(\mathrm{G})\Big) = 0$ and $\mathcal{P}(\mathrm{F}) \cup^+ \mathcal{P}(\mathrm{G}) = \{\emptyset\} \to \mathrm{H}^*_{\mathcal{Q}}\Big(\mathcal{P}(\mathrm{F}) \cup^+ \mathcal{P}(\mathrm{G})\Big) = +\infty$;

(β) $H^*_{\mathcal{Q}}\Big(\mathcal{P}(\mathrm{F}) \cup^+ \mathcal{P}(\mathrm{G})\Big) = H^*_{\mathcal{Q}}\Big(\mathcal{P}(\mathrm{G}) \cup^+ \mathcal{P}(\mathrm{F})\Big)$;

(γ) $\mathcal{P}'(\mathrm{F}) \preceq \mathcal{P}(\mathrm{F}) \Rightarrow$

$$H^*_{\mathcal{Q}}\Big(\mathcal{P}'(\mathrm{F}) \cup^+ \mathcal{P}(\mathrm{G})\Big) \leq H^*_{\mathcal{Q}}\Big(\mathcal{P}(\mathrm{F}) \cup^+ \mathcal{P}(\mathrm{G})\Big),$$

(δ) $H^*_{\mathcal{Q}}\Big((\mathcal{P}(\mathrm{F}) \cup^+ \mathcal{P}(\mathrm{G}))\nabla\mathcal{R}\Big) =$

$$H^*_{\mathcal{Q}}\Big(\mathcal{P}(\mathrm{F}) \cup^+ \mathcal{P}(\mathrm{G})\Big) + H^*(\mathcal{R}),$$

if $\mathcal{R}$ is algebraically independent from $\mathcal{P}(\mathrm{F})$ and $\mathcal{P}(\mathrm{G})$ and it is not conditioned by $\mathcal{Q}$; the (δ) is a consequence of (II) and (III) seen above.

4 Statement of the problem

Taking into account the locality principle fuzzy partitions [7]:

$$H^*\Big(\mathcal{P}(\mathrm{F}) \cup^+ \mathcal{P}(\mathrm{G})\Big) - H^*\Big(\mathcal{P}(\mathrm{F}) \cup^+ \{\mathrm{G}\}\Big) =$$

$$H^*\Big(\{F\} \cup^+ \mathcal{P}(\mathrm{G})\Big) - H^*\Big(\{F\} \cup^+ \{G\}\Big). \tag{4}$$

we put

$$H^*_{\mathcal{Q}}\Big(\mathcal{P}(\mathrm{F}) \cup^+ \mathcal{P}(\mathrm{G})\Big) = \Psi\Big[\mathrm{H}^*\Big((\mathcal{P}(\mathrm{F}) \cup^+ \{\mathrm{G}\})\nabla\mathcal{Q}\Big),$$

$$H^*\Big((\{F\} \cup^+ \mathcal{P}(\mathrm{G}))\nabla\mathcal{Q}\Big), H^*\Big((\{F\} \cup^+ \{G\})\nabla\mathcal{Q}\Big), H^*(\mathcal{Q})\Big] \tag{5}$$

where $\mathcal{P}(\mathrm{F}), \mathcal{P}'(\mathrm{F}), \mathcal{P}(\mathrm{G}) \in \mathcal{H}, \mathcal{Q}, \mathcal{R} \in \mathcal{K}$, with $H^*(\mathcal{Q}) \neq +\infty$ and $\Psi(\cdot) = [0, +\infty] \times [0, +\infty] \times [0, +\infty] \times [0, +\infty] \to [0, +\infty]$.

From $[(\alpha) - (\delta)]$, we have:

(A) $\Psi\Big[H^*\Big(\{X\}\nabla\mathcal{Q}\Big), H^*\Big(\{X\}\nabla\mathcal{Q}\Big),$

$$H^*\Big(\{X\}\nabla\mathcal{Q}\Big), H^*(\mathcal{Q})\Big] = 0;$$

(B) $\Psi\Big[H^*\Big((\mathcal{P}(\mathrm{F}) \cup^+ \{\mathrm{G}\})\nabla\mathcal{Q}\Big), H^*\Big((\{F\} \cup^+ \mathcal{P}(\mathrm{G}))\nabla\mathcal{Q}\Big),$

$$H^*\Big((\{F\} \cup^+ \{G\})\nabla\mathcal{Q}\Big), H^*(\mathcal{Q})\Big] =$$

$$\Psi\Big[H^*\Big((\mathcal{P}(\mathrm{G})\cup^+\{\mathrm{F}\})\nabla\mathcal{Q}\Big), H^*\Big((\{G\}\cup^+\mathcal{P}(\mathrm{F}))\nabla\mathcal{Q}\Big),$$

$$H^*\Big((\{G\}\cup^+\{F\})\nabla\mathcal{Q}\Big), H^*(\mathcal{Q})\Big];$$

(C) $\Psi\Big[H^*\Big((\mathcal{P}'(\mathrm{F})\cup^+\{\mathrm{G}\})\nabla\mathcal{Q}\Big),$

$$H^*\Big((\{F\}\cup^+\mathcal{P}(\mathrm{G}))\nabla\mathcal{Q}\Big),$$

$$H^*\Big((\{F\}\cup^+\{G\})\nabla\mathcal{Q}\Big), H^*(\mathcal{Q})\Big]$$

$$\leq\Psi\Big[H^*\Big((\mathcal{P}(\mathrm{F})\cup^+\{\mathrm{G}\})\nabla\mathcal{Q}\Big), H^*\Big((\{F\}\cup^+\mathcal{P}(\mathrm{G}))\nabla\mathcal{Q}\Big),$$

$$H^*\Big((\{F\}\cup^+\{G\})\nabla\mathcal{Q}\Big), H^*(\mathcal{Q})\Big]$$

if $\mathcal{P}'(\mathrm{F})\preceq\mathcal{P}(\mathrm{F})$;

(D) $\Psi\Big[H^*\Big(((\mathcal{P}(\mathrm{F})\cup^+\{\mathrm{G}\})\nabla\mathcal{Q})\nabla\mathcal{R}\Big),$

$$H^*\Big(((\{F\}\cup^+\mathcal{P}(\mathrm{G}))\nabla\mathcal{Q})\nabla\mathcal{R}\Big),$$

$$H^*\Big(((\{F\}\cup^+\{G\})\nabla\mathcal{Q})\nabla\mathcal{R}\Big), H^*(\mathcal{Q})\Big] =$$

$$\Psi\Big[H^*\Big((\mathcal{P}(\mathrm{F})\cup^+\{\mathrm{G}\})\nabla\mathcal{Q}\Big),$$

$$H^*\Big((\{F\}\cup^+\mathcal{P}(\mathrm{G}))\nabla\mathcal{Q}\Big),$$

$$H^*\Big((\{F\}\cup^+\{G\})\nabla\mathcal{Q}\Big), H^*(\mathcal{Q})\Big] + H^*(\mathcal{R}),$$

if $\mathcal{R}\in\mathcal{K}$ is algebraically independent from $\mathcal{P}(\mathrm{F})$ and $\mathcal{P}(\mathrm{G})$ and it is not conditioned by $\mathcal{Q}$.

Setting:

$$H^*\Big((\{F\}\cup^+\mathcal{P}(\mathrm{F}))\nabla\mathcal{Q}\Big) = x,$$

$$H^*\Big((\mathcal{P}(\mathrm{F})\cup^+\{\mathrm{G}\})\nabla\mathcal{Q}\Big) = y,$$

$$H^*\Big((\{F\}\cup^+\{G\})\nabla\mathcal{Q}\Big) = z,$$

$$H^*\Big((\mathcal{P}'(\mathrm{F})\cup^+\{\mathrm{G}\})\nabla\mathcal{Q}\Big) = y',$$

$$H^*(\mathcal{R}) = s, H^*(\mathcal{Q}) = t,$$

with $x, y, z, y' \in [0, +\infty], x \leq z, y \leq z$ and $s, t \in (0, +\infty)$. From $(A) - (D)$, we get the following system of functional equations:

$$\begin{cases} (a)\ \Psi(t,t,t,t) = 0\ , \\ (b)\ \Psi(x,y,z,t) = \Psi(y,x,z,t)\ , \\ (c)\ \Psi(x,y',z,t) \leq \Psi(x,y,z,t) \text{ if } y' \leq y\ , \\ (d)\ \Psi(x+s,y+s,z+s,t) = \Psi(x,y,z,t) + s\ . \end{cases}$$

5 Solution of the problem

In this paragraph we shall solve the system of functional equations seen above.

We look for a function Ψ continuous as a universal law, in the sense that the equations and the inequality about Ψ must be satisfied for all values of the variables in their proper spaces.

Proposition 5.1 *Let h be any bijettive and strictly increasing function, differentiable with its inverse, and with $h(0) = 0$; then every function Ψ of the type*

$$\Psi(x,y,z,t) = \frac{1}{3}\Big[h^{-1}\Big(h(x) - h(t)\Big) +$$

$$h^{-1}\Big(h(y) - h(t)\Big) + h^{-1}\Big(h(z) - h(t)\Big)\Big]\ , \qquad (6)$$

enjoys the equations $(a), (b)$ and (c); it satisfies also (d) if and only if

$$\Psi(x,y,z,t) = \frac{x+y+z}{3} - t\ . \qquad (7)$$

Proof

The conditions $(a), (b), (c)$ are clearly satisfied by any function satisfying (6).

Moreover, it is obvious that the function (7) satisfies all the conditions $(a) - (d)$, and is of the form (6), for h = identity map.

So, it remains to prove that every function Ψ of the form (6), satisfying (d), must be of the type (7).

By (6) the condition (δ) becomes

$$\frac{1}{3}\Big[h^{-1}\Big(h(x+s) - h(t)\Big) + h^{-1}\Big(h(y+s) - h(t)\Big)$$

$$+h^{-1}\Big(h(z+s) - h(t)\Big)\Big] = \frac{1}{3}\Big[h^{-1}\Big(h(x) - h(t)\Big) +$$

$$h^{-1}\Big(h(y) - h(t)\Big) + h^{-1}\Big(h(z) - h(t)\Big)\Big] + s\ . \qquad (8)$$

Putting

$$h^{-1}\Big(h(x) - h(t)\Big) = \varphi_t(x), \qquad (9)$$

then (8) becomes

$$\frac{1}{3}\Big[\varphi_t(x+s) + \varphi_t(y+s) + \varphi_t(z+s)\Big]$$

$$= \frac{1}{3}\Big[\varphi_t(x)+\varphi_t(y)+\varphi_t(z)\Big] + s, \quad \text{i.e.}$$

$$\varphi_t(x+s)+\varphi_t(y+s)+\varphi_t(z+s)$$

$$= \varphi_t(x)+\varphi_t(y)+\varphi_t(z)+3s \; . \qquad (10)$$

Choosing $y = x+s$, $z = y+s$, (10) gives rise to $\varphi_t(z+s) = \varphi_t(x)+3s$, i.e.

$$\varphi_t(x+3s) = \varphi_t(x)+3s. \qquad (11)$$

Now, we calculate the derivative of (11) with respect to s :

$$3\frac{\partial\varphi_t}{\partial s}(x+3s) = 3, \quad \frac{\partial\varphi_t}{\partial s} = 1$$

i.e., by arbitrariness of x and s,

$$\varphi_t(s) = s + c(t) \; , \qquad (12)$$

where $c(t)$ is an arbitrary function, which we can calculate by the derivative of the function (11) with respect to t. We get

$$\frac{\partial\varphi_t}{\partial t}(x+3s) = \frac{\partial\varphi_t}{\partial t}(x) \;, \forall x, s \text{ so } \frac{\partial\varphi_t}{\partial t} = (\text{cost}) = c_1$$

and from (12) $c'(t) = c_1$. The last equality implies $c(t) = c_1 t + c_2$, where c_2 is another constant. So, from (12), we have obtained the function

$$\varphi_t(x) = x + c_1 t + c_2. \qquad (13)$$

By substituting (13) in (9), we get:

$$h(x) - h(t) = h(x + c_1 t + c_2), \forall x; \qquad (14)$$

in particular for $x = t$ we obtain

$$0 = h((1+c_1)t + c_2) \; \forall t. \qquad (15)$$

Now (15) implies two possibilities: 1) $h(t) = 0, \forall t$ which we eliminate, 2) $c_1 = -1$ and $c_2 = 0$. From (14) we deduce $h(x) - h(t) = h(x-t)$, so the function h is linear: $h(x) = \alpha x$. In conclusion, from (6) it follows (7):

$$\Psi(x,y,z,t) = \frac{x+y+z}{3} - t \; .$$

6 Conditional entropy for crisp partitions

For crisp partitions [2], we can use the following locality property:

$$H\Big(\pi_A \cup \pi_B\Big) - H\Big(\pi_A \cup \{B\}\Big) =$$

$$H\Big(\{A\} \cup \pi_B\Big) - H\Big(\{A\} \cup \{B\}\Big) \quad , \qquad (16)$$

and we put

$$H_{\pi'}(\pi_A \cup \pi_B) = \Phi\Bigg[H\Big((\pi_A \cup \{B\}) \cap \pi'\Big),$$

$$H\Big((\{A\} \cup \pi_B) \cap \pi'\Big),\; H\Big((\{A\} \cup \{B\}) \cap \pi'\Big),\; H(\pi')\Bigg] \quad (17)$$

$\pi_A, \pi_B, \pi' \in \mathcal{E}$, $\mathcal{E}$ is the family of all partitions of $A, B \subset X$, π' is the partition which conditiones $\pi_A \cup \pi_B$ and $H(\pi') \neq +\infty$, and $\Phi(\cdot) = [0,+\infty] \times [0,+\infty] \times [0,+\infty] \times [0,+\infty] \to [0,+\infty]$.

The system of functional equations are the same, so, also in the crisp setting, we get the same solution (7).

7 Conclusion

Taking into account the locality principle and the independence axiom, the measures of conditional entropy are:

- from (5) and (7), for fuzzy partitions:

$$H_{\mathcal{Q}}(\mathcal{P}(\mathrm{F}) \cup^+ \mathcal{P}(\mathrm{G})) = \frac{1}{3}\Bigg[H^*\Big((\{F\} \cup^+ \mathcal{P}(\mathrm{F}))\nabla\mathcal{Q}\Big) +$$

$$H^*\Big((\mathcal{P}(\mathrm{F}) \cup^+ \{\mathrm{G}\})\nabla\mathcal{Q}\Big) + H^*\Big((\{F\} \cup^+ \{G\})\nabla\mathcal{Q}\Big)\Bigg] - \mathrm{H}(\mathcal{Q})$$

- from (17) and (7), for crisp partitions:

$$H_{\pi'}(\pi_A \cup \pi_B)) = \frac{1}{3}\Bigg[H\Big((\{A\} \cup \pi_B) \cap \pi'\Big) + H\Big((\pi_A \cup \{B\}) \cap \pi'\Big) +$$

$$H\Big((\{A\} \cup \{B\}) \cap \pi'\Big)\Bigg] - H(\pi').$$

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