The Swaption Cube

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Introduction

- A standard European swaption is an option to enter into a fixed versus floating forward starting interest rate swap at a predetermined rate on the fixed leg
- Understanding the swaption market is important from a practical perspective...
  - By some measures (such as the notional amount of outstanding options) the world's largest derivatives market
  - Many standard fixed income securities, such as fixed rate mortgage-backed securities and callable agency securities, embed swaption-like options
  - Many exotic interest rate derivatives and structured products also embed swaptions
  - Many large corporations are active in the swaption market either directly or indirectly (through the issuance and swapping of callable debt)
- and from an academic perspective...
  - Swaptions contain valuable information about interest rate distributions

Many empirical studies on the swaption market. This paper differs from these in two important ways

- All existing studies are limited to only using data on at-the-money (ATM) swaptions. In contrast, we analyze a proprietary data set consisting of *swaption cubes*, which sheds new light on the swaption market.
- Existing studies are mostly concerned with the pricing and hedging of swaptions using reduced-form models. Key objective of the paper is to understand the fundamental drivers of prices and risk premia in the swaption market.

- Vast literature on interest derivatives
- Paper is related to a growing literature linking the term structure of interest rates to macro factors (Ang and Piazzesi (2003), Gallmeyer, Hollifield, and Zin (2005), Ang, Piazzesi, and Wei (2006), Smith and Taylor (2009), Bekaert, Cho, and Moreno (2010), Bikbov and Chernov (), Chun (2010), and Joslin, Priebsch, and Singleton (2010))
- This literature is mainly based on Gaussian models and, consequently, primarily concerns itself with the determinants of the conditional mean of interest rates
- We complement this literature by studying the determinants of the conditional volatility and skewness of interest rates, which are critical for derivatives prices
- While existing macro-term structure papers focus on USD market, we consider both USD and EUR markets to see if differences in monetary policy objectives has effect on the relative importance of various macro drivers

Agenda

- The swaption cube data
- A model independent analysis of the swaption cube
- A dynamic term structure model for swap rates
- Fundamental drivers of the swap rate distributions

The swaption cube data

The swaption cube is an object that shows how swaption prices vary along three dimensions:

- The maturities of the underlying swaps (2, 5, 10, 20, and 30 years)
- The expiries of the options (1, 3, 6, 9 months and 1, 2, 5, and 10 years)
- The option strikes (15 different degrees of moneyness measured as the strike minus the at-the-money level of the swaption ($\pm$ 400, $\pm$ 300, $\pm$ 200, $\pm$ 150, $\pm$ 100, $\pm$ 50, $\pm$ 25, and 0 basis points))

- Consider both USD and EUR markets, which are by far the most liquid markets.
- For the USD market, the data is from December 19, 2001 to January 27, 2010 (419 weeks, total of 172,658 quotes). For the EUR market, the data is from June 6, 2001 to January 27, 2010 (449 weeks, total of 172,500 quotes)
- Data is from ICAP plc., which is the largest inter-dealer broker in the interest rate derivatives market and as such provides the most accurate quotations.

A model independent analysis of the swaption cube

- For a given option expiry and swap maturity, conditional moments of the swap rate distribution (under the appropriate pricing measure) at a time horizon equal to the option expiry can be inferred from the implied volatility smile
- We analyze how conditional moments vary with option expiry, swap maturity, and across time in order to establish a set of robust stylized facts regarding the swaption cube

- Consider a fixed versus floating interest rate swap for the period T_m to T_n with a fixed rate of K. At every time T_j in a pre-specified set of dates $T_{m+1}, ..., T_n$, the fixed leg pays $\tau_{j-1}K$, where τ_{j-1} is the year-fraction between times T_{j-1} and T_j.
- The value of the swap at time $t < T_m$ (assuming a notional of one) is given by

  $$V_{m,n}(t) = P(t, T_m) - P(t, T_n) - KA_{m,n}(t),$$

  where

  $$A_{m,n}(t) = \sum_{j=m+1}^{n} \tau_{j-1} P(t, T_j)$$

  and $P(t, T)$ denotes the time-t price of a zero-coupon bond maturing at time T.

- The time-t forward swap rate, $S_{m,n}(t)$, is the rate on the fixed leg that makes the present value of the swap equal to zero, and is given by

$$S_{m,n}(t) = \frac{P(t, T_m) - P(t, T_n)}{A_{m,n}(t)}.$$

- A payer swaption is an option to enter into an interest rate swap, paying the fixed leg at a predetermined rate and receiving the floating leg. Let $\mathcal{P}_{m,n}(t, K)$ denote the time-t value of a European payer swaption expiring at T_m with strike K on a swap for the period T_m to T_n. At expiration, the swaption has a payoff of

$$\begin{aligned} V_{m,n}(T_m)^+ &= (1 - P(T_m, T_n) - KA_{m,n}(T_m))^+ \\ &= A_{m,n}(T_m)\,(S_{m,n}(T_m) - K)^+ . \end{aligned}$$

- At time $t < T_m$, its price is given by

$$\begin{aligned} \mathcal{P}_{m,n}(t, K) &= E_t^{\mathbb{Q}}\left[e^{-\int_t^{T_m} r(s)ds} A_{m,n}(T_m)\,(S_{m,n}(T_m) - K)^+\right] \\ &= A_{m,n}(t) E_t^{\mathbb{A}}\left[(S_{m,n}(T_m) - K)^+\right], \end{aligned}$$

where $\mathbb{Q}$ denotes expectation under the risk-neutral measure and $\mathbb{A}$ denotes expectation under the annuity measure associated with using $A_{m,n}(t)$ as numeraire.

- From Bakshi and Madan (2000), Carr and Madan (2001), and Bakshi, Kapadia, and Madan (2003) it follows that for any fixed Z, we can write any twice continuously differentiable function of $S_{m,n}(T_m)$, $g(S_{m,n}(T_m))$, as

$$\begin{aligned} g(S_{m,n}(T_m)) &= g(Z) + g'(Z)(S_{m,n}(T_m) - Z) \\ &\quad + \int_Z^\infty g''(K)(S_{m,n}(T_m) - K)^+ dK + \int_0^Z g''(K)(K - S_{m,n}(T_m))^+ dK. \end{aligned}$$

- Taking expectations under the annuity measure and setting $Z = S_{m,n}(t)$, we obtain an expression in terms of prices of out-of-the-money receiver and payer swaptions

$$\begin{aligned} E_t^{\mathbb{A}}\left[g(S_{m,n}(T_m))\right] &= g(S_{m,n}(t)) \\ &\quad + \frac{1}{A_{m,n}(t)}\left(\int_{S_{m,n}(t)}^\infty g''(K)\mathcal{P}_{m,n}(t,K)dK + \int_0^{S_{m,n}(t)} g''(K)\mathcal{R}_{m,n}(t,K)dK\right). \end{aligned}$$

We can use this result to compute conditional moments of the swap rate distribution at a time horizon equal to the expiry of the option.

- Conditional mean

$$\mu_t \equiv E_t^{\mathbb{A}}\left[S_{m,n}(T_m)\right] = S_{m,n}(t).$$

- Conditional variance

$$\mathrm{Var}_t^{\mathbb{A}}\left(S_{m,n}(T_m)\right) = E_t^{\mathbb{A}}\left[\left(S_{m,n}(T_m) - \mu_t\right)^2\right] =$$
$$\frac{2}{A_{m,n}(t)}\Big(\int_{S_{m,n}(t)}^{\infty} \mathcal{P}_{m,n}(t,K)dK + \int_0^{S_{m,n}(t)} \mathcal{R}_{m,n}(t,K)dK\Big)$$

- Conditional skewness

$$\mathrm{Skew}_t^{\mathbb{A}}\left(S_{m,n}(T_m)\right) = \frac{E_t^{\mathbb{A}}\left[\left(S_{m,n}(T_m) - \mu_t\right)^3\right]}{\mathrm{Var}_t^{\mathbb{A}}\left(S_{m,n}(t,T_m)\right)^{3/2}} =$$

$$\frac{\frac{6}{A_{m,n}(t)}\Big(\int_{S_{m,n}(t)}^{\infty}\left(K - S_{m,n}(t)\right)\mathcal{P}_{m,n}(t,K)dK + \int_0^{S_{m,n}(t)}\left(K - S_{m,n}(t)\right)\mathcal{R}_{m,n}(t,K)dK\Big)}{\mathrm{Var}_t^{\mathbb{A}}\left(S_{m,n}(t,T_m)\right)^{3/2}}$$

- Conditional kurtosis

$$\mathrm{Kurt}_t^{\mathbb{A}}\left(S_{m,n}(T_m)\right) = \frac{E_t^{\mathbb{A}}\left[\left(S_{m,n}(T_m) - \mu_t\right)^4\right]}{\mathrm{Var}_t^{\mathbb{A}}\left(S_{m,n}(t,T_m)\right)^{2}} =$$

$$\frac{\frac{12}{A_{m,n}(t)}\Big(\int_{S_{m,n}(t)}^{\infty}\left(K - S_{m,n}(t)\right)^2\mathcal{P}_{m,n}(t,K)dK + \int_0^{S_{m,n}(t)}\left(K - S_{m,n}(t)\right)^2\mathcal{R}_{m,n}(t,K)dK\Big)}{\mathrm{Var}_t^{\mathbb{A}}\left(S_{m,n}(t,T_m)\right)^{2}}$$

Tenor	Option expiry							
	1 mth	3 mths	6 mths	9 mths	1 yr	2 yrs	5 yrs	10 yrs
				Panel A: USD market				
2 yrs	110.1 (35.5)	112.0 (31.5)	114.4 (27.9)	117.4 (26.9)	120.7 (26.6)	123.6 (24.6)	116.9 (18.0)	98.2 (12.0)
5 yrs	122.4 (37.4)	122.2 (33.5)	121.8 (29.4)	121.1 (27.4)	121.3 (26.3)	120.4 (23.2)	111.7 (16.5)	93.2 (10.4)
10 yrs	115.9 (36.6)	115.8 (32.7)	115.4 (28.7)	114.7 (26.5)	114.4 (25.0)	113.3 (22.0)	104.7 (15.0)	86.8 (9.2)
20 yrs	106.9 (36.1)	105.7 (31.4)	104.0 (26.7)	102.7 (24.0)	101.8 (21.8)	99.5 (18.6)	89.9 (12.3)	74.1 (7.9)
30 yrs	103.5 (37.6)	101.7 (31.5)	99.9 (26.3)	98.6 (23.4)	97.5 (20.9)	95.1 (17.1)	85.4 (10.8)	69.6 (6.3)
				Panel B: EUR market				
2 yrs	80.7 (29.2)	81.2 (24.6)	81.6 (20.2)	81.3 (17.2)	80.8 (15.3)	80.6 (12.9)	78.0 (8.7)	72.1 (6.3)
5 yrs	83.6 (25.3)	82.4 (21.1)	80.9 (17.1)	79.6 (14.8)	78.6 (13.4)	77.1 (11.5)	74.2 (8.5)	69.1 (7.2)
10 yrs	74.7 (23.1)	74.7 (20.5)	74.3 (17.6)	73.7 (15.9)	73.3 (15.0)	73.4 (13.6)	71.9 (9.9)	67.1 (7.6)
20 yrs	72.3 (30.0)	72.0 (26.4)	71.2 (21.8)	70.4 (19.3)	70.0 (18.0)	69.3 (15.4)	67.2 (10.9)	61.9 (8.2)
30 yrs	71.6 (36.8)	71.2 (32.5)	70.3 (26.7)	69.3 (23.0)	68.7 (20.8)	67.9 (17.6)	65.3 (12.9)	59.8 (9.2)

Notes: The table shows average conditional volatilities (annualized and in basis points) of the future swap rate distributions under the annuity measure $\mathbb{A}$. Standard deviations of conditional volatilities are in parentheses. In USD, each statistic is computed on the basis of 419 weekly observations from December 19, 2001 to January 27, 2010. In EUR, each statistic is computed on the basis of 449 weekly observations from June 6, 2001 to January 27, 2010.

Table 1: Volatility (annualized) of swap rate distributions

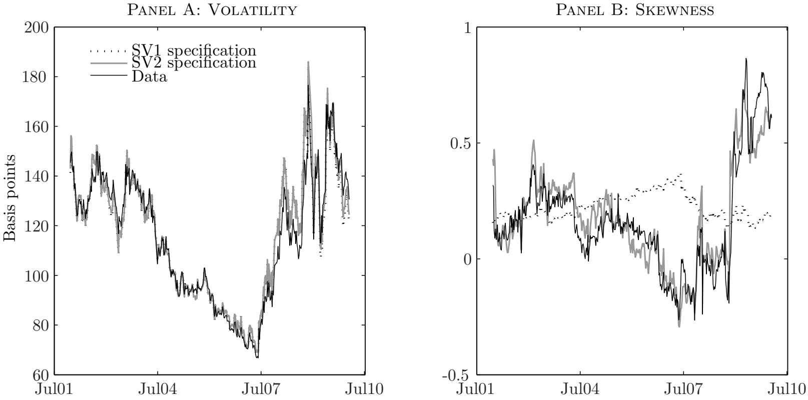

Figure 1: Time-series of volatility and skewness of the conditional 1-year ahead distribution of the USD 10-year swap rate

Notes: Panel A displays conditional volatility, measured in basis points, and Panel B displays conditional skewness. The moments are computed under the annuity measure $\mathbb{A}$ The time-series consist of 419 weekly observations from December 19th, 2001 to January 27th, 2010.

Tenor	Option expiry							
	1 mth	3 mths	6 mths	9 mths	1 yr	2 yrs	5 yrs	10 yrs
	Panel A: USD market							
2 yrs	0.00 (0.30)	0.20 (0.36)	0.24 (0.43)	0.24 (0.40)	0.27 (0.42)	0.30 (0.41)	0.43 (0.33)	0.42 (0.29)
5 yrs	0.01 (0.20)	0.17 (0.23)	0.19 (0.28)	0.20 (0.25)	0.21 (0.28)	0.23 (0.34)	0.33 (0.35)	0.37 (0.30)
10 yrs	−0.00 (0.17)	0.15 (0.18)	0.16 (0.22)	0.15 (0.20)	0.16 (0.23)	0.18 (0.30)	0.29 (0.34)	0.32 (0.31)
20 yrs	−0.03 (0.13)	0.12 (0.13)	0.13 (0.16)	0.12 (0.16)	0.13 (0.19)	0.18 (0.25)	0.27 (0.32)	0.30 (0.36)
30 yrs	−0.04 (0.14)	0.10 (0.13)	0.12 (0.15)	0.11 (0.15)	0.12 (0.17)	0.15 (0.23)	0.22 (0.30)	0.27 (0.33)
	Panel B: EUR market							
2 yrs	−0.00 (0.15)	0.17 (0.21)	0.27 (0.28)	0.31 (0.27)	0.36 (0.29)	0.47 (0.35)	0.60 (0.33)	0.62 (0.38)
5 yrs	−0.15 (0.38)	−0.03 (0.41)	0.04 (0.40)	0.09 (0.33)	0.12 (0.35)	0.22 (0.30)	0.39 (0.27)	0.46 (0.25)
10 yrs	−0.21 (0.37)	−0.11 (0.39)	−0.06 (0.36)	−0.00 (0.29)	0.02 (0.30)	0.12 (0.25)	0.29 (0.25)	0.36 (0.25)
20 yrs	−0.29 (0.28)	−0.16 (0.27)	−0.11 (0.24)	−0.05 (0.21)	−0.02 (0.22)	0.05 (0.22)	0.24 (0.24)	0.35 (0.27)
30 yrs	−0.32 (0.29)	−0.18 (0.27)	−0.13 (0.25)	−0.07 (0.22)	−0.04 (0.24)	0.00 (0.24)	0.22 (0.28)	0.33 (0.34)

Notes: The table shows average conditional skewness of the future swap rate distributions under the annuity measure $\mathbb{A}$. Standard deviations of conditional skewness are in parentheses. In USD, each statistic is computed on the basis of 419 weekly observations from December 19, 2001 to January 27, 2010. In EUR, each statistic is computed on the basis of 449 weekly observations from June 6, 2001 to January 27, 2010.

Table 2: Skewness of swap rate distributions

A dynamic term structure model for swap rates

- Propose and estimate a dynamic term structure model capable of matching the dynamics of the conditional moments of the swap rate distributions under the annuity measure, $\mathbb{A}$
- Infer the conditional moments under the risk-neutral measure, $\mathbb{Q}$, and physical measure, $\mathbb{P}$

Dynamics under the risk-neutral measure, $\mathbb{Q}$

- Dynamics of zero-coupon bond prices

$$
\begin{aligned}
\frac{dP(t,T)}{P(t,T)} &= r(t)dt + \sum_{i=1}^{N} \sigma_{P,i}(t,T)\left(\sqrt{v_1(t)}dW_i^{\mathbb{Q}}(t) + \sqrt{v_2(t)}d\overline{W}_i^{\mathbb{Q}}(t)\right) \\
dv_1(t) &= (\eta_1 - \kappa_1 v_1(t) - \kappa_{12} v_2(t))dt + \sigma_{v1}\sqrt{v_1(t)}dZ^{\mathbb{Q}}(t) \\
dv_2(t) &= (\eta_2 - \kappa_{21} v_1(t) - \kappa_2 v_2(t))dt + \sigma_{v2}\sqrt{v_2(t)}d\overline{Z}^{\mathbb{Q}}(t),
\end{aligned}
$$

- Allow for correlation between $Z^{\mathbb{Q}}(t)$ and $W_i^{\mathbb{Q}}(t)$, $i = 1, ..., N$, (denoted by ρ_i), and between $\overline{Z}^{\mathbb{Q}}(t)$ and $\overline{W}_i^{\mathbb{Q}}(t)$, $i = 1, ..., N$, (denoted by $\overline{\rho}_i$).
- Most general correlation structure that preserves the tractability of the model

- Dynamics of the forward swap rate

$$
\begin{aligned}
dS_{m,n}(t) &= \left(-\sum_{i=1}^{N} \sigma_{S,i}(t, T_m, T_n)\sigma_{A,i}(t, T_m, T_n)(v_1(t) + v_2(t))\right) dt + \\
&\quad \sum_{i=1}^{N} \sigma_{S,i}(t, T_m, T_n)\left(\sqrt{v_1(t)}dW_i^{\mathbb{Q}}(t) + \sqrt{v_2(t)}d\overline{W}_i^{\mathbb{Q}}(t)\right),
\end{aligned}
$$

where

$$
\begin{aligned}
\sigma_{S,i}(t, T_m, T_n) &= \sum_{j=m}^{n} \zeta_j(t)\sigma_{P,i}(t, T_j) \\
\sigma_{A,i}(t, T_m, T_n) &= \sum_{j=m+1}^{n} \chi_j(t)\sigma_{P,i}(t, T_j)
\end{aligned}
$$

Dynamics under the physical measure, $\mathbb{P}$

- Dynamics under $\mathbb{P}$ are obtained by specifying the market prices of risk that link the Wiener processes under $\mathbb{Q}$ and $\mathbb{P}$
- Market prices of risk specification

$$
\begin{aligned}
dW_i^{\mathbb{P}}(t) &= dW_i^{\mathbb{Q}}(t) - \lambda_i \sqrt{v_1(t)}dt, \quad i = 1, ..., N \\
d\overline{W}_i^{\mathbb{P}}(t) &= d\overline{W}_i^{\mathbb{Q}}(t) - \overline{\lambda}_i \sqrt{v_2(t)}dt, \quad i = 1, ..., N \\
dZ^{\mathbb{P}}(t) &= dZ^{\mathbb{Q}}(t) - \nu \sqrt{v_1(t)}dt, \\
d\overline{Z}^{\mathbb{P}}(t) &= d\overline{Z}^{\mathbb{Q}}(t) - \overline{\nu} \sqrt{v_2(t)}dt.
\end{aligned}
$$

- Dynamics of the forward swap rate

$$
\begin{aligned}
dS_{m,n}(t) &= \Bigg(-\sum_{i=1}^{N} \sigma_{S,i}(t, T_m, T_n)\Big((\sigma_{A,i}(t, T_m, T_n) + \lambda_i) v_1(t) \\
&\quad + (\sigma_{A,i}(t, T_m, T_n) + \overline{\lambda}_i) v_2(t) \Big) \Bigg) dt \\
&\quad + \sum_{i=1}^{N} \sigma_{S,i}(t, T_m, T_n) \left(\sqrt{v_1(t)} dW_i^{\mathbb{P}}(t) + \sqrt{v_2(t)} d\overline{W}_i^{\mathbb{P}}(t) \right),
\end{aligned}
$$

with

$$
\begin{aligned}
dv_1(t) &= \left(\eta_1 - \kappa_1^{\mathbb{P}} v_1(t) - \kappa_{12} v_2(t) \right) dt + \sigma_{v1} \sqrt{v_1(t)} dZ^{\mathbb{P}}(t) \\
dv_2(t) &= \left(\eta_2 - \kappa_{21} v_1(t) - \kappa_2^{\mathbb{P}} v_2(t) \right) dt + \sigma_{v2} \sqrt{v_2(t)} d\overline{Z}^{\mathbb{P}}(t)
\end{aligned}
$$

Dynamics under the annuity measure, $\mathbb{A}$

- Dynamics of the forward swap rate

$$dS_{m,n}(t) \;=\; \sum_{i=1}^{N} \sigma_{S,i}(t, T_m, T_n) \left(\sqrt{v_1(t)} dW_i^{\mathbb{A}}(t) + \sqrt{v_2(t)} d\overline{W}_i^{\mathbb{A}}(t) \right),$$

where

$$\begin{aligned} dv_1(t) &= \left(\eta_1 - \kappa_1^{\mathbb{A}} v_1(t) - \kappa_{12} v_2(t) \right) dt + \sigma_{v1} \sqrt{v_1(t)} dZ^{\mathbb{A}}(t) \\ dv_2(t) &= \left(\eta_2 - \kappa_{21} v_1(t) - \kappa_2^{\mathbb{A}} v_2(t) \right) dt + \sigma_{v2} \sqrt{v_2(t)} d\overline{Z}^{\mathbb{A}}(t), \end{aligned}$$

- Leads to a fast and accurate Fourier-based pricing formula for swaptions

Specifications

- Three term structure factors ($N = 3$) with factor loadings

$$\begin{aligned}
\sigma_{f,1}(t,T) &= \alpha_1 e^{-\xi(T-t)} \\
\sigma_{f,2}(t,T) &= \alpha_2 e^{-\gamma(T-t)} \\
\sigma_{f,3}(t,T) &= \alpha_3 (T-t) e^{-\gamma(T-t)}
\end{aligned}$$

- "Level" (in the limit as $\xi \to 0$), "slope", and "curvature" factor loadings proposed by Nelson and Siegel (1987)
- Leads to finite dimensional affine state vector
- SV1 specification has one volatility factor, $v_1(t)$
- SV2 specification has two volatility factors, $v_1(t)$ and $v_2(t)$, but we impose that $\eta_1 = \eta_2$, $\kappa_1 = \kappa_2$, $\kappa_{12} = \kappa_{21}$, and $\sigma_{v1} = \sigma_{v2}$, in which case the volatility factors only differ in terms of the their correlation with the term structure factors

Stochastic skewness

- Consider the case of $N = 1$, where the correlation between innovations to the forward swap rate, $S_{m,n}(t)$, and its instantaneous variance, $\sigma_{S,1}(t, T_m, T_n)(v_1(t) + v_2(t))$, is given by

$$\rho_1 \frac{v_1(t)}{v_1(t) + v_2(t)} + \overline{\rho}_1 \frac{v_2(t)}{v_1(t) + v_2(t)},$$

- Weighted average of ρ_1 and $\overline{\rho}_1$, with stochastic weights determined by the relative size of $v_1(t)$ and $v_2(t)$.
- Skewness of the future swap rate distribution, which depends on this correlation, is therefore stochastic. In particular, if ρ_1 and $\overline{\rho}_1$ have opposite signs, skewness may switch sign.

Estimation

- Maximum-likelihood in conjunction with Kalman filtering
- Measurement equation includes all available swap rates and swaptions
- Due to the non-linearities in the relationship between observations and state variables, we apply the non-linear unscented Kalman filter.

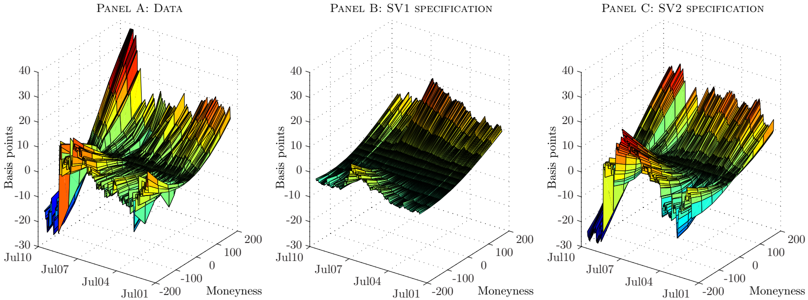

Figure 2: Time-series of the USD normal implied volatility smile of the 1-year option on 10-year swap rate

Notes: Panel A displays the data, Panel B displays the smiles obtained in the SV1 specification, and Panel C displays the smiles obtained in the SV2 specification. The normal implied volatilities are measured in basis points. The time-series consist of 419 weekly observations from December 19th, 2001 to January 27th, 2010.

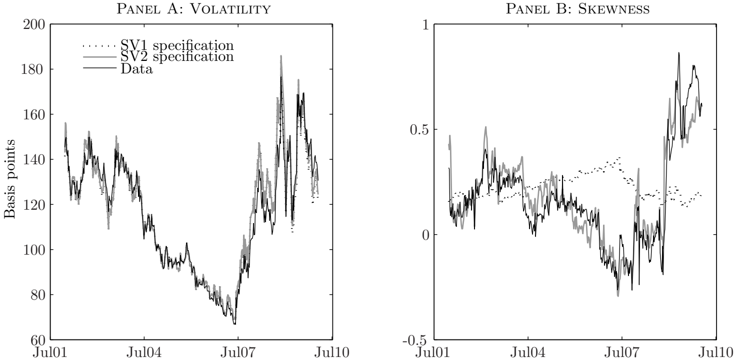

Figure 1: Time-series of volatility and skewness of the conditional 1-year ahead distribution of the USD 10-year swap rate

Notes: Panel A displays conditional volatility, measured in basis points, and Panel B displays conditional skewness. The moments are computed under the annuity measure $\mathbb{A}$ The time-series consist of 419 weekly observations from December 19th, 2001 to January 27th, 2010.

	USD market		EUR market	
	SV1	SV2	SV1	SV2
α_1	0.0072 (0.0001)	0.0065 (0.0001)	0.0090 (0.0001)	0.0069 (0.0001)
α_2	0.0021 (0.0000)	0.0028 (0.0000)	0.0016 (0.0000)	0.0061 (0.0001)
α_3	0.0049 (0.0001)	0.0045 (0.0001)	0.0020 (0.0000)	0.0022 (0.0000)
ξ	0.0104 (0.0002)	0.0086 (0.0001)	0.0115 (0.0001)	0.0005 (0.0000)
γ	0.2471 (0.0046)	0.2435 (0.0034)	0.2795 (0.0042)	0.1670 (0.0023)
φ	0.0479 (0.0018)	0.0464 (0.0018)	0.0435 (0.0018)	0.0441 (0.0017)
ρ_1	0.2546 (0.0202)	-0.1843 (0.0270)	0.0887 (0.0188)	-0.0446 (0.0265)
ρ_2	0.5752 (0.0199)	-0.0945 (0.0298)	0.5263 (0.0247)	0.3736 (0.0207)
ρ_3	-0.0019 (0.0204)	-0.3208 (0.0207)	-0.0509 (0.0201)	-0.5463 (0.0211)
$\overline{\rho}_1$		0.4753 (0.0230)		0.3736 (0.0216)
$\overline{\rho}_2$		0.5748 (0.0229)		0.5582 (0.0178)
$\overline{\rho}_3$		0.5247 (0.0201)		0.5199 (0.0264)
$\kappa_1 = \kappa_2$	0.5045 (0.0066)	0.4156 (0.0067)	0.5300 (0.0072)	0.6948 (0.0109)
$\eta_1 = \eta_2$	0.4957 (0.0075)	0.2404 (0.0029)	0.2797 (0.0039)	0.1981 (0.0022)
ν	-0.2169 (0.0823)	-0.0873 (0.0225)	-0.2014 (0.0681)	-0.1320 (0.0427)
$\overline{\nu}$		-0.2992 (0.0909)		-0.4377 (0.1329)
$\sigma_{rates} \times 10^4$	6.3465 (0.0714)	6.3676 (0.0751)	5.4623 (0.0826)	5.4587 (0.1001)
$\sigma_{swaptions} \times 10^4$	5.5125 (0.0775)	4.5998 (0.0861)	4.8212 (0.0540)	4.3356 (0.0550)
Log-likelihood $\times 10^4$	-27.1280	-25.9420	-23.6951	-23.2632

	USD market			*EUR market*		
	SV1	SV2	SV2-SV1	SV1	SV2	SV2-SV1
Swaptions	5.44	4.58	-0.87^{***} (-6.04)	4.72	4.31	-0.41^{***} (-5.37)
Volatility	4.96	4.83	-0.13^{**} (-1.97)	3.83	3.76	-0.07^{*} (-1.88)
Skewness	0.21	0.05	-0.16^{***} (-6.71)	0.24	0.08	-0.17^{***} (-10.15)
Kurtosis	0.34	0.21	-0.13^{*} (-1.81)	0.47	0.30	-0.17^{***} (-2.64)

Notes: The table compares the SV1 and SV2 specifications in terms of their ability to match the normal implied volatilities (in basis points) as well as conditional volatility (annualized and in basis points), skewness, and kurtosis of the future swap rate distributions under the annuity measure $\mathbb{A}$. It reports means of RMSE time series of implied volatilities and swap rate moments. It also reports mean differences in RMSEs between the two model specifications. T-statistics, corrected for serial correlation up to 26 lags (i.e., two quarters), are in parentheses. In USD, each statistic is computed on the basis of 419 weekly observations from December 19, 2001 to January 27, 2010. In EUR, each statistic is computed on the basis of 449 weekly observations from June 6, 2001 to January 27, 2010. *, **, and *** denote significance at the 10, 5, and 1 percent levels, respectively.

Table 5: Overall comparison between models

- Use dynamic term structure model is to infer the conditional swap rate distributions under the risk-neutral measure as well as the physical measure
- This allows us to study the pricing of risk in the swaption market
- We show that the risk-neutral swap rate distributions on average exhibit higher volatility and are more skewed towards higher rates than the swap rate distributions under the physical measure.

Fundamental drivers of the swap rate distributions

- Regress volatility and skewness of the physical swap rate distributions as well as volatility and skewness risk premia on a number of explanatory variables
- Primarily interested in the effects of macro-economic uncertainty
- Control for other factors that may have an effect on swap rate distributions, including moments of the equity index return distribution, a measure of market-wide liquidity, and a measure of refinancing activity

Moments of agents' probability distributions for future real GDP growth and inflation

- A number of equilibrium pricing models, primarily related to equity derivatives, imply that volatility and volatility risk premia are increasing in uncertainty and/or disagreement among agents about fundamentals.
- We investigate the extent to which agents' perceptions about macro-economic risks (risks to future real GDP growth and inflation) affect swap rate distributions
- Use the quarterly survey of professional forecasters, where participants are asked to assign a probability distribution to their forecasts for real GDP growth and inflation
- Aggregate the probability distributions of the individual respondents and compute dispersion (i.e., standard deviation) and skewness of the aggregate distributions of future real GDP growth and inflation.

Moments of the equity index return distribution

- Numerous papers have documented that equity and fixed-income markets are interconnected.
- We investigate the extent to which characteristics of the equity index return distribution have an impact on the swap rate distributions.
- Specifically, we consider the S&P 500 index in the USD market and the Eurostoxx 50 in the EUR market,
- Compute volatility and skewness of the risk-neutral return distributions in a model independent way using the formulas in Bakshi, Kapadia, and Madan (2003).

Market-wide liquidity

- A number of papers show that liquidity affects derivatives prices.
- We investigate the effect of liquidity at the market-wide level
- As proxy for market-wide liquidity, we use the spread between the 3-month overnight index swap (OIS) rate and the 3-month Treasury bill yield (for the EUR market, we use the German counterpart to the 3-month Treasury bill)
- OIS is a measure the expected average overnight rate during the life of the swap and is virtually free of credit and counterparty risk, so the spread is a fairly clean proxy for liquidity

Refinancing activity

- Several papers find that derivatives prices are affected by supply and demand.
- In the USD market, an important demand for swaptions comes from investors in MBSs who actively hedge the negative convexity risk which stems from the prepayment option embedded in fixed rate mortgages.
- As a proxy for MBS hedging flows, we use Mortgage Bankers Association (MBA) Refinancing Index, which is a weekly measure of refinancing activity.

Issue 1:

- In principle we could run regressions for volatility, skewness, and associated risk premia in each tenor – option expiry category
- As these quantities are highly correlated across the swaption matrix, we instead run regressions using cross-sectional averages of volatility, skewness, and associated risk premia
- In other words, our focus is on understanding the overall time-series variation, rather than the cross-sectional variation.

Issue 2:

- Our proxies for macro-economic uncertainty are only available at a quarterly frequency, while the remaining variables are available at a weekly frequency.
- To make use of all the information in the data, we run MIDAS-type regressions

$$
\begin{aligned}
y_t = {} & \beta_0 + \beta_1 f(\theta,\tau) GDPvol_{t_q} + \beta_2 f(\theta,\tau) GDPskew_{t_q} + \\
& \beta_3 f(\theta,\tau) INFvol_{t_q} + \beta_4 f(\theta,\tau) INFskew_{t_q} + \\
& \beta_5 EQvol_t + \beta_6 EQskew_t + \beta_7 LIQ_t + \beta_8 REFI_t + \epsilon_t,
\end{aligned}
$$

  where $\tau = t - t_q$ is the time between the weekly observation at t and the most recent quarterly observation at t_q, and y_t is the cross-sectional average of either physical volatility, volatility risk premia, physical skewness, or skewness risk premia.
- $f(\theta,\tau) = \exp(-\theta\tau)$ weighs the quarterly observations according to their distance from t
- MIDAS regression model is estimated by non-linear least squares

	$GDPvol$	$GDPskew$	$INFvol$	$INFskew$	$EQvol$	$EQskew$	LIQ	$REFI$	R^2
vol	21.428^{***} (4.888)	8.401^{**} (2.064)	30.536 (0.854)	−6.701 (−0.690)					0.404
vol	19.027^{***} (6.345)	0.892 (0.199)	51.982 (1.390)	9.763 (1.427)	77.317^{***} (4.619)	127.823^{***} (3.069)	11.401^{**} (2.456)	2.142^{**} (2.077)	0.585
$volPrem$	-3.707^{***} (−9.589)	-0.736^{***} (−2.628)	-4.900^{**} (−2.063)	0.496 (0.691)					0.540
$volPrem$	-3.397^{***} (−10.613)	−0.288 (−1.002)	−2.758 (−1.021)	−0.171 (−0.292)	-7.050^{***} (−5.980)	-6.225^{*} (−1.880)	-0.897^{**} (−2.015)	-0.162^{*} (−1.761)	0.641
$skew$	0.290^{*} (1.907)	0.352^{***} (2.773)	−0.726 (−1.139)	−0.060 (−0.304)					0.413
$skew$	0.199^{*} (1.677)	0.284^{**} (2.373)	−0.421 (−0.615)	0.123 (0.985)	0.729^{*} (1.719)	3.971^{***} (3.174)	−0.192 (−1.241)	0.017 (0.810)	0.544
$skewPrem$	0.023^{*} (1.741)	0.011^{*} (1.814)	0.074 (1.208)	-0.035^{*} (−1.786)					0.134
$skewPrem$	0.008^{*} (1.760)	0.007 (1.319)	−0.004 (−0.088)	−0.015 (−1.495)	0.045^{*} (1.892)	-0.203^{*} (−1.931)	0.010 (1.447)	−0.002 (−1.301)	0.248

Notes: The table reports estimates of the MIDAS regression specification (28) in which the cross-sectional average of USD physical volatility (vol), volatility risk premia ($volPrem$), physical skewness ($skew$), or skewness risk premia ($skewPrem$) is regressed on a constant, dispersion and skewness of agents' belief distributions for future U.S. real GDP growth and inflation ($GDPvol$, $GDPskew$, $INFvol$, and $INFskew$), volatility and skewness of the risk-neutral S&P 500 index return distribution ($EQvol$ and $EQskew$), the spread between the 3-month OIS rate and the 3-month Treasury bill yield (LIQ), and the MBA Refinancing Index ($REFI$). Physical volatility and volatility risk premia are measured in basis points, and the MBA Refinancing Index is divided by 1000. Estimation is by non-linear least squares. T-statistics, corrected for heteroscedasticity and serial correlation up to 26 lags (i.e., two quarters), are in parentheses. The sample period is December 19, 2001 to January 27, 2010. *, **, and *** denote significance at the 10, 5, and 1 percent levels, respectively.

Table 9: Fundamental drivers of USD swap rate distributions

	$GDPvol$	$GDPskew$	$INFvol$	$INFskew$	$EQvol$	$EQskew$	LIQ	$REFI$	R^2
vol	-2.193 (-0.602)	3.153^{*} (1.759)	106.612^{***} (6.296)	10.677^{**} (2.183)					0.630
vol	2.730 (1.092)	2.176^{*} (1.856)	60.941^{***} (4.924)	4.057 (1.017)	25.633^{***} (3.817)	3.357 (0.174)	4.232^{*} (1.882)	0.409 (0.749)	0.698
$volPrem$	-0.781^{**} (-2.063)	-0.447^{*} (-1.660)	-7.063^{***} (-3.937)	-0.254 (-0.364)					0.411
$volPrem$	-0.704^{**} (-2.300)	-0.425 (-1.588)	-4.759^{***} (-3.160)	-0.337 (-0.492)	-2.665^{***} (-2.791)	-4.165 (-1.426)	-0.520 (-1.587)	-0.067 (-0.884)	0.434
$skew$	0.256^{**} (2.468)	0.047 (0.468)	-1.071^{**} (-2.141)	0.483^{**} (2.548)					0.269
$skew$	0.114 (1.226)	0.126 (1.338)	-0.877^{*} (-1.718)	0.351^{*} (1.846)	-0.767^{*} (-1.951)	2.911^{***} (2.780)	0.134 (1.266)	0.009 (0.351)	0.462
$skewPrem$	-0.044^{*} (-1.847)	0.003 (0.158)	0.237^{**} (2.248)	-0.098^{***} (-2.638)					0.228
$skewPrem$	-0.035^{*} (-1.733)	-0.011 (-0.584)	0.191^{*} (1.715)	-0.071^{**} (-1.971)	0.159^{**} (2.080)	-0.396^{*} (-1.806)	-0.037 (-1.577)	-0.002 (-0.376)	0.329

Notes: The table reports estimates of the MIDAS regression specification (28) in which the cross-sectional average of EUR physical volatility (vol), volatility risk premia ($volPrem$), physical skewness ($skew$), or skewness risk premia ($skewPrem$) is regressed on a constant, dispersion and skewness of agents' belief distributions for future Eurozone real GDP growth and inflation ($GDPvol$, $GDPskew$, $INFvol$, and $INFskew$), volatility and skewness of the risk-neutral Eurostoxx 50 index return distribution ($EQvol$ and $EQskew$), the spread between the 3-month OIS rate and the 3-month German Bubill yield (LIQ), and the MBA Refinancing Index ($REFI$). Physical volatility and volatility risk premia are measured in basis points, and the MBA Refinancing Index is divided by 1000. Estimation is by non-linear least squares. T-statistics, corrected for heteroscedasticity and serial correlation up to 26 lags (i.e., two quarters), are in parentheses. The sample period is June 6, 2001 to January 27, 2010. *, **, and *** denote significance at the 10, 5, and 1 percent levels, respectively.

Table 10: Fundamental drivers of EUR swap rate distributions

Main results

- Physical volatility and skewness as well as volatility risk premia and skewness risk premia are significantly related to the characteristics of agents' belief distributions for the macro-economy
- GDP beliefs the more important factor in the USD market and inflation beliefs the more important factor in the EUR market
- Consistent with differences in monetary policy objectives in the two economies
- Results hold true controlling for other factors that may have an effect on swap rate distributions, including moments of the equity index return distribution, market-wide liquidity, and refinancing activity

Robustness test

- Run the regressions in quarterly differences, i.e.

$$\begin{aligned}\Delta y_{t_q} &= \beta_0 + \beta_1 \Delta GDPvol_{t_q} + \beta_2 \Delta GDPskew_{t_q} + \\ &\quad \beta_3 \Delta INFvol_{t_q} + \beta_4 \Delta INFskew_{t_q} + \\ &\quad \beta_5 \Delta EQvol_{t_q} + \beta_6 \Delta EQskew_{t_q} + \beta_7 \Delta LIQ_{t_q} + \beta_8 \Delta REFI_{t_q} + \epsilon_{t_q},\end{aligned}$$

  where Δy_{t_q} is the quarterly change in the cross-sectional average of either physical volatility, volatility risk premia, physical skewness, or skewness risk premia.

- This entails discarding information, it may be more robust than the MIDAS specification

	$\Delta GDPvol$	$\Delta GDPskew$	$\Delta INFvol$	$\Delta INFskew$	$\Delta EQvol$	$\Delta EQskew$	ΔLIQ	$\Delta REFI$	R^2
Δvol	23.591^{**} (2.343)	-0.474 (-0.083)	8.335 (0.221)	8.059 (1.175)					0.176
Δvol	21.784^{**} (2.139)	1.790 (0.244)	12.096 (0.322)	11.926 (1.513)	18.723 (0.556)	-94.012 (-0.944)	12.153^{**} (2.013)	2.154 (1.103)	0.311
$\Delta volPrem$	-1.742^{*} (-1.692)	-0.741 (-1.267)	-0.210 (-0.055)	-1.049 (-1.495)					0.187
$\Delta volPrem$	-2.125^{*} (-1.885)	-0.803 (-0.989)	-1.842 (-0.443)	-0.635 (-0.728)	-0.203 (-0.055)	17.462 (1.585)	0.082 (0.122)	0.033 (0.153)	0.280
$\Delta skew$	-0.077 (-0.269)	0.436^{***} (2.672)	1.483 (1.380)	-0.117 (-0.595)					0.258
$\Delta skew$	-0.041 (-0.151)	0.433^{**} (2.196)	1.931^{*} (1.910)	-0.282 (-1.331)	0.585 (0.646)	-0.337 (-0.126)	-0.414^{**} (-2.548)	0.045 (0.853)	0.502
$\Delta skewPrem$	0.018 (0.593)	-0.013 (-0.765)	-0.181 (-1.569)	0.038^{*} (1.817)					0.141
$\Delta skewPrem$	0.039 (1.541)	-0.002 (-0.110)	-0.151 (-1.609)	0.037^{*} (1.852)	-0.146^{*} (-1.724)	-1.021^{***} (-4.095)	0.023 (1.497)	0.003 (0.669)	0.472

Notes: The table reports estimates of the regression specification (29) in which the quarterly change in the cross-sectional average of USD physical volatility (vol), volatility risk premia ($volPrem$), physical skewness ($skew$), or skewness risk premia ($skewPrem$) is regressed on a constant and the quarterly changes in the dispersion and skewness of agents' belief distributions for future U.S. real GDP growth and inflation ($GDPvol$, $GDPskew$, $INFvol$, and $INFskew$), the volatility and skewness of the risk-neutral S&P 500 index return distribution ($EQvol$ and $EQskew$), the spread between the 3-month OIS rate and the 3-month Treasury bill yield (LIQ), and the MBA Refinancing Index ($REFI$). Physical volatility and volatility risk premia are measured in basis points, and the MBA Refinancing Index is divided by 1000. Estimation is by ordinary least squares. T-statistics, corrected for heteroscedasticity and serial correlation up to 2 lags (i.e., two quarters), are in parentheses. The sample period is January, 2002 to January, 2010. *, **, and *** denote significance at the 10, 5, and 1 percent levels, respectively.

Table 11: Fundamental drivers of USD swap rate distributions, regression in changes

	$\Delta GDPvol$	$\Delta GDPskew$	$\Delta INFvol$	$\Delta INFskew$	$\Delta EQvol$	$\Delta EQskew$	ΔLIQ	$\Delta REFI$	R^2
Δvol	-0.703	5.926^{*}	66.321^{**}	-1.567					0.169
	(-0.146)	(1.883)	(2.122)	(-0.249)					
Δvol	-1.506	6.155^{*}	67.924^{**}	-4.409	34.517^{*}	-43.332	5.718	-1.282	0.308
	(-0.305)	(1.744)	(2.055)	(-0.676)	(1.737)	(-0.807)	(1.597)	(-1.108)	
$\Delta volPrem$	-0.317^{*}	-0.206^{*}	-2.516^{**}	0.250					0.226
	(-1.676)	(-1.671)	(-2.055)	(1.013)					
$\Delta volPrem$	-0.350^{*}	-0.265^{*}	-2.940^{**}	0.281	-0.577	2.546	-0.016	0.036	0.285
	(-1.714)	(-1.818)	(-2.155)	(1.043)	(-0.703)	(1.148)	(-0.105)	(0.762)	
$\Delta skew$	0.193^{*}	0.055	0.984	0.294^{**}					0.257
	(1.776)	(0.769)	(1.398)	(2.073)					
$\Delta skew$	0.224^{**}	0.070	1.052	0.297^{**}	-0.867^{*}	1.056	0.083	0.025	0.357
	(1.975)	(0.863)	(1.386)	(1.981)	(-1.901)	(0.855)	(1.008)	(0.932)	
$\Delta skewPrem$	-0.051^{*}	0.003	-0.202	-0.061^{*}					0.309
	(-1.882)	(0.164)	(-1.158)	(-1.732)					
$\Delta skewPrem$	-0.052^{*}	-0.003	-0.205	-0.063^{*}	0.242^{**}	0.108	-0.027	-0.006	0.439
	(-1.891)	(-0.161)	(-1.124)	(-1.763)	(2.204)	(0.363)	(-1.346)	(-0.924)	

Notes: The table reports estimates of the regression specification (29) in which the quarterly change in the cross-sectional average of EUR physical volatility (vol), volatility risk premia ($volPrem$), physical skewness ($skew$), or skewness risk premia ($skewPrem$) is regressed on a constant and the quarterly changes in the dispersion and skewness of agents' belief distributions for future Eurozone real GDP growth and inflation ($GDPvol$, $GDPskew$, $INFvol$, and $INFskew$), the volatility and skewness of the risk-neutral Eurostoxx 50 index return distribution ($EQvol$ and $EQskew$), the spread between the 3-month OIS rate and the 3-month German Bubill yield (LIQ), and the MBA Refinancing Index ($REFI$). Physical volatility and volatility risk premia are measured in basis points, and the MBA Refinancing Index is divided by 1000. Estimation is by ordinary least squares. T-statistics, corrected for heteroscedasticity and serial correlation up to 2 lags (i.e., two quarters), are in parentheses. The sample period is July, 2001 to January, 2010. *, **, and *** denote significance at the 10, 5, and 1 percent levels, respectively.

Table 12: Fundamental drivers of EUR swap rate distributions, regression in changes

Conclusion

- In recent years, a number of equilibrium models for the term structure of interest rates have been proposed. (Piazzesi and Schneider (2007), Bansal and Shaliastovich (2009), Le and Singleton (2010), and Xiong and Yan (2010))
- A key challenge for future fixed income research is developing successful equilibrium models for interest rate derivatives.
- By investigating the fundamental determinants of volatility and skewness of interest rate distributions, our paper provides the first step in this direction.