INTERNATIONAL JOURNAL OF
NUMERICAL ANALYSIS AND MODELING, *SERIES B*
Volume 4, Number 3, Pages 235–255

FINITE ELEMENT ANALYSIS FOR STOKES AND NAVIER-STOKES EQUATIONS DRIVEN BY THRESHOLD SLIP BOUNDARY CONDITIONS

J.K. DJOKO AND M. MBEHOU

Abstract. This paper is devoted to the study of finite element approximations of variational inequalities with a special nonlinearity coming from boundary conditions. After re-writing the problems in the form of variational inequalities, a fixed point strategy is used to show existence of solutions. Next we prove that the finite element approximations for the Stokes and Navier Stokes equations converge respectively to the solutions of each continuous problems. Finally, Uzawa's algorithm is formulated and convergence of the procedure is shown, and numerical validation test is achieved.

Key words. Stokes/Navier-Stokes equations, nonlinear slip boundary conditions, variational inequality, finite element method, error estimate, Uzawa's algorithm.

1. Introduction

This work is devoted to the finite element analysis of the Stokes and Navier Stokes equations driven by threshold slip boundary conditions. The Stokes systems of equations for stationary flows of incompressible Newtonian fluids we considered satisfies

$$-\nu\Delta\boldsymbol{u} + \nabla p = \boldsymbol{f} \quad \text{in } \Omega, \tag{1}$$

$$\operatorname{div}\boldsymbol{u} = 0 \quad \text{in } \Omega, \tag{2}$$

we assume the homogeneous Dirichlet boundary condition on Γ, that is

$$\boldsymbol{u} = \boldsymbol{0} \quad \text{on } \Gamma, \tag{3}$$

with the impermeability boundary condition

$$u_n = \boldsymbol{u}\cdot\boldsymbol{n} = 0 \quad \text{on } S, \tag{4}$$

and the slip boundary condition [1, 2]

$$\left.\begin{array}{l} |(\boldsymbol{\sigma n})_{\boldsymbol{\tau}}| \le g, \\ |(\boldsymbol{\sigma n})_{\boldsymbol{\tau}}| < g \Rightarrow \boldsymbol{u_\tau} = \boldsymbol{0}, \\ |(\boldsymbol{\sigma n})_{\boldsymbol{\tau}}| = g \Rightarrow \boldsymbol{u_\tau} \neq \boldsymbol{0}\ ,\ -(\boldsymbol{\sigma n})_{\boldsymbol{\tau}} = (g + k|\boldsymbol{u_\tau}|)\dfrac{\boldsymbol{u_\tau}}{|\boldsymbol{u_\tau}|} \end{array}\right\} \text{ on } S. \tag{5}$$

Here $\Omega \subset \mathbb{R}^d$ (d=2,3) is a bounded domain, with boundary $\partial\Omega$. It is assumed that $\partial\Omega$ is made of two components S, and Γ with $\overline{\partial\Omega} = \overline{S \cup \Gamma}$, and $S \cap \Gamma = \emptyset$. ν is a positive quantity representing the viscosity coefficient, k is the "friction" coefficient assume to be positive, and $g : S \to (0, \infty)$ is the barrier or threshold function. The velocity of the fluid is $\boldsymbol{u}$ and p stands for the pressure, while $\boldsymbol{f}$ is the external force. Furthermore, $\boldsymbol{n}$ is the outward unit normal to the boundary $\partial\Omega$ of Ω, $\boldsymbol{u_\tau} = \boldsymbol{u} - u_n\boldsymbol{n}$ is the tangential component of the velocity $\boldsymbol{u}$, and $(\boldsymbol{\sigma n})_{\boldsymbol{\tau}} = \boldsymbol{\sigma n} - (\boldsymbol{n}\cdot\boldsymbol{\sigma n})\boldsymbol{n}$ is the tangential traction. Of course, $\boldsymbol{\sigma} = -p\boldsymbol{I} + 2\nu\boldsymbol{D}(\boldsymbol{u})$ is the Cauchy stress tensor,

Received by the editors May 28, 2013 and, in revised form, August 10, 2013.

2000 *Mathematics Subject Classification.* 35L70, 65N30, 76D06.

where $\boldsymbol{I}$ is the identity matrix, and $\boldsymbol{D}(\boldsymbol{u}) = \frac{1}{2}(\nabla \boldsymbol{u} + (\nabla \boldsymbol{u})^T)$. It should quickly be mentioned that (5) is equivalent following [3] to

$$(\boldsymbol{\sigma n})_{\boldsymbol{\tau}} \cdot \boldsymbol{u}_{\boldsymbol{\tau}} + (g + k|\boldsymbol{u}_{\boldsymbol{\tau}}|)|\boldsymbol{u}_{\boldsymbol{\tau}}| = 0 \quad \text{on } S, \tag{6}$$

which is rewritten with the use of sub-differential as

$$-(\boldsymbol{\sigma n})_{\boldsymbol{\tau}} \in (g + k|\boldsymbol{u}_{\boldsymbol{\tau}}|)\partial|\boldsymbol{u}_{\boldsymbol{\tau}}| \quad \text{on } S, \tag{7}$$

where the symbol $\partial|.|$ is the sub-differential of the real value function $|.|$, with $|\boldsymbol{u}|^2 = \boldsymbol{u} \cdot \boldsymbol{u}$. We recall that if X is the Hilbert space with $x_0 \in X$, then

$$y \in \partial\Psi(x_0) \Leftrightarrow \Psi(x) - \Psi(x_0) \geq y \cdot (x - x_0) \quad \text{for all} \quad x \in X \ . \tag{8}$$

The Stokes system can be considered a simplification of the Navier Stokes equations when convection is negligible. That is (1) is replaced by

$$-\nu \, \Delta \boldsymbol{u} + (\boldsymbol{u} \cdot \nabla)\boldsymbol{u} + \nabla p = \boldsymbol{f} \quad \text{in} \quad \Omega, \tag{9}$$

with (2),(3), (4) and (5) unchanged, and the nonlinear term in (9) is the convection term given as

$$(\boldsymbol{u} \cdot \nabla)\boldsymbol{u} = \sum_{i=1}^{d} u_i \frac{\partial \boldsymbol{u}}{\partial x_i}.$$

Over the past few years a remarkable progress has been achieved in the field of computational contact mechanics. One of the key ingredients in this phenomenal growth is attributed to the better mathematical understanding of problems. The formulation by means of variational inequalities (see [3, 4, 5, 6, 7, 8, 9]) and the finite element method have contributed to the development of reliable frameworks for the numerical treatment of such problems. Despite such advances in the modeling and numerical treatment of contact problems with friction, it should be mentioned that most works reported in the literature are still restricted to solid mechanics. The numerical analysis works dealing with fluids flow are concerned with the standard Amontons-Coulomb law of perfect friction [10, 11, 12, 13, 14, 15, 16, 17, 18], replacing (5) by

$$\left.\begin{aligned} &|(\boldsymbol{\sigma n})_{\boldsymbol{\tau}}| \leq g, \\ &|(\boldsymbol{\sigma n})_{\boldsymbol{\tau}}| < g \Rightarrow \boldsymbol{u}_{\boldsymbol{\tau}} = \boldsymbol{0}, \\ &|(\boldsymbol{\sigma n})_{\boldsymbol{\tau}}| = g \Rightarrow \boldsymbol{u}_{\boldsymbol{\tau}} \neq \boldsymbol{0} \ , \ -(\boldsymbol{\sigma n})_{\boldsymbol{\tau}} = g\frac{\boldsymbol{u}_{\boldsymbol{\tau}}}{|\boldsymbol{u}_{\boldsymbol{\tau}}|} \end{aligned}\right\} \text{ on } \quad S. \tag{10}$$

As pointed out by C. Leroux [1], such a theory can represent only a limited range of possible situations. The purpose of this work is to numerically analyze by means of finite element approximation equations (1)–(5), and (2)–(5),(9). At this juncture, it is important to recall that this type of nonlinear slip boundary conditions as far as fluid flows are concerned was first introduced by Fujita in [19, 20]. This is in continuation of a series of investigations aimed at the analysis of Stokes and Navier Stokes equations driven by nonlinear slip boundary conditions of friction type (see [10, 11, 12]). The principal goal is to analyze from the numerical analysis viewpoint the solvability, stability and convergence of the resulting variational inequalities of such problems. In order to provide a background for a better mathematical understanding of the problems, we shall introduce in Section 2 some needed tools, and quickly indicate how the problems are solvable. At this step, we recall that in C. Leroux and Tani [1, 2] a fixed point argument is used to establish the solvability of a class of problems similar to what we want to study. It is re-introduced here because of its usefulness in the finite element analysis and to make this paper self-contained.

Hence one can see a sort of "continuum" between the continuous and discrete analysis. The finite element formulations for both Stokes and Navier Stokes equations are derived in Section 3. The finite elements are defined on conforming triangular mesh as introduced in [21], and in each triangle the velocity and pressure are taken so that the Babuska-Brezzi's condition [22, 23] is satisfied. In our work, we do not use penalty method, or pressure stabilized method to enforce the incompressibility condition. Instead we use a direct method and sufficient conditions of existence of solutions are employed to derive a priori error estimates in Section 3. In Section 4, Uzawa's algorithm is formulated and analyzed for solving the Stokes and Navier-Stokes finite element discretization. It is shown that the Uzawa's algorithm converges. In Section 5 numerical simulations that confirm the predictions of the theory are exhibited, and concluding remarks are drawn in Section 6.

2. Preliminaries and Variational Formulations

In this section, we introduce notation and some results that will be used throughout our work. We also formulate various weak formulations and discuss (recall) some existence results.

2.1. Notations and Preliminaries. The Lebesgue space is denoted as $L^r(\Omega)$, $1 \le r \le \infty$, with norms $\|\cdot\|_{L^r}$ (except the $L^2(\Omega)$-norm which is denoted by $\|\cdot\|$). For any non-negative integer m and real number $r \ge 1$, the classical Sobolev spaces [24]:

$$W^{m,r}(\Omega) = \{v \in L^r(\Omega);\ \partial^\alpha v \in L^r(\Omega) \text{ for all } \quad |\alpha| \le m\},$$

is equipped with the seminorm

$$|v|_{m,r} = \left\{ \sum_{|\alpha|=m} \int_\Omega |\partial^\alpha v|^r \mathrm{dx} \right\}^{1/r}, \tag{11}$$

and norm

$$\|v\|_{m,r} = \left\{ \sum_{0\le|\alpha|=m} |v|^r_{W^{k,r}(\Omega)} \right\}^{1/r}, \tag{12}$$

with the usual extension when $r = \infty$. When $r = 2$, $W^{m,r}(\Omega)$ is the Hilbert space $H^m(\Omega)$ with the scalar product

$$((v, w))_m = \sum_{|\alpha|\le m} (\partial^\alpha v, \partial^\alpha w).$$

It should be mentioned that ∂^α stands for the derivative in the sense of distribution, while $\alpha = (\alpha_1, \dots, \alpha_d)$ denote a multi-index of length $|\alpha| = \alpha_1 + \cdots + \alpha_d$. For the analysis of (1)–(5), and (2)–(9), we introduce

$$\boldsymbol{V} = \{\boldsymbol{v} \in H^1(\Omega)^d,\ \boldsymbol{v}|_\Gamma = 0,\ \boldsymbol{v}\cdot\mathbf{n}|_S = 0\}, \quad \boldsymbol{V}_0 = \boldsymbol{H}^1_0(\Omega),$$
$$\boldsymbol{V}_{div} = \{\boldsymbol{v} \in \boldsymbol{V},\ \operatorname{div}\boldsymbol{v} = 0\}, \quad M = L^2_0(\Omega).$$

From Poincaré-Fredrichs's inequality, there exists a positive constant C, such that

$$\int_\Omega |\boldsymbol{v}|^2 \mathrm{dx} \le C \int_\Omega |\nabla \boldsymbol{v}|^2 \mathrm{dx} \text{ for all } \boldsymbol{v} \in \boldsymbol{V}, \tag{13}$$

which implies that on $\boldsymbol{V}$, the semi-norm (11) defines a norm which is equivalent to the norm in (12). Also, of importance in this work is the Korn's inequality which reads; there exists a positive constant C, such that

$$\int_{\Omega} |\nabla \boldsymbol{v}|^2 \mathrm{dx} \leq C \int_{\Omega} |\boldsymbol{D}(\boldsymbol{v})|^2 \mathrm{dx} \text{ for all } \boldsymbol{v} \in \boldsymbol{V}, \tag{14}$$

which implies that we can equip $\boldsymbol{V}$ with $\|\cdot\|_V = \|D(\cdot)\|$ which is equivalent to $\|\cdot\|_1$. We now recall classical operators associated with the formulation of the Stokes problems (1)–(5), and Navier-Stokes problem (2)–(9) (see [22, 23]).
We first introduce bilinear forms $a(\cdot,\cdot)$ and $b(\cdot,\cdot)$ defined as follows

$$a : \boldsymbol{V} \times \boldsymbol{V} \to \mathbb{R} \quad \text{with} \quad a(\boldsymbol{u}, \boldsymbol{v}) = 2\nu \int_{\Omega} \boldsymbol{D}(\boldsymbol{u}) : \boldsymbol{D}(\boldsymbol{v}) dx$$

$$b : \boldsymbol{V} \times M \to \mathbb{R} \quad \text{with} \quad b(\boldsymbol{u}, p) = \int_{\Omega} p \operatorname{div} \boldsymbol{u} dx.$$

Let $d(\cdot,\cdot,\cdot)$ be the trilinear form defined as follows

$$d : \boldsymbol{V} \times \boldsymbol{V} \times \boldsymbol{V} \to \mathbb{R} \quad \text{with} \quad d(\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w}) = \int_{\Omega} (\boldsymbol{u} \cdot \nabla) \boldsymbol{v} \cdot \boldsymbol{w} \, dx.$$

The trilinear form $d(\cdot,\cdot,\cdot)$ is continuous on $\boldsymbol{V} \times \boldsymbol{V} \times \boldsymbol{V}$, i.e., there exists a positive constant C_d such that

$$|d(\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w})| \leq C_d \|\boldsymbol{u}\|_V \|\boldsymbol{v}\|_V \|\boldsymbol{w}\|_V \quad \text{for all} \quad \boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w} \in \boldsymbol{V}$$

Moreover, for all $\boldsymbol{u} \in \boldsymbol{V}_{div}$ and $\boldsymbol{v}, \boldsymbol{w} \in \boldsymbol{V}$

$$d(\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w}) = -d(\boldsymbol{u}, \boldsymbol{w}, \boldsymbol{v}), \tag{15}$$

$$d(\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{v}) = 0. \tag{16}$$

The bilinear form $b(\cdot,\cdot)$ satisfies the inf-sup condition, i.e., there exists a positive constant β such that

$$\beta \|p\| \leq \sup_{\boldsymbol{u} \in \boldsymbol{V}} \frac{b(\boldsymbol{u}, p)}{\|\boldsymbol{u}\|_V} \qquad \text{for all} \quad p \in L_0^2(\Omega). \tag{17}$$

As a readily obtainable consequence of Korn's inequality (14), $a(\cdot,\cdot)$ is coercive on $\boldsymbol{V}$, that is

$$a(\boldsymbol{v}, \boldsymbol{v}) = 2\nu \|\boldsymbol{v}\|_V^2 \quad \text{for all} \quad \boldsymbol{v} \in \boldsymbol{V}. \tag{18}$$

The coercivity of $a(\cdot,\cdot)$ will allow us to apply the following classical existence and uniqueness result for elliptic variational inequalities of the second kind [3].

Lemma 2.1. *Suppose that $\boldsymbol{V}$ is a Hilbert space, $a : \boldsymbol{V} \times \boldsymbol{V} \to \mathbb{R}$ is bilinear, continuous and coercive, $J : \boldsymbol{V} \to \mathbb{R} \cup \infty$ is convex, lower semi-continuous and proper, and $\boldsymbol{f} \in \boldsymbol{V}'$. Then there exists a unique $\boldsymbol{u} \in \boldsymbol{V}$ such that*

$$a(\boldsymbol{u}, \boldsymbol{v} - \boldsymbol{u}) + J(\boldsymbol{v}) - J(\boldsymbol{u}) \geq \langle \boldsymbol{f}, \boldsymbol{v} - \boldsymbol{u} \rangle \quad \text{for all} \quad \boldsymbol{v} \in \boldsymbol{V}.$$

2.1.1. Mixed Variational formulation of (1)–(5). Suppose that $\boldsymbol{f} \in \boldsymbol{H}^{-1}(\Omega)$ and $g \in L^2(S)$ with $g \geq 0$ on S. We multiply the equation (1) by $\boldsymbol{v} - \boldsymbol{u}$ for all $\boldsymbol{v} \in \boldsymbol{V}$ and integrate the resulting equation over Ω. After application of Green's formula, we obtain

$$a(\boldsymbol{u}, \boldsymbol{v} - \boldsymbol{u}) - b(\boldsymbol{v} - \boldsymbol{u}, p) - \int_S \boldsymbol{\sigma} n \cdot (\boldsymbol{v} - \boldsymbol{u}) \, ds = \int_{\Omega} \boldsymbol{f} \cdot (\boldsymbol{v} - \boldsymbol{u}) \, dx. \tag{19}$$

Next, we briefly recall that

$$\boldsymbol{\sigma n} = \boldsymbol{\sigma}_N \boldsymbol{n} + \boldsymbol{\sigma}_{\boldsymbol{\tau}}, \quad \boldsymbol{v} - \boldsymbol{u} = (v_N - u_N)\boldsymbol{n} + (\boldsymbol{v}_{\boldsymbol{\tau}} - \boldsymbol{u}_{\boldsymbol{\tau}}),$$

then we have

$$\int_S \boldsymbol{\sigma n}\cdot(\mathbf{v}-\mathbf{u})\,ds \;=\; \int_S \boldsymbol{\sigma_\tau}(\boldsymbol{v_\tau}-\boldsymbol{u_\tau})\,ds, \quad \text{since} \quad v_N-u_N|_\Gamma=0. \tag{20}$$

On the other hand, it follows from boundary conditions (5) which is equivalent to (7) that after using the definition (8)

$$\int_S (g+k|\boldsymbol{u_\tau}|)(|\boldsymbol{v_\tau}|-|\boldsymbol{u_\tau}|)ds \geq -\int_S \boldsymbol{\sigma_\tau}(\boldsymbol{v_\tau}-\boldsymbol{u_\tau})\text{ds}. \tag{21}$$

We now define the functional

$$J:\boldsymbol{H}^1(\Omega)\times\boldsymbol{H}^1(\Omega)\to[0,\infty) \quad \text{with} \quad J(\boldsymbol{u},\boldsymbol{v})=\int_S (g+k|\boldsymbol{u_\tau}|)|\boldsymbol{v_\tau}|dx. \tag{22}$$

Together with (19)-(22), the following weak formulation is obtained: Find $(\boldsymbol{u},p)\in\boldsymbol{V}\times M$ such that

$$\begin{array}{rcl} & & \text{for all } \boldsymbol{v},q\in\boldsymbol{V}\times M,\\ a(\boldsymbol{u},\boldsymbol{v}-\boldsymbol{u})-b(\boldsymbol{v}-\boldsymbol{u},p)+J(\boldsymbol{u},\boldsymbol{v})-J(\boldsymbol{u},\boldsymbol{u}) & \geq & \langle\boldsymbol{f},\boldsymbol{v}-\boldsymbol{u}\rangle\\ b(\boldsymbol{u},q) & = & 0.\end{array} \tag{23}$$

Note that since the bilinear form $b(\cdot,\cdot)$ satisfies the inf-sup condition (17), the variational inequality problem (23) is equivalent to

$$\begin{cases}\text{Find } \boldsymbol{u}\in\boldsymbol{V}_{\text{div}} \text{ such that}\\ a(\boldsymbol{u},\boldsymbol{w}-\boldsymbol{u})+J(\boldsymbol{u},\boldsymbol{w})-J(\boldsymbol{u},\boldsymbol{u})\geq\langle\boldsymbol{f},\boldsymbol{w}-\boldsymbol{u}\rangle \text{ for all } \boldsymbol{w}\in\boldsymbol{V}_{\text{div}}.\end{cases} \tag{24}$$

Proposition 2.1. *The functional J satisfies:*

(a) *for all $\boldsymbol{v}\in\boldsymbol{H}^1(\Omega)$, $J(\boldsymbol{v},\cdot)$ is convex and nonnegative continuous on $\boldsymbol{H}^1(\Omega)$.*

(b) *for all $\boldsymbol{v}_1$, $\boldsymbol{v}_2$, $\boldsymbol{\zeta}_1$, $\boldsymbol{\zeta}_2\in\boldsymbol{H}^1(\Omega)$, there exists C_0 such that*

$$J(\boldsymbol{v}_1,\boldsymbol{\zeta}_2)-J(\boldsymbol{v}_1,\boldsymbol{\zeta}_1)+J(\boldsymbol{v}_2,\boldsymbol{\zeta}_1)-J(\boldsymbol{v}_2,\boldsymbol{\zeta}_2)\leq C_0k\|\boldsymbol{v}_1-\boldsymbol{v}_2\|_V\|\boldsymbol{\zeta}_1-\boldsymbol{\zeta}_2\|_V. \tag{25}$$

Proof (a) is readily obtained. For (b), note that:

$$\begin{array}{rl} & J(\boldsymbol{v}_1,\boldsymbol{\zeta}_2)-J(\boldsymbol{v}_1,\boldsymbol{\zeta}_1)+J(\boldsymbol{v}_2,\boldsymbol{\zeta}_1)-J(\boldsymbol{v}_2,\boldsymbol{\zeta}_2)\\ = & \displaystyle\int_S k(|\boldsymbol{v}_1|-|\boldsymbol{v}_2|)(|\boldsymbol{\zeta}_2|-|\boldsymbol{\zeta}_1|)ds\\ \leq & \displaystyle\int_S k(|\boldsymbol{v}_1-\boldsymbol{v}_2|)(|\boldsymbol{\zeta}_1-\boldsymbol{\zeta}_2|)ds\\ \leq & C_0k\|\boldsymbol{v}_1-\boldsymbol{v}_2\|_V\|\boldsymbol{\zeta}_1-\boldsymbol{\zeta}_2\|_V.\end{array}$$

The main result of this subsection is the following

Theorem 2.1. *Suppose that*

$$0<\frac{C_0k}{2\nu}<1. \tag{26}$$

Then the mixed variational problem (23) admits a unique solution $(\boldsymbol{u},p)\in\boldsymbol{V}\times M$, which satisfies the following bound

$$\|\boldsymbol{u}\|_V \;\leq\; C(\|\boldsymbol{f}\|_{-1}+\|g\|_{L^2(S)}), \tag{27}$$

$$\|p\| \;\leq\; C(\|\boldsymbol{f}\|_{-1}+\|g\|_{L^2(S)}). \tag{28}$$

It is clear from our condition (26), that we either need; a large enough viscosity or a small friction coefficient.
The proof of Theorem 2.1 is based on fixed point arguments and established in two steps in [1] where similar proofs can be found. To derive the a priori estimate (27), let $\boldsymbol{w} = \boldsymbol{0}$ and $\boldsymbol{w} = 2\boldsymbol{u}$ in (24), one has

$$a(\boldsymbol{u},\boldsymbol{u}) + J(\boldsymbol{u},\boldsymbol{u}) = \langle \boldsymbol{f}, \boldsymbol{u}\rangle,$$

which from (18), and trace's inequality gives

$$2\nu\|\boldsymbol{u}\|_V \leq \|\boldsymbol{f}\|_{-1} + C\|g\|_{L^2(S)}.$$

Next, we derive the a priori bound for the pressure. For that purpose, let $\boldsymbol{w} \in \mathbf{V}_0$, and replace $\boldsymbol{v}$ in (23) successively by $\boldsymbol{u}+\boldsymbol{w}$ and $\boldsymbol{u}-\boldsymbol{w}$, and observe that $J(\boldsymbol{u},\boldsymbol{v}) = J(\boldsymbol{u},\boldsymbol{u}\pm\boldsymbol{w}) = J(\boldsymbol{u},\boldsymbol{u})$. Then one obtains

$$a(\boldsymbol{u},\boldsymbol{w}) - b(\boldsymbol{w},p) = \langle \boldsymbol{f},\boldsymbol{w}\rangle \quad \text{for all} \quad \boldsymbol{w} \in \boldsymbol{V}_0. \tag{29}$$

Next, from the compatibility condition (17) and (29), one has

$$\begin{aligned} \beta\|p\| &\leq \sup_{\boldsymbol{w}\in\boldsymbol{V}_0} \frac{b(\boldsymbol{w},p)}{\|\boldsymbol{w}\|_V} = \sup_{\boldsymbol{w}\in\boldsymbol{V}_0} \frac{|a(\boldsymbol{u},\boldsymbol{w}) - \langle \mathbf{f},\boldsymbol{w}\rangle|}{\|\boldsymbol{w}\|_V} \\ &\leq 2\nu\|\boldsymbol{u}\|_V + \|\boldsymbol{f}\|_{-1}, \end{aligned}$$

and the use of the bound on $\boldsymbol{u}$ leads to the desired estimate.

2.1.2. Mixed Variational formulation (2)–(5) and (9). Suppose that $\boldsymbol{f} \in \boldsymbol{H}^{-1}(\Omega)$ and $g \in L^2(S)$ with $g \geq 0$ on S. We multiply (9) by $\boldsymbol{v}-\boldsymbol{u}$ for all $\boldsymbol{v} \in \boldsymbol{V}$, integrate the resulting equation over Ω, and apply Green's formula to obtain

$$a(\boldsymbol{u},\boldsymbol{v}-\boldsymbol{u}) + d(\boldsymbol{u},\boldsymbol{u},\boldsymbol{v}-\boldsymbol{u}) - b(\boldsymbol{v}-\boldsymbol{u},p) - \int_S \boldsymbol{\sigma}\cdot(\boldsymbol{v}-\boldsymbol{u})\,ds = \langle \boldsymbol{f},\boldsymbol{v}-\boldsymbol{u}\rangle.$$

According to the relations (20), (21) and (22), the weak formulation of (2)-(9) can be written as follows: Find $(\boldsymbol{u},p) \in \boldsymbol{V}\times M$ such that
(30)

$$\begin{aligned} &\text{for all } (\boldsymbol{v},q) \in \boldsymbol{V}\times M \\ a(\boldsymbol{u},\boldsymbol{v}-\boldsymbol{u}) + d(\boldsymbol{u},\boldsymbol{u},\boldsymbol{v}-\boldsymbol{u}) - b(\boldsymbol{v}-\boldsymbol{u},p) + J(\boldsymbol{u},\boldsymbol{v}) - J(\boldsymbol{u},\boldsymbol{u}) &\geq \langle \boldsymbol{f},\boldsymbol{v}-\boldsymbol{u}\rangle \\ b(\boldsymbol{u},q) &= 0. \end{aligned}$$

Since the bilinear form $b(\cdot,\cdot)$ satisfies the inf-sup condition (17), the variational inequality problem (30) is equivalent to

$$\begin{cases} \text{Find } \boldsymbol{u} \in \boldsymbol{V}_{\text{div}} \text{ such that for all } \boldsymbol{w} \in \boldsymbol{V}_{\text{div}} \\ a(\boldsymbol{u},\boldsymbol{w}-\boldsymbol{u}) + d(\boldsymbol{u},\boldsymbol{u},\boldsymbol{w}-\boldsymbol{u}) + J(\boldsymbol{u},\boldsymbol{w}) - J(\boldsymbol{u},\boldsymbol{u}) \geq \langle \boldsymbol{f},\boldsymbol{w}-\boldsymbol{u}\rangle. \end{cases} \tag{31}$$

By the contraction mapping principle, we can prove the following existence and uniqueness theorem.

Theorem 2.2. *If the following conditions hold:*

$$0 < \frac{C_d C_1(\|\boldsymbol{f}\|_{-1} + \|g\|_{L^2(S)})}{\nu^2} < 1, \tag{32}$$

$$0 < \frac{C_0 k}{2\nu} < 1 - \frac{C_d C_1(\|\boldsymbol{f}\|_{-1} + \|g\|_{L^2(S)})}{2\nu^2}, \tag{33}$$

then the mixed variational problem (31) admits a unique solution $(\boldsymbol{u}, p) \in \boldsymbol{K}_{\text{div}} \times M$, *which satisfies the following bound*

$$\|\boldsymbol{u}\|_V \leq C(\|\boldsymbol{f}\|_{-1} + \|g\|_{L^2(S)}), \tag{34}$$

$$\|p\| \leq C(\|\boldsymbol{f}\|_{-1} + \|g\|_{L^2(S)} + \|\boldsymbol{f}\|_{-1}^2 + \|g\|_{L^2(S)}^2). \tag{35}$$

where C_1 *satisfies*

$$\left|\langle \boldsymbol{f}, \boldsymbol{v}\rangle - \int_S g|\boldsymbol{v}_{\boldsymbol{\tau}}|ds\right| \leq C_1(\|\boldsymbol{f}\|_{-1} + \|g\|_{L^2(S)})\|\boldsymbol{v}\|_V \qquad \textit{for all} \quad \boldsymbol{v} \in \boldsymbol{V}$$

and

$$\boldsymbol{K}_{\text{div}} = \{\boldsymbol{v} \in \boldsymbol{V}_{\text{div}} \ , \quad \|\boldsymbol{v}\|_V \leq \frac{C_1}{\nu}\left(\|\boldsymbol{f}\|_{-1} + \|g\|_{L^2(S)}\right)\}.$$

Proof The proof of Theorem 2.2 follows along the same lines as the proof of Theorem 2.1, but it is more involved because of the additional nonlinear convection term.

First, for a fixed $\boldsymbol{v} \in \boldsymbol{K}_{\text{div}}$, consider the following variational inequality problem:

$$\begin{cases} \text{Find } \boldsymbol{\eta_v} \in \boldsymbol{K}_{\text{div}} \text{ such that for all } \boldsymbol{w} \in \boldsymbol{V}_{\text{div}}, \\ a(\boldsymbol{\eta_v}, \boldsymbol{w} - \boldsymbol{\eta_v}) + d(\boldsymbol{\eta_v}, \boldsymbol{\eta_v}, \boldsymbol{w} - \boldsymbol{\eta_v}) + J_{\boldsymbol{v}}(\boldsymbol{w}) - J_{\boldsymbol{v}}(\boldsymbol{\eta_v}) \geq \langle \boldsymbol{f}, \boldsymbol{w} - \boldsymbol{\eta_v}\rangle. \end{cases} \tag{36}$$

Lemma 2.2. *Assume that the condition (32) holds, then there exists a unique solution* $\boldsymbol{\eta_v} \in \boldsymbol{K}_{\text{div}}$ *to the problem (36).*

Proof The proof can be found in [16, Theorem 2.1, P553] where similar condition is needed.
Next, let consider the mapping $\Phi : \boldsymbol{K}_{\text{div}} \to \boldsymbol{K}_{\text{div}}$ defined as follows

$$\Phi(\boldsymbol{v}) = \boldsymbol{\eta_v}$$

where $\boldsymbol{\eta_v}$ is a unique solution of problem (36). It is obvious that the fixed point of Φ if exists will be the solution of (31).

Lemma 2.3. *Under the assumption of Theorem 2.2, the operator* Φ *will be a contraction on* $\boldsymbol{K}_{\text{div}}$.

Proof Let $\boldsymbol{v}_1$, $\boldsymbol{v}_2 \in \boldsymbol{K}_{\text{div}}$ and set $\boldsymbol{\eta}_1 = \Phi(\boldsymbol{v}_1)$, $\boldsymbol{\eta}_2 = \Phi(\boldsymbol{v}_2)$ then, we have

$$a(\boldsymbol{\eta}_1, \boldsymbol{w} - \boldsymbol{\eta}_1) + d(\boldsymbol{\eta}_1, \boldsymbol{\eta}_1, \boldsymbol{w} - \boldsymbol{\eta}_1) + J_{\boldsymbol{v}1}(\boldsymbol{w}) - J_{\boldsymbol{v}1}(\boldsymbol{\eta}_1) \geq \langle \mathbf{f}, \boldsymbol{w} - \boldsymbol{\eta}_1\rangle \text{ for all } \boldsymbol{w} \in \boldsymbol{V}_{\text{div}}\ , \tag{37}$$

and

$$a(\boldsymbol{\eta}_2, \boldsymbol{w} - \boldsymbol{\eta}_2) + d(\boldsymbol{\eta}_2, \boldsymbol{\eta}_2, \boldsymbol{w} - \boldsymbol{\eta}_2) + J_{\boldsymbol{v}2}(\boldsymbol{w}) - J_{\boldsymbol{v}2}(\boldsymbol{\eta}_2) \geq \langle \mathbf{f}, \boldsymbol{w} - \boldsymbol{\eta}_2\rangle \text{ for all } \boldsymbol{w} \in \boldsymbol{V}_{\text{div}}\ . \tag{38}$$

Taking $\boldsymbol{w} = \boldsymbol{\eta}_2$ in (37) and $\boldsymbol{w} = \boldsymbol{\eta}_1$ in (38) and add the resultant equations, we obtain

$$\begin{aligned} & a(\boldsymbol{\eta}_2 - \boldsymbol{\eta}_1, \boldsymbol{\eta}_2 - \boldsymbol{\eta}_1) \\ \leq\ & d(\boldsymbol{\eta}_2, \boldsymbol{\eta}_2, \boldsymbol{\eta}_2 - \boldsymbol{\eta}_1) - d(\boldsymbol{\eta}_1, \boldsymbol{\eta}_1, \boldsymbol{\eta}_2 - \boldsymbol{\eta}_1) + J(\boldsymbol{v}_1, \boldsymbol{\eta}_2) - J(\boldsymbol{v}_1, \boldsymbol{\eta}_1) + J(\boldsymbol{v}_2, \boldsymbol{\eta}_1) - J(\boldsymbol{v}_2, \boldsymbol{\eta}_2) \end{aligned}$$

In other hand, since from (16) $d(\boldsymbol{\eta}_2, \boldsymbol{\eta}_2 - \boldsymbol{\eta}_1, \boldsymbol{\eta}_2 - \boldsymbol{\eta}_1) = 0$,

$$\begin{aligned} & d(\boldsymbol{\eta}_2, \boldsymbol{\eta}_2, \boldsymbol{\eta}_2 - \boldsymbol{\eta}_1) - d(\boldsymbol{\eta}_1, \boldsymbol{\eta}_1, \boldsymbol{\eta}_2 - \boldsymbol{\eta}_1) \\ =\ & d(\boldsymbol{\eta}_2, \boldsymbol{\eta}_2 - \boldsymbol{\eta}_1, \boldsymbol{\eta}_2 - \boldsymbol{\eta}_1) + d(\boldsymbol{\eta}_2 - \boldsymbol{\eta}_1, \boldsymbol{\eta}_1, \boldsymbol{\eta}_2 - \boldsymbol{\eta}_1) \\ =\ & d(\boldsymbol{\eta}_2 - \boldsymbol{\eta}_1, \boldsymbol{\eta}_1, \boldsymbol{\eta}_2 - \boldsymbol{\eta}_1), \end{aligned}$$

then

$$\begin{aligned} & a(\boldsymbol{\eta}_2-\boldsymbol{\eta}_1,\boldsymbol{\eta}_2-\boldsymbol{\eta}_1) \\ \leq \ & d(\boldsymbol{\eta}_2-\boldsymbol{\eta}_1,\boldsymbol{\eta}_1,\boldsymbol{\eta}_2-\boldsymbol{\eta}_1)+J(\boldsymbol{v}_1,\boldsymbol{\eta}_2)-J(\boldsymbol{v}_1,\boldsymbol{\eta}_1)+J(\boldsymbol{v}_2,\boldsymbol{\eta}_1)-J(\boldsymbol{v}_2,\boldsymbol{\eta}_2). \end{aligned}$$

It follows from (18), (25) and the continuity of $d(\cdot,\cdot,\cdot)$ that

$$2\nu\|\boldsymbol{\eta}_2-\boldsymbol{\eta}_1\|_V^2 \leq C_d\|\boldsymbol{\eta}_1\|_V\|\boldsymbol{\eta}_2-\boldsymbol{\eta}_1\|_V^2+C_0k\|\boldsymbol{v}_2-\boldsymbol{v}_1\|_V\|\boldsymbol{\eta}_2-\boldsymbol{\eta}_1\|_V,$$

and due to the fact that $\boldsymbol{\eta}_1\in\boldsymbol{K}_{div}$, $\|\boldsymbol{\eta}_1\|_V\leq\frac{C_1}{\nu}(\|\boldsymbol{f}\|_{-1}+\|g\|_{L^2(S)})$,
we have

$$\|\boldsymbol{\eta}_2-\boldsymbol{\eta}_1\|_V\leq\frac{C_0k/2\nu}{\left(1-\dfrac{C_dC_1(\|\boldsymbol{f}\|_{-1}+\|g\|_{L^2(S)})}{2\nu^2}\right)}\|\boldsymbol{v}_2-\boldsymbol{v}_1\|_V,$$

that is

$$\|\Phi(\boldsymbol{v}_2)-\Phi(\boldsymbol{v}_1)\|_V\leq L\|\boldsymbol{v}_2-\boldsymbol{v}_1\|_V,$$

with $0<L:=\dfrac{C_0k/2\nu}{\left(1-\dfrac{C_dC_1(\|\boldsymbol{f}\|_{-1}+\|g\|_{L^2(S)})}{2\nu^2}\right)}<1.$

The derivation of the a priori estimates for the velocity and pressure are similar to the one obtained for the Stokes equations and will not be repeated here. □

Remark 2.1. (a) *It should be noted that (32) is only needed for (36), while (32) and (33) are required for (31).*

(b) *It is manifest that in both conditions (32), and (33), we need smallness of the applied forces or large enough kinematic viscosity. In fact such requirement are not new, and are similar to those needed for Navier Stokes equations with classical Dirichlet boundary conditions* [22].

3. Finite element approximations

We assume that $\mathcal{T}_h$ is a regular partition of Ω in the sense introduced by Ciarlet [21]. The diameter of an element $K\in\mathcal{T}_h$ is denoted by h_K, and the mesh size h is defined by $h=\max_{K\in\mathcal{T}_h}h_K$. Let introduce the following subspaces:

$$\begin{aligned} M_h &= \{q_h\in M\cap\mathcal{C}(\Omega),\ q_h|_K\in\mathcal{P}_l(K)\text{ for all }K\in\mathcal{T}_h\}, \\ \boldsymbol{V}_h &= \{\boldsymbol{v}_h\in\boldsymbol{V}\cap\mathcal{C}(\Omega)^d,\ \boldsymbol{v}_h|_K\in\mathcal{P}_k(K)\text{ for all }K\in\mathcal{T}_h\}, \\ \boldsymbol{W}_h &= \{\boldsymbol{v}_h\in\boldsymbol{V}_h\ ,\quad b(\boldsymbol{v}_h,q_h)=0\quad\text{for all}\quad q_h\in M_h\}, \\ \boldsymbol{V}_{0h} &= \boldsymbol{V}_h\cap\boldsymbol{V}_0, \end{aligned}$$

where k,l are non-negative integers that will be made precise later, and $\mathcal{P}_l(K)$ the space of polynomial functions of two variables in K with degree less than or equal to l. In fact the integers k,l are such that the discrete counterpart of the inf-sup condition (17) holds with its constant β_h independent of h (see [22, 23] for more discussions and concrete examples).

3.1. Finite element approximation of the variational inequality (23).

3.1.1. Existence and uniqueness of solution. With the finite dimensional spaces $\boldsymbol{V}_h$ and M_h introduced, the finite element discretization of the variational inequality (23) reads: Find $(\boldsymbol{u}_h,p_h)\in\boldsymbol{V}_h\times M_h$ such that

(39)

$$\begin{aligned} &\text{for all }(\boldsymbol{v}_h,q_h)\in\boldsymbol{V}_h\times M_h, \\ a(\boldsymbol{u}_h,\boldsymbol{v}_h-\boldsymbol{u}_h)-b(\boldsymbol{v}_h-\boldsymbol{u}_h,p_h)+J(\boldsymbol{u}_h,\boldsymbol{v}_h)-J(\boldsymbol{u}_h,\boldsymbol{u}_h) &\geq \langle\boldsymbol{f},\boldsymbol{v}_h-\boldsymbol{u}_h\rangle, \\ b(\boldsymbol{u}_h,q_h) &= 0, \end{aligned}$$

which is equivalent to

$$(40)\qquad \begin{cases} \text{Find } \boldsymbol{u}_h \in \boldsymbol{W}_h \text{ such that for all } \boldsymbol{w}_h \in \boldsymbol{W}_h \\ a(\boldsymbol{u}_h, \boldsymbol{w}_h - \boldsymbol{u}_h) + J(\boldsymbol{u}_h, \boldsymbol{w}_h) - J(\boldsymbol{u}_h, \boldsymbol{u}_h) \geq \langle \boldsymbol{f}, \boldsymbol{w}_h - \boldsymbol{u}_h \rangle. \end{cases}$$

The existence of solutions of (40) follows the same procedure as the existence result for (23), and thus it holds that

Theorem 3.1. *Suppose that* $0 < \dfrac{C_0 k}{2\nu} < 1$. *Then the mixed variational problem (39) admits a unique solution* $(\boldsymbol{u}_h, p_h) \in \boldsymbol{V}_h \times M_h$, *which satisfies the following bound*

$$(41)\qquad \|\boldsymbol{u}_h\|_V + \|p_h\| \leq C(\|\boldsymbol{f}\|_{-1} + \|g\|_{L^2(S)}).$$

3.1.2. A priori error estimate. One of the main contribution of this work is the following result

Theorem 3.2. *Suppose that* $0 < \dfrac{C_0 k}{2\nu} < 1$. *Let* $(\boldsymbol{u}, p)$ *be the unique solution of (23), and* $(\boldsymbol{u}_h, p_h)$ *the unique solution of (39). Then there exists a generic positive constant* C *independent on* h *such that for all* $\boldsymbol{v}_h \in \boldsymbol{V}_h$ *and* $q_h \in M_h$,

$$(42)\qquad \|\boldsymbol{u} - \boldsymbol{u}_h\|_V \leq C\left\{\|\boldsymbol{u} - \boldsymbol{v}_h\|_V + \|p - q_h\| + \|\boldsymbol{u} - \boldsymbol{v}_h\|^{1/2}_{L^2(S)}\right\},$$

$$(43)\qquad \|p - p_h\| \leq C\left\{\|\boldsymbol{u} - \boldsymbol{v}_h\|_V + \|p - q_h\| + \|\boldsymbol{u} - \boldsymbol{v}_h\|^{1/2}_{L^2(S)}\right\}.$$

Proof For $\boldsymbol{w} \in \boldsymbol{V}_0$, replace $\boldsymbol{v}$ in (23) by $\boldsymbol{u}+\boldsymbol{w}$ and $\boldsymbol{u}-\boldsymbol{w}$, and putting together the resulting equations, one gets

$$(44)\qquad a(\boldsymbol{u}, \boldsymbol{w}) - b(\boldsymbol{w}, p) = \langle \boldsymbol{f}, \boldsymbol{w} \rangle \quad \text{for all } \boldsymbol{w} \in \boldsymbol{V}_0.$$

Likewise with (39) and $\boldsymbol{w}_h \in \boldsymbol{V}_{0h}$, one arrives at

$$(45)\qquad a(\boldsymbol{u}_h, \boldsymbol{w}_h) - b(\boldsymbol{w}_h, p_h) = \langle \boldsymbol{f}, \boldsymbol{w}_h \rangle \quad \text{for all } \boldsymbol{w}_h \in \boldsymbol{V}_{0h}.$$

Let $\boldsymbol{w} = \boldsymbol{w}_h$, then (44) and (45) give

$$a(\boldsymbol{u} - \boldsymbol{u}_h, \boldsymbol{w}_h) - b(\boldsymbol{w}_h, p - p_h) = 0 \quad \text{for all} \quad \boldsymbol{w}_h \in \boldsymbol{V}_{0h},$$

which is re-written as

$$b(\boldsymbol{w}_h, p_h - q_h) = a(\boldsymbol{u} - \boldsymbol{u}_h, \boldsymbol{w}_h) + b(\boldsymbol{w}_h, p - q_h).$$

Now, the equality together with the discrete version of the inf-sup condition (17) and the continuity of $a(\cdot,\cdot)$ and $b(\cdot,\cdot)$ gives

$$\begin{aligned} \beta\|p_h - q_h\| \leq \sup_{\boldsymbol{w}_h \in \boldsymbol{V}_{0h}} \frac{b(\boldsymbol{w}_h, p_h - q_h)}{\|\boldsymbol{w}_h\|_V} &= \sup_{\boldsymbol{w}_h \in \boldsymbol{V}_{0h}} \frac{|a(\boldsymbol{u} - \boldsymbol{u}_h, \boldsymbol{w}_h) + b(\boldsymbol{w}_h, p - q_h)|}{\|\boldsymbol{w}_h\|_V} \\ &\leq (2\nu\|\boldsymbol{u} - \boldsymbol{u}_h\|_V + \|p - q_h\|), \end{aligned}$$

so that,

$$(46)\qquad \|p - p_h\| \leq \|p - q_h\| + \|q_h - p_h\| \leq C\left\{\|\boldsymbol{u} - \boldsymbol{u}_h\|_V + \|p - q_h\|\right\}.$$

Next, let $\boldsymbol{v}_h \in \boldsymbol{V}_h$, replacing successively $\boldsymbol{v}$ in $(23)_1$ by $\boldsymbol{v} = \boldsymbol{u}_h$ and $\boldsymbol{v} = 2\boldsymbol{u} - \boldsymbol{v}_h$ and putting together the resulting inequalities, yields

(47)

$$a(\boldsymbol{u}, \boldsymbol{u}_h - \boldsymbol{v}_h) - b(\boldsymbol{u}_h - \boldsymbol{v}_h, p) + J(\boldsymbol{u}, 2\boldsymbol{u} - \boldsymbol{v}_h) + J(\boldsymbol{u}, \boldsymbol{u}_h) - 2J(\boldsymbol{u}, \boldsymbol{u}) \geq \langle \boldsymbol{f}, \boldsymbol{u}_h - \boldsymbol{v}_h \rangle.$$

Note that the inequality $(39)_1$ can be recast as

$$(48)\qquad -a(\boldsymbol{u}_h, \boldsymbol{u}_h - \boldsymbol{v}_h) + b(\boldsymbol{u}_h - \boldsymbol{v}_h, p_h) + J(\boldsymbol{u}_h, \boldsymbol{v}_h) - J(\boldsymbol{u}_h, \boldsymbol{u}_h) \geq -\langle \boldsymbol{f}, \boldsymbol{u}_h - \boldsymbol{v}_h \rangle$$

Next, (47)+(48) yields

$$a(\boldsymbol{u}-\boldsymbol{u}_h,\boldsymbol{u}_h-\boldsymbol{v}_h)-b(\boldsymbol{u}_h-\boldsymbol{v}_h,p-p_h)+J(\boldsymbol{u},2\boldsymbol{u}-\boldsymbol{v}_h)\quad + \tag{49}$$
$$J(\boldsymbol{u},\boldsymbol{u}_h)-2J(\boldsymbol{u},\boldsymbol{u})+J(\boldsymbol{u}_h,\boldsymbol{v}_h)-J(\boldsymbol{u}_h,\boldsymbol{u}_h)\geq 0.$$

By linearity of $a(\cdot,\cdot)$,

$$a(\boldsymbol{u}-\boldsymbol{u}_h,\boldsymbol{u}_h-\boldsymbol{v}_h)=a(\boldsymbol{u}-\boldsymbol{v}_h,\boldsymbol{u}_h-\boldsymbol{v}_h)-a(\boldsymbol{u}_h-\boldsymbol{v}_h,\boldsymbol{u}_h-\boldsymbol{v}_h). \tag{50}$$

Using $(23)_2$ and $(39)_2$, one has

$$\begin{aligned} b(\boldsymbol{u}_h-\boldsymbol{v}_h,p-p_h) &= b(\boldsymbol{u}_h-\boldsymbol{u},p-q_h)+b(\boldsymbol{u}_h-\boldsymbol{u},q_h-p_h)+b(\boldsymbol{u}-\boldsymbol{v}_h,p-p_h)\\ &= b(\boldsymbol{u}_h-\boldsymbol{u},p-q_h)+b(\boldsymbol{u}-\boldsymbol{v}_h,p-p_h). \end{aligned} \tag{51}$$

Returning to (49) with (50) and (51), we obtain

$$2\nu\|\boldsymbol{u}_h-\boldsymbol{v}_h\|_V^2\leq a(\boldsymbol{u}_h-\boldsymbol{v}_h,\boldsymbol{u}_h-\boldsymbol{v}_h)\leq I_1+I_2, \tag{52}$$

where

$$I_1 = a(\boldsymbol{u}-\boldsymbol{v}_h,\boldsymbol{u}_h-\boldsymbol{v}_h)-b(\boldsymbol{u}_h-\boldsymbol{u},p-q_h)-b(\boldsymbol{u}-\boldsymbol{v}_h,p-p_h), \tag{53}$$
$$I_2 = J(\boldsymbol{u},2\boldsymbol{u}-\boldsymbol{v}_h)+J(\boldsymbol{u},\boldsymbol{u}_h)-2J(\boldsymbol{u},\boldsymbol{u})+J(\boldsymbol{u}_h,\boldsymbol{v}_h)-J(\boldsymbol{u}_h,\boldsymbol{u}_h). \tag{54}$$

By simple algebra manipulation, one obtains

$$\begin{aligned} & I_2\\ = \;& J(\boldsymbol{u},\boldsymbol{u}_h)-J(\boldsymbol{u},\boldsymbol{v}_h)+J(\boldsymbol{u}_h,\boldsymbol{v}_h)-J(\boldsymbol{u}_h,\boldsymbol{u}_h)\\ & +J(\boldsymbol{u},2\boldsymbol{u}-\boldsymbol{v}_h)-2J(\boldsymbol{u},\boldsymbol{u})+J(\boldsymbol{u},\boldsymbol{v}_h), \end{aligned}$$

which from (25), and application of the triangle's inequality gives

$$I_2\leq C_0k\|\boldsymbol{u}-\boldsymbol{u}_h\|_V\|\boldsymbol{u}_h-\boldsymbol{v}_h\|_V+2(\|g\|_{L^2(S)}+C_0k\|\boldsymbol{u}\|_V)\|\boldsymbol{u}-\boldsymbol{v}_h\|_{L^2(S)}. \tag{55}$$

Next, applying the continuity of both bilinear forms $a(\cdot,\cdot)$ and $b(\cdot,\cdot)$, we get

$$I_1\leq 2\nu\|\boldsymbol{u}-\boldsymbol{v}_h\|_V\|\boldsymbol{u}_h-\boldsymbol{v}_h\|_V+\|p-q_h\|\|\boldsymbol{u}-\boldsymbol{u}_h\|_V+\|p-p_h\|\|\boldsymbol{u}-\boldsymbol{v}_h\|_V,$$

together with (55) and (52) gives

$$\begin{aligned} & 2\nu\|\boldsymbol{u}_h-\boldsymbol{v}_h\|_V^2\\ \leq & \left\{\begin{array}{l} 2\nu\|\boldsymbol{u}-\boldsymbol{v}_h\|_V\|\boldsymbol{u}_h-\boldsymbol{v}_h\|_V+\|p-q_h\|\|\boldsymbol{u}-\boldsymbol{u}_h\|_V+\|p-p_h\|\|\boldsymbol{u}-\boldsymbol{v}_h\|_V\\ +C_0k\|\boldsymbol{u}-\boldsymbol{u}_h\|_V\|\boldsymbol{u}_h-\boldsymbol{v}_h\|_V+2(\|g\|_{L^2(S)}+C_0k\|\boldsymbol{u}\|_V)\|\boldsymbol{u}-\boldsymbol{v}_h\|_{L^2(S)} \end{array}\right\}, \end{aligned}$$

which together with Young's inequality, the a priori estimate (34), (46), and the triangle inequality

$$\|\boldsymbol{u}-\boldsymbol{u}_h\|_V\leq\|\boldsymbol{u}-\boldsymbol{v}_h\|_V+\|\boldsymbol{v}_h-\boldsymbol{u}_h\|_V,$$

gives the desired bound (42), whereas (43) is a consequence of (42) and (46). □

3.2. Finite element approximation of the variational inequality (30).

3.2.1. Existence and uniqueness of solution. The finite element approximation of the variational inequality (30) reads:
Find $(\boldsymbol{u}_h,p_h)\in\boldsymbol{V}_h\times M_h$ such that

$$\begin{cases} \text{for all } \boldsymbol{v}_h,q_h\in\boldsymbol{V}_h\times M_h\\ a(\boldsymbol{u}_h,\boldsymbol{v}_h-\boldsymbol{u}_h)+d(\boldsymbol{u}_h,\boldsymbol{u}_h,\boldsymbol{v}_h-\boldsymbol{u}_h)-b(p_h,\boldsymbol{v}_h-\boldsymbol{u}_h)\\ +J(\boldsymbol{u}_h,\boldsymbol{v}_h)-J(\boldsymbol{u}_h,\boldsymbol{u}_h)\geq\langle\boldsymbol{f},\boldsymbol{v}_h-\boldsymbol{u}_h\rangle\\ b(\boldsymbol{u}_h,q_h)=0, \end{cases} \tag{56}$$

which is equivalent to: Find $\boldsymbol{u}_h \in \boldsymbol{W}_h$ such that
(57)
$$\begin{cases} \text{for all } \boldsymbol{v}_h \in \boldsymbol{V}_h \\ a(\boldsymbol{u}_h, \boldsymbol{v}_h - \boldsymbol{u}_h) + d(\boldsymbol{u}_h, \boldsymbol{u}_h, \boldsymbol{v}_h - \boldsymbol{u}_h) + J(\boldsymbol{u}_h, \boldsymbol{v}_h) - J(\boldsymbol{u}_h, \boldsymbol{u}_h) \geq \langle \boldsymbol{f}, \boldsymbol{v}_h - \boldsymbol{u}_h \rangle. \end{cases}$$

As far as the existence of solutions of (56) is concerned, one claim that

Theorem 3.3. *If the following conditions hold:*

$$0 < \frac{C_d C_1(\|\boldsymbol{f}\|_{-1} + \|g\|_{L^2(S)})}{\nu^2} < 1 \tag{58}$$

$$0 < \frac{C_0 k}{2\nu} < 1 - \frac{C_d C_1(\|\boldsymbol{f}\|_{-1} + \|g\|_{L^2(S)})}{2\nu^2}. \tag{59}$$

Then the mixed finite variational problem (56) admits a unique solution $(\boldsymbol{u}_h, p_h) \in \boldsymbol{K}_h \times M_h$*, which satisfies the following bound*

$$\|\boldsymbol{u}_h\|_V \leq C(\|\boldsymbol{f}\|_{-1} + \|g\|_{L^2(S)}) \tag{60}$$

$$\|p_h\| \leq C(\|\boldsymbol{f}\|_{-1} + \|g\|_{L^2(S)} + \|\boldsymbol{f}\|_{-1}^2 + \|g\|_{L^2(S)}^2). \tag{61}$$

where

$$\boldsymbol{K}_h = \{\boldsymbol{v}_h \in \boldsymbol{W}_h, \quad \|\boldsymbol{v}_h\|_1 \leq \frac{C_1}{\nu}(\|\boldsymbol{f}\|_{-1} + \|g\|_{L^2(S)})\}.$$

The proof goes along the same lines as the proof of Theorem 2.2, and hence will not be repeated here.

3.2.2. A priori error estimate.

Theorem 3.4. *If conditions (58) and (59) are satisfied with* $\boldsymbol{f} \in \boldsymbol{H}^{-1}(\Omega)$*,* $g \in L^2(S)$*, and* $g > 0$*, then there exists a generic positive constant* C *independent of* h *such that for all* $\boldsymbol{v}_h \in \boldsymbol{V}_h$ *and* $q_h \in M_h$*,*

$$\|\boldsymbol{u} - \boldsymbol{u}_h\|_V \leq C\left\{\|\boldsymbol{u} - \boldsymbol{v}_h\|_V + \|p - q_h\| + \|\boldsymbol{u} - \boldsymbol{v}_h\|_{L^2(S)}^{1/2}\right\}, \tag{62}$$

$$\|p - p_h\| \leq C\left\{\|\boldsymbol{u} - \boldsymbol{v}_h\|_V + \|p - q_h\| + \|\boldsymbol{u} - \boldsymbol{v}_h\|_{L^2(S)}^{1/2}\right\}. \tag{63}$$

Proof Let $\boldsymbol{w} \in \boldsymbol{V}_0$. Replacing $\boldsymbol{v}$ in (30) by $\boldsymbol{u} - \boldsymbol{w}$ and $\boldsymbol{u} + \boldsymbol{w}$ and adding the resulting equations gives

$$a(\boldsymbol{u}, \boldsymbol{w}) + d(\boldsymbol{u}, \boldsymbol{u}, \boldsymbol{w}) - b(\boldsymbol{w}, p) = \langle \boldsymbol{f}, \boldsymbol{w} \rangle \quad \text{for all } \boldsymbol{w} \in \boldsymbol{V}_0.$$

Next, let $\boldsymbol{w}_h \in \boldsymbol{V}_{0h}$, and replace $\boldsymbol{v}_h$ in (56) by $\boldsymbol{u}_h - \boldsymbol{w}_h$ and $\boldsymbol{u}_h + \boldsymbol{w}_h$, adding the resulting equations, one gets

$$a(\boldsymbol{u}_h, \boldsymbol{w}_h) + d(\boldsymbol{u}_h, \boldsymbol{u}_h, \boldsymbol{w}_h) - b(\boldsymbol{w}_h, p_h) = \langle \boldsymbol{f}, \boldsymbol{w}_h \rangle \quad \text{for all } \boldsymbol{w}_h \in \boldsymbol{V}_{0h}.$$

Putting together the former and later equations for $\boldsymbol{w} = \boldsymbol{w}_h$, gives
(64)
$$a(\boldsymbol{u} - \boldsymbol{u}_h, \boldsymbol{w}_h) + d(\boldsymbol{u}, \boldsymbol{u}, \boldsymbol{w}_h) - d(\boldsymbol{u}_h, \boldsymbol{u}_h, \boldsymbol{w}_h) - b(\boldsymbol{w}_h, p - p_h) = 0 \quad \text{for all} \quad \boldsymbol{w}_h \in \boldsymbol{V}_{0h}.$$

From the linearity

$$\begin{aligned} b(\boldsymbol{w}_h, p_h - q_h) &= b(\boldsymbol{w}_h, p_h - p) + b(\boldsymbol{w}_h, p - q_h), \\ d(\boldsymbol{u}, \boldsymbol{u}, \boldsymbol{w}_h) - d(\boldsymbol{u}_h, \boldsymbol{u}_h, \boldsymbol{w}_h) &= d(\boldsymbol{u}, \boldsymbol{u} - \boldsymbol{u}_h, \boldsymbol{w}_h) + d(\boldsymbol{u} - \boldsymbol{u}_h, \boldsymbol{u}_h, \boldsymbol{w}_h), \end{aligned}$$

which together with the inf-sup on $b(\cdot,\cdot)$ and (64) gives

$$
\begin{aligned}
& \beta\|p_h - q_h\| \\
\leq \; & \sup_{\boldsymbol{w}\in \boldsymbol{V}_{0h}} \frac{b(\boldsymbol{w}_h, p_h - q_h)}{\|\boldsymbol{w}_h\|_V} \\
= \; & \sup_{\boldsymbol{w}_h\in \boldsymbol{V}_{0h}} \frac{|a(\boldsymbol{u}-\boldsymbol{u}_h, \boldsymbol{w}_h) + d(\boldsymbol{u}, \boldsymbol{u}-\boldsymbol{u}_h, \boldsymbol{w}_h) + d(\boldsymbol{u}-\boldsymbol{u}_h, \boldsymbol{u}_h, \boldsymbol{w}_h) + b(\boldsymbol{w}_h, p - q_h)|}{\|\boldsymbol{w}_h\|_V} \\
\leq \; & (2\nu\|\boldsymbol{u}-\boldsymbol{u}_h\|_V + C_d\|\boldsymbol{u}\|_V\|\boldsymbol{u}-\boldsymbol{u}_h\|_V + C_d\|\boldsymbol{u}_h\|_V\|\boldsymbol{u}-\boldsymbol{u}_h\|_V + \|p - q_h\|) \\
\leq \; & (2\nu\|\boldsymbol{u}-\boldsymbol{u}_h\|_V + \frac{2C_dC_1}{\nu}(\|\boldsymbol{f}\|_{-1} + \|g\|_{L^2(S)})\|\boldsymbol{u}-\boldsymbol{u}_h\|_V + \|p - q_h\|) \\
& \text{since} \quad \boldsymbol{u} \in \boldsymbol{K}_{div} \quad \text{and} \quad \boldsymbol{u}_h \in \boldsymbol{K}_h.
\end{aligned}
$$

Hence

$$\|p - p_h\| \leq \|p - q_h\| + \|q_h - p_h\| \leq C\left\{\|\boldsymbol{u}-\boldsymbol{u}_h\|_V + \|p - q_h\|\right\}. \tag{65}$$

Next, we take $\boldsymbol{v}_h \in \boldsymbol{V}_h$, replacing successively $\boldsymbol{v}$ in $(30)_1$ by $\boldsymbol{v} = \boldsymbol{u}_h$ and $\boldsymbol{v} = 2\boldsymbol{u}-\boldsymbol{v}_h$, one gets

$$a(\boldsymbol{u}, \boldsymbol{u}_h-\boldsymbol{u}) + d(\boldsymbol{u}, \boldsymbol{u}, \boldsymbol{u}_h-\boldsymbol{u}) - b(\boldsymbol{u}_h-\boldsymbol{u}, p) + J(\boldsymbol{u}, \boldsymbol{u}_h) - J(\boldsymbol{u}, \boldsymbol{u}) \geq \langle \boldsymbol{f}, \boldsymbol{u}_h-\boldsymbol{u}\rangle, \tag{66}$$

and

$$a(\boldsymbol{u}, \boldsymbol{u}-\boldsymbol{v}_h) + d(\boldsymbol{u}, \boldsymbol{u}, \boldsymbol{u}-\boldsymbol{v}_h) - b(\boldsymbol{u}-\boldsymbol{v}_h, p) + J(\boldsymbol{u}, 2\boldsymbol{u}-\boldsymbol{v}_h) - J(\boldsymbol{u}, \boldsymbol{u}) \geq \langle \boldsymbol{f}, \boldsymbol{u}-\boldsymbol{v}_h\rangle. \tag{67}$$

(66)+(67) yields

$$
\begin{aligned}
& a(\boldsymbol{u}, \boldsymbol{u}_h-\boldsymbol{v}_h) + d(\boldsymbol{u}, \boldsymbol{u}, \boldsymbol{u}_h-\boldsymbol{v}_h) - b(\boldsymbol{u}_h-\boldsymbol{v}_h, p) \\
& + J(\boldsymbol{u}, 2\boldsymbol{u}-\boldsymbol{v}_h) + J(\boldsymbol{u}, \boldsymbol{u}_h) - 2J(\boldsymbol{u}, \boldsymbol{u}) \geq \langle \boldsymbol{f}, \boldsymbol{u}_h-\boldsymbol{v}_h\rangle.
\end{aligned} \tag{68}
$$

Note that the inequality $(56)_1$ can be written as

$$-a(\boldsymbol{u}_h, \boldsymbol{u}_h-\boldsymbol{v}_h) - d(\boldsymbol{u}_h, \boldsymbol{u}_h, \boldsymbol{u}_h-\boldsymbol{v}_h) + b(\boldsymbol{u}_h-\boldsymbol{v}_h, p_h) + J(\boldsymbol{u}_h, \boldsymbol{v}_h) - J(\boldsymbol{u}_h, \boldsymbol{u}_h) \geq -\langle \boldsymbol{f}, \boldsymbol{u}_h-\boldsymbol{v}_h\rangle,$$

which together with (68) leads to

$$
\begin{aligned}
& a(\boldsymbol{u}-\boldsymbol{u}_h, \boldsymbol{u}_h-\boldsymbol{v}_h) + d(\boldsymbol{u}, \boldsymbol{u}, \boldsymbol{u}_h-\boldsymbol{v}_h) - d(\boldsymbol{u}_h, \boldsymbol{u}_h, \boldsymbol{u}_h-\boldsymbol{v}_h) - b(\boldsymbol{u}_h-\boldsymbol{v}_h, p-p_h) \\
& + J(\boldsymbol{u}, 2\boldsymbol{u}-\boldsymbol{v}_h) + J(\boldsymbol{u}, \boldsymbol{u}_h) - 2J(\boldsymbol{u}, \boldsymbol{u}) + J(\boldsymbol{u}_h, \boldsymbol{v}_h) - J(\boldsymbol{u}_h, \boldsymbol{u}_h) \geq 0.
\end{aligned} \tag{69}
$$

Note that

$$d(\boldsymbol{u}, \boldsymbol{u}, \boldsymbol{u}_h-\boldsymbol{v}_h) - d(\boldsymbol{u}_h, \boldsymbol{u}_h, \boldsymbol{u}_h-\boldsymbol{v}_h) = d(\boldsymbol{u}, \boldsymbol{u}-\boldsymbol{u}_h, \boldsymbol{u}_h-\boldsymbol{v}_h) + d(\boldsymbol{u}-\boldsymbol{u}_h, \boldsymbol{u}_h, \boldsymbol{u}_h-\boldsymbol{v}_h). \tag{70}$$

Substituting equations (50), (51) and (70) into (69) yields

$$
\begin{aligned}
& 2\nu\|\boldsymbol{u}_h-\boldsymbol{v}_h\|_V^2 && (71)\\
\leq \; & a(\boldsymbol{u}_h-\boldsymbol{v}_h, \boldsymbol{u}_h-\boldsymbol{v}_h) \\
\leq \; & a(\boldsymbol{u}-\boldsymbol{v}_h, \boldsymbol{u}_h-\boldsymbol{v}_h) - b(\boldsymbol{u}_h-\boldsymbol{u}, p-q_h) - b(\boldsymbol{u}-\boldsymbol{v}_h, p-p_h \\
& + d(\boldsymbol{u}, \boldsymbol{u}-\boldsymbol{u}_h, \boldsymbol{u}_h-\boldsymbol{v}_h) + d(\boldsymbol{u}-\boldsymbol{u}_h, \boldsymbol{u}_h, \boldsymbol{u}_h-\boldsymbol{v}_h) && (72)\\
& + J(\boldsymbol{u}, 2\boldsymbol{u}-\boldsymbol{v}_h) + J(\boldsymbol{u}, \boldsymbol{u}_h) - 2J(\boldsymbol{u}, \boldsymbol{u}) + J(\boldsymbol{u}_h, \boldsymbol{v}_h) - J(\boldsymbol{u}_h, \boldsymbol{u}_h).
\end{aligned}
$$

Using standard inequalities and (55), (72) becomes

$$
\begin{array}{l}
2\nu\|\boldsymbol{u}_h-\boldsymbol{v}_h\|_V^2 \\
\leq \left\{\begin{array}{l}
2\nu\|\boldsymbol{u}-\boldsymbol{v}_h\|_V\|\boldsymbol{u}_h-\boldsymbol{v}_h\|_V+\|p-q_h\|\|\boldsymbol{u}-\boldsymbol{u}_h\|_V+\|p-p_h\|\|\boldsymbol{u}-\boldsymbol{v}_h\|_V \\
+C_d\|\boldsymbol{u}\|_V\|\boldsymbol{u}-\boldsymbol{u}_h\|_V\|\boldsymbol{u}_h-\boldsymbol{v}_h\|_V+C_d\|\boldsymbol{u}_h\|_V\|\boldsymbol{u}-\boldsymbol{u}_h\|_V\|\boldsymbol{u}_h-\boldsymbol{v}_h\|_V \\
+C_0k\|\boldsymbol{u}-\boldsymbol{u}_h\|_V\|\boldsymbol{u}_h-\boldsymbol{v}_h\|_V+2(\|g\|_{L^2(S)}+C_0k\|\boldsymbol{u}\|_V)\|\boldsymbol{u}-\boldsymbol{v}_h\|_{L^2(S)}
\end{array}\right\} \\
\leq \left\{\begin{array}{c}
2\nu\|\boldsymbol{u}-\boldsymbol{v}_h\|_V\|\boldsymbol{u}_h-\boldsymbol{v}_h\|_V+\|p-q_h\|\|\boldsymbol{u}-\boldsymbol{u}_h\|_V+\|p-p_h\|\|\boldsymbol{u}-\boldsymbol{v}_h\|_V \\
+\frac{2C_dC_1}{\nu}(\|\boldsymbol{f}\|_{-1}+\|g\|_{L^2(S)})\|\boldsymbol{u}-\boldsymbol{u}_h\|_V\|\boldsymbol{u}_h-\boldsymbol{v}_h\|_V+C_0k\|\boldsymbol{u}-\boldsymbol{u}_h\|_V\|\boldsymbol{u}_h-\boldsymbol{v}_h\|_V \\
+2(\|g\|_{L^2(S)}+\frac{C_0kC_1}{\nu}(\|\boldsymbol{f}\|_{-1}+\|g\|_{L^2(S)}))\|\boldsymbol{u}-\boldsymbol{v}_h\|_{L^2(S)}
\end{array}\right\} \\
\text{since} \quad \boldsymbol{u}\in\boldsymbol{K}_{div} \quad \boldsymbol{u}_h\in\boldsymbol{K}_h
\end{array}
$$

Hence using the triangle inequalities, the Young's inequality, and the relations (58), (59) and (65), we obtain

$$\|\boldsymbol{u}-\boldsymbol{u}_h\|_V^2 \leq C\left\{\|\boldsymbol{u}-\boldsymbol{v}_h\|_V^2+\|p-q_h\|^2+\|\boldsymbol{u}-\boldsymbol{v}_h\|_{L^2(S)}\right\},$$

which automatically gives (62), while (63) is a consequence of (62) and (65). □

Remark 3.1. *It should be mentioned that specific choice of $\boldsymbol{V}_h$ and M_h leads derivation of particular rate of convergence in Theorem 3.2 and Theorem 3.4. (see* [22, 23]*).*

4. Numerical Algorithm

In this section, we present and analyze the algorithms for the implementation of (39) and (56). Next, we present some numerical computations related to the algorithms described.

4.1. Numerical algorithm for Stokes variational inequality (39).

Let consider the following problem: Given $\boldsymbol{u}_h^0 \in \boldsymbol{V}_h$, find $(\boldsymbol{u}_h^n, p_h^n) \in \boldsymbol{V}_h \times M_h$ such that

(73)
$$
\begin{array}{rcl}
& \text{for all } \boldsymbol{v}_h, q_h \in \boldsymbol{V}_h\times M_h & \\
a(\boldsymbol{u}_h^n,\boldsymbol{v}_h-\boldsymbol{u}_h^n)-b(\boldsymbol{v}_h-\boldsymbol{u}_h^n,p_h^n)+J(\boldsymbol{u}_h^{n-1},\boldsymbol{v}_h)-J(\boldsymbol{u}_h^{n-1},\boldsymbol{u}_h^n) & \geq & \langle\boldsymbol{f},\boldsymbol{v}_h-\boldsymbol{u}_h^n\rangle \\
b(\boldsymbol{u}_h^n,q_h) & = & 0,
\end{array}
$$

which is also equivalent to; given $\boldsymbol{u}_h^0 \in \boldsymbol{W}_h$, find $(\boldsymbol{u}_h^n, p_h^n) \in \boldsymbol{W}_h$ such that

$$
\begin{array}{rcl}
& \text{for all } \boldsymbol{v}_h\in\boldsymbol{W}_h & \\
a(\boldsymbol{u}_h^n,\boldsymbol{v}_h-\boldsymbol{u}_h^n)+J(\boldsymbol{u}_h^{n-1},\boldsymbol{v}_h)-J(\boldsymbol{u}_h^{n-1},\boldsymbol{u}_h^n) & \geq & \langle\boldsymbol{f},\boldsymbol{v}_h-\boldsymbol{u}_h^n\rangle.
\end{array} \tag{74}
$$

About the convergence of the algorithm (73), one can claim the following

Theorem 4.1. *Suppose that $0 < \dfrac{C_0k}{2\nu} < 1$, problem (73) admits a unique solution $(\boldsymbol{u}_h^n, p_h^n) \in \boldsymbol{V}_h \times M_h$. Moreover let $(\boldsymbol{u}_h, p_h) \in \boldsymbol{V}_h \times M_h$ be the solution of problem (39). Then the iterative solution $(\boldsymbol{u}_h^n, p_h^n)$ converges to $(\boldsymbol{u}_h, p_h)$ in $\boldsymbol{V}_h \times M_h$ as $n \to \infty$. More precisely,*

$$\|\boldsymbol{u}_h^n-\boldsymbol{u}_h\|_V \leq \left(\frac{C_0k}{2\nu}\right)^n\|\boldsymbol{u}_h^0-\boldsymbol{u}_h\|_V \tag{75}$$

$$\|p_h^n-p_h\| \leq C\|\boldsymbol{u}_h^n-\boldsymbol{u}_h\|_V\ . \tag{76}$$

Proof. For the solvability of (73), note that knowing $\boldsymbol{u}_h^{n-1} \in \boldsymbol{V}_h$, computing $(\boldsymbol{u}_h^n, p_h^n) \in \boldsymbol{V}_h \times M_h$ in (73) is to solve the variational inequality of the second kind with g replaced by $g + k|\boldsymbol{u}_{\boldsymbol{\tau}h}^{n-1}|$.
Next, let $\boldsymbol{v}_h = \boldsymbol{u}_h^n$ in (40) and $\boldsymbol{v}_h = \boldsymbol{u}_h$ in (74), adding the resulting equations, we obtain:

$$a(\boldsymbol{u}_h^n-\boldsymbol{u}_h,\boldsymbol{u}_h^n-\boldsymbol{u}_h) \leq J(\boldsymbol{u}_h^{n-1},\boldsymbol{u}_h)-J(\boldsymbol{u}_h^{n-1},\boldsymbol{u}_h^n)+J(\boldsymbol{u}_h,\boldsymbol{u}_h^n)-J(\boldsymbol{u}_h,\boldsymbol{u}_h). \tag{77}$$

(77) is treated using the coercivity (18) on the left, whereas its right hand side is bounded using the inequality (25). We then obtain

$$2\nu\|\boldsymbol{u}_h^n - \boldsymbol{u}_h\|_V^2 \leq C_0 k \|\boldsymbol{u}_h^{n-1} - \boldsymbol{u}_h\|_V \|\boldsymbol{u}_h^n - \boldsymbol{u}_h\|_V,$$

which gives

$$\|\boldsymbol{u}_h^n - \boldsymbol{u}_h\|_V \leq \frac{C_0 k}{2\nu}\|\boldsymbol{u}_h^{n-1} - \boldsymbol{u}_h\|_V.$$

By induction one has (75).
For the convergence of p_h^n, let $\boldsymbol{w}_h \in \boldsymbol{W}_h$ and replace $\boldsymbol{v}_h$ in $(73)_1$ successively by $\boldsymbol{u}_h^n + \boldsymbol{w}_h$ and $\boldsymbol{u}_h^n - \boldsymbol{w}_h$. Observe that $J(\boldsymbol{u}_h^{n-1}, \boldsymbol{v}_h) = J(\boldsymbol{u}_h^{n-1}, \boldsymbol{u}_h^n \pm \boldsymbol{w}_h) = J(\boldsymbol{u}_h^{n-1}, \boldsymbol{u}_h^n)$, then

$$a(\boldsymbol{u}_h^n, \boldsymbol{w}_h) - b(\boldsymbol{w}_h, p_h^n) = \langle \boldsymbol{f}, \boldsymbol{w}_h \rangle \text{ for all } \boldsymbol{w}_h \in \boldsymbol{W}_h. \tag{78}$$

Likewise, one obtains

$$a(\boldsymbol{u}_h, \boldsymbol{w}_h) - b(\boldsymbol{w}_h, p_h) = \langle \boldsymbol{f}, \boldsymbol{w}_h \rangle \text{ for all } \boldsymbol{w}_h \in \boldsymbol{W}_h,$$

which together with (78) gives

$$a(\boldsymbol{u}_h - \boldsymbol{u}_h^n, \boldsymbol{w}_h) - b(\boldsymbol{w}_h, p_h - p_h^n) = 0 \text{ for all } \boldsymbol{w}_h \in \boldsymbol{W}_h.$$

That relation together with the discrete inf-sup condition on $b(\cdot,\cdot)$ leads to,

$$\beta\|p_h^n - p_h\| \leq \sup_{\boldsymbol{v}_h \in \boldsymbol{V}_h} \frac{b(\boldsymbol{v}_h, p_h - p_h^n)}{\|\boldsymbol{v}_h\|_V} = \sup_{\boldsymbol{v}_h \in \boldsymbol{V}_h} \frac{a(\boldsymbol{u}_h - \boldsymbol{u}_h^n, \boldsymbol{w}_h)}{\|\boldsymbol{w}_h\|_1} \leq 2\nu\|\boldsymbol{u}_h^n - \boldsymbol{u}_h\|_V,$$

and the proof is terminated using (75). □

Remark 4.1. *Knowing $\boldsymbol{u}_h^{n-1} \in \boldsymbol{V}_h$, computing $(\boldsymbol{u}_h^n, p_h^n) \in \boldsymbol{V}_h \times M_h$ in (73) is to solve the variational inequality of the second kind with g replaced by $g + k|\boldsymbol{u}_{\boldsymbol{\tau}h}^{n-1}|$ which can be solved numerically using Uzawa iteration method (see* [4, 14, 18, 22]*).*

Then we construct the following Uzawa iteration algorithm to solve (39) via (73)
Algorithm 1:

$$\boldsymbol{u}_h^0 \in \boldsymbol{V}_h, \quad \lambda_h^1 \in \Lambda_h \quad \text{arbitrary given} \tag{79}$$

where $\Lambda = \{\lambda \in L^2(S) : \ |\lambda(x)| \leq 1 \ a.e. \text{ on } S\}$ and $\Lambda_h \subset \Lambda$ is the finite element space.
Step 1: knowing $(\boldsymbol{u}_h^{n-1}, \lambda_h^n) \in \boldsymbol{V}_h \times \Lambda_h$, compute $(\boldsymbol{u}_h^n, p_h^n) \in \boldsymbol{V}_h \times M_h$ by
(80)

$$\begin{cases} a(\boldsymbol{u}_h^n, \boldsymbol{v}_h) - b(\boldsymbol{v}_h, p_h^n) = \langle \boldsymbol{f}, \boldsymbol{v}_h \rangle - \int_S \lambda_h^n (g + k|\boldsymbol{u}_{\boldsymbol{\tau}h}^{n-1}|)\boldsymbol{v}_{\boldsymbol{\tau}h} ds \text{ for all } \boldsymbol{v}_h \in \boldsymbol{V}_h \\ b(\boldsymbol{u}_h^n, q_h) = 0 \text{ for all } q_h \in M_h \ , \end{cases}$$

Step 2: Renew $\lambda_h^{n+1} \in \Lambda_h$

$$\lambda_h^{n+1} = P_{\Lambda_h}(\lambda_h^n + \rho(g + k|\boldsymbol{u}_{\boldsymbol{\tau}h}^{n-1}|)\boldsymbol{u}_{\boldsymbol{\tau}h}^n) \tag{81}$$

where $P_{\Lambda_h}(\mu) = \sup(-1, \inf(1, \mu))$ for all $\mu \in L^2(S)$ and $\rho > 0$

Remark 4.2. *The unique existence of $(\boldsymbol{u}_h^n, p_h^n) \in \boldsymbol{V}_h \times M_h$ satisfying (80) is guaranteed by the discrete inf-sup condition on $b(\cdot,\cdot)$.*

4.2. Numerical algorithm for Navier-Stokes variational inequality (56). Let consider the following problem: Given $\boldsymbol{u}_h^0 \in \boldsymbol{V}_h$, find $(\boldsymbol{u}_h^n, p_h^n) \in \boldsymbol{V}_h \times M_h$ such that

(82)

$$\begin{array}{rcl} \text{for all } \boldsymbol{v}_h, q_h \in \boldsymbol{V}_h \times M_h & & \\ a(\boldsymbol{u}_h^n, \boldsymbol{v}_h - \boldsymbol{u}_h^n) + d(\boldsymbol{u}_h^{n-1}, \boldsymbol{u}_h^n, \boldsymbol{v}_h - \boldsymbol{u}_h^n) - b(\boldsymbol{v}_h - \boldsymbol{u}_h^n, p_h^n) & & \\ +J(\boldsymbol{u}_h^{n-1}, \boldsymbol{v}_h) - J(\boldsymbol{u}_h^{n-1}, \boldsymbol{u}_h^n) & \geq & \langle \boldsymbol{f}, \boldsymbol{v}_h - \boldsymbol{u}_h^n \rangle, \\ b(\boldsymbol{u}_h^n, q_h) & = & 0, \end{array}$$

which is equivalent to

$$(83) \qquad \begin{cases} \text{Knowing } \boldsymbol{u}_h^0 \in \boldsymbol{V}_h, \text{ Find } \boldsymbol{u}_h^n \in \boldsymbol{W}_h \text{ such that} \\ a(\boldsymbol{u}_h^n, \boldsymbol{v}_h - \boldsymbol{u}_h^n) + d(\boldsymbol{u}_h^{n-1}, \boldsymbol{u}_h^n, \boldsymbol{v}_h - \boldsymbol{u}_h^n) \\ +J(\boldsymbol{u}_h^{n-1}, \boldsymbol{v}_h) - J(\boldsymbol{u}_h^{n-1}, \boldsymbol{u}_h^n) \geq \langle \boldsymbol{f}, \boldsymbol{v}_h - \boldsymbol{u}_h^n \rangle \text{ for all } \boldsymbol{v}_h \in \boldsymbol{W}_h. \end{cases}$$

About the convergence of the algorithm (82), we claim that

Theorem 4.2. *Assume (32), and (33). Then the problem (82) admits a unique solution $(\boldsymbol{u}_h^n, p_h^n) \in \boldsymbol{K}_h \times M_h$. Moreover let $(\boldsymbol{u}_h, p_h) \in \boldsymbol{V}_h \times M_h$ be the solution of problem (56). Then the iterative solution $(\boldsymbol{u}_h^n, p_h^n)$ converges to $(\boldsymbol{u}_h, p_h)$ in $\boldsymbol{V}_h \times M_h$ as $n \to \infty$. More precisely,*

$$(84) \qquad \|\boldsymbol{u}_h^n - \boldsymbol{u}_h\|_V \leq \left(\frac{C_0 k}{2\nu} + \frac{C_d C_1(\|\boldsymbol{f}\|_{-1} + \|g\|_{L^2(S)})}{2\nu^2} \right)^n \|\boldsymbol{u}_h^0 - \boldsymbol{u}_h\|_V$$

$$(85) \qquad \|p_h^n - p_h\| \leq C(\|\boldsymbol{u}_h^n - \boldsymbol{u}_h\|_V + \|\boldsymbol{u}_h^{n-1} - \boldsymbol{u}_h\|_V) .$$

Proof. We start by proving that $\|\boldsymbol{u}_h^n\|_V \leq \frac{C_1}{\nu}(\|\boldsymbol{f}\|_{-1} + \|g\|_{L^2(S)})$ i.e, $\boldsymbol{u}_h^n \in \boldsymbol{K}_h$. Let $\boldsymbol{w}_h = \boldsymbol{0}$ and $\boldsymbol{w}_h = 2\boldsymbol{u}_h^n$ in (83), since $d(\boldsymbol{u}_h^{n-1}, \boldsymbol{u}_h^n, \boldsymbol{u}_h^n) = 0$, one has

$$\begin{array}{rcl} 2\nu\|\boldsymbol{u}_h^n\|_V^2 \leq \int_S k|\boldsymbol{u}_{\boldsymbol{\tau}h}^{n-1}||\boldsymbol{u}_{\boldsymbol{\tau}h}^n|ds + a(\boldsymbol{u}_h^n, \boldsymbol{u}_h^n) & = & \langle \boldsymbol{f}, \boldsymbol{u}_h^n \rangle - \int_S g|\boldsymbol{u}_{\boldsymbol{\tau}h}^n|ds \\ & \leq & |\langle \boldsymbol{f}, \boldsymbol{u}_h^n \rangle - \int_S g|\boldsymbol{u}_{\boldsymbol{\tau}h}^n|ds| \\ & \leq & C_1(\|\boldsymbol{f}\|_{-1} + \|g\|_{L^2(S)})\|\boldsymbol{u}_h^n\|_V, \end{array}$$

hence

$$\|\boldsymbol{u}_h^n\|_V \leq \frac{C_1}{2\nu}(\|\boldsymbol{f}\|_{-1} + \|g\|_{L^2(S)}) \leq \frac{C_1}{\nu}(\|\boldsymbol{f}\|_{-1} + \|g\|_{L^2(S)}).$$

Next, setting $\boldsymbol{v}_h = \boldsymbol{u}_h^n$ in (57) and $\boldsymbol{v}_h = \boldsymbol{u}_h$ in (83) and adding the resulting equations, we obtain:

$$\begin{array}{l} -a(\boldsymbol{u}_h^n - \boldsymbol{u}_h, \boldsymbol{u}_h^n - \boldsymbol{u}_h) + d(\boldsymbol{u}_h, \boldsymbol{u}_h, \boldsymbol{u}_h^n - \boldsymbol{u}_h) - d(\boldsymbol{u}_h^{n-1}, \boldsymbol{u}_h^n, \boldsymbol{u}_h^n - \boldsymbol{u}_h) \\ \quad +J(\boldsymbol{u}_h^{n-1}, \boldsymbol{u}_h) - J(\boldsymbol{u}_h^{n-1}, \boldsymbol{u}_h^n) + J(\boldsymbol{u}_h, \boldsymbol{u}_h^n) - J(\boldsymbol{u}_h, \boldsymbol{u}_h) \geq 0 \end{array}$$

Note that since $d(\boldsymbol{u}_h, \boldsymbol{u}_h^n - \boldsymbol{u}_h, \boldsymbol{u}_h^n - \boldsymbol{u}_h) = 0$

$$d(\boldsymbol{u}_h, \boldsymbol{u}_h, \boldsymbol{u}_h^n - \boldsymbol{u}_h) - d(\boldsymbol{u}_h^{n-1}, \boldsymbol{u}_h^n, \boldsymbol{u}_h^n - \boldsymbol{u}_h) = -d(\boldsymbol{u}_h^{n-1} - \boldsymbol{u}_h, \boldsymbol{u}_h^n, \boldsymbol{u}_h^n - \boldsymbol{u}_h)$$

thus

$$\begin{array}{rl} & 2\nu\|\boldsymbol{u}_h^n - \boldsymbol{u}_h\|_V^2 \\ \leq & a(\boldsymbol{u}_h^n - \boldsymbol{u}_h, \boldsymbol{u}_h^n - \boldsymbol{u}_h) \\ \leq & -d(\boldsymbol{u}_h^{n-1} - \boldsymbol{u}_h, \boldsymbol{u}_h^n, \boldsymbol{u}_h^n - \boldsymbol{u}_h) \\ + & J(\boldsymbol{u}_h^{n-1}, \boldsymbol{u}_h) - J(\boldsymbol{u}_h^{n-1}, \boldsymbol{u}_h^n) + J(\boldsymbol{u}_h, \boldsymbol{u}_h^n) - J(\boldsymbol{u}_h, \boldsymbol{u}_h) \\ \leq & C_d\|\boldsymbol{u}_h^n\|_V\|\boldsymbol{u}_h^{n-1} - \boldsymbol{u}_h\|_V\|\boldsymbol{u}_h^n - \boldsymbol{u}_h\|_V + C_0 k\|\boldsymbol{u}_h^{n-1} - \boldsymbol{u}_h\|_V\|\boldsymbol{u}_h^n - \boldsymbol{u}_h\|_V \end{array}$$

which together with $\|\boldsymbol{u}_h^n\|_V \leq \frac{C_1}{\nu}(\|\boldsymbol{f}\|_{-1} + \|g\|_{L^2(S)})$, yields

$$\|\boldsymbol{u}_h^n - \boldsymbol{u}_h\|_V \leq \left(\frac{C_0 k}{2\nu} + \frac{C_d C_1(\|\boldsymbol{f}\|_{-1} + \|g\|_{L^2(S)})}{2\nu^2}\right)\|\boldsymbol{u}_h^{n-1} - \boldsymbol{u}_h\|_V,$$

and (84) follows by induction.
For the convergence of p_h^n, let $\boldsymbol{w}_h \in \boldsymbol{W}_h$ and replace $\boldsymbol{v}_h$ in $(82)_1$ successively by $\boldsymbol{u}_h^n + \boldsymbol{w}_h$ and $\boldsymbol{u}_h^n - \boldsymbol{w}_h$. Observing that $J(\boldsymbol{u}_h^{n-1}, \boldsymbol{v}_h) = J(\boldsymbol{u}_h^{n-1}, \boldsymbol{u}_h^n \pm \boldsymbol{w}_h) = J(\boldsymbol{u}_h^{n-1}, \boldsymbol{u}_h^n)$, one has

$$a(\boldsymbol{u}_h^n, \boldsymbol{w}_h) + d(\boldsymbol{u}_h^{n-1}, \boldsymbol{u}_h^n, \boldsymbol{w}_h) - b(\boldsymbol{w}_h, p_h^n) = \langle \boldsymbol{f}, \boldsymbol{w}_h\rangle \text{ for all } \boldsymbol{w}_h \in \boldsymbol{W}_h. \tag{86}$$

Likewise, one has

$$a(\boldsymbol{u}_h, \boldsymbol{w}_h) + d(\boldsymbol{u}_h, \boldsymbol{u}_h, \boldsymbol{w}_h) - b(\boldsymbol{w}_h, p_h) = \langle \boldsymbol{f}, \boldsymbol{w}_h\rangle \text{ for all } \boldsymbol{w}_h \in \boldsymbol{W}_h,$$

which when combined with (86) gives

$$a(\boldsymbol{u}_h^n - \boldsymbol{u}_h, \boldsymbol{w}_h) + d(\boldsymbol{u}_h^{n-1} - \boldsymbol{u}_h, \boldsymbol{u}_h^n, \boldsymbol{w}_h) - d(\boldsymbol{u}_h, \boldsymbol{u}_h^n - \boldsymbol{u}_h, \boldsymbol{w}_h) - b(\boldsymbol{w}_h, p_h^n - p_h) = 0 \text{ for all } \boldsymbol{w}_h \in \boldsymbol{W}_h.$$

That relation together with the discrete inf-sup condition on $b(\cdot,\cdot)$ gives

$$\begin{aligned}
&\beta\|p_h^n - p_h\| \\
\leq\ & \sup_{\boldsymbol{w}_h \in \boldsymbol{V}_{\sigma h}} \frac{b(\boldsymbol{w}_h, p_h^n - p_h)}{\|\boldsymbol{w}_h\|_V} \\
\leq\ & \sup \frac{|a(\boldsymbol{u}_h^n - \boldsymbol{u}_h, \boldsymbol{w}_h)| + d(\boldsymbol{u}_h^{n-1} - \boldsymbol{u}_h, \boldsymbol{u}_h^n, \boldsymbol{w}_h) - d(\boldsymbol{u}_h, \boldsymbol{u}_h^n - \boldsymbol{u}_h, \boldsymbol{w}_h)|}{\|\boldsymbol{w}_h\|_V} \\
\leq\ & 2\nu\|\boldsymbol{u}_h^n - \boldsymbol{u}_h\|_V + C_d\|\boldsymbol{u}_h^n\|_V\|\boldsymbol{u}_h^{n-1} - \boldsymbol{u}_h\|_V + C_d\|\boldsymbol{u}_h\|_V\|\boldsymbol{u}_h^n - \boldsymbol{u}_h\|_V
\end{aligned}$$

therefore since $\boldsymbol{u}_h \in \boldsymbol{K}_h$ and $\boldsymbol{u}_h^n \in \boldsymbol{K}_h$, we claim (85). □

Remark 4.3. (a) *The convergence factor* $\left(\frac{C_0 k}{2\nu} + \frac{C_d C_1(\|\boldsymbol{f}\|_{-1} + \|g\|_{L^2(S)})}{2\nu^2}\right)^n$ *is strictly less than one as one can see from (33).*
(b) *It should be observed that similar condition is obtained in* [26] *for Navier Stokes equations under Dirichlet boundary conditions.*

As in Stokes formulation (39), we construct the following Uzawa iteration algorithm to solve (56) via (82) .
Algorithm 2:

$$\boldsymbol{u}_h^0 \in \boldsymbol{V}_h, \quad \lambda_h^1 \in \Lambda_h \quad \text{arbitrary given} \tag{87}$$

Step 1: knowing $(\boldsymbol{u}_h^{n-1}, \lambda_h^n) \in \boldsymbol{V}_h \times \Lambda_h$, compute $(\boldsymbol{u}_h^n, p_h^n) \in \boldsymbol{V}_h \times M_h$ by
(88)
$$\begin{cases} a(\boldsymbol{u}_h^n, \boldsymbol{v}_h) + d(\boldsymbol{u}_h^{n-1}, \boldsymbol{u}_h^n, \boldsymbol{v}_h) - b(\boldsymbol{v}_h, p_h^n) = \langle \boldsymbol{f}, \boldsymbol{v}_h\rangle - \int_S \lambda_h^n (g + k|\boldsymbol{u}_{\boldsymbol{\tau}h}^{n-1}|)\boldsymbol{v}_{\boldsymbol{\tau}h} ds \text{ for all } \boldsymbol{v}_h \in \boldsymbol{V}_h \\ b(\boldsymbol{u}_h^n, q_h) = 0 \text{ for all } q_h \in M_h, \end{cases}$$

Step 2: Renew $\lambda_h^{n+1} \in \Lambda_h$

$$\lambda_h^{n+1} = P_{\Lambda_h}(\lambda_h^n + \rho(g + k|\boldsymbol{u}_{\boldsymbol{\tau}h}^{n-1}|)\boldsymbol{u}_{\boldsymbol{\tau}h}^n) \tag{89}$$

where $P_{\Lambda_h}(\mu) = \sup(-1, \inf(1, \mu))$ for all $\mu \in L^2(S)$ and $\rho > 0$.

Remark 4.4. *The unique existence of* $(\boldsymbol{u}_h^n, p_h^n) \in \boldsymbol{V}_h \times M_h$ *satisfying (88) is guaranteed by the inf-sup condition (17).*
The initialization of the flow defined by (80) and (88) is important. Let us observe

that since one has well-posedness of (39) and (56), in order to consolidate the convergence of (39) and (56), we suggest the solution of Stokes equations

$$\begin{cases} a(\boldsymbol{u}_h, \boldsymbol{v}_h) - b(\boldsymbol{v}_h, p_h) = (\boldsymbol{f}, \boldsymbol{v}_h) \;\; \forall \boldsymbol{v}_h \in \boldsymbol{V}_h \\ b(\boldsymbol{u}_h, q_h) = 0, \;\; \forall q_h \in M_h \;, \end{cases} \tag{90}$$

as initial condition for our algorithms (80) and (88).

5. Numerical experiments

Let us explain our numerical experiments. We assume $\Omega = (0,1)^2$, the boundary of which consists of two portions Γ and S given by:

$$\Gamma = \{(0,y)/0 < y < 1\} \cup \{(x,0)/0 < x < 1\} \cup \{(1,y)/0 < y < 1\} \tag{91}$$

$$S = \{(x,1)/0 < x < 1\} \tag{92}$$

For the triangulation $\mathcal{T}_h$ of $\overline{\Omega}$, we employ a uniform $N \times N$ mesh, where N denotes the division number of each side of the domain. The implementation is done by extending the Matlab code developed in [27, 28]. In all the examples presented, the velocity and pressure will be approximated by $P2 - P1$ element.
We recall that the different steps of our algorithm are as follows: Choosing the parameter ρ (here we choose $\rho = 0.5$),

(a) Starting with $\boldsymbol{u}_h^0$, solution of (90) and $\lambda_h^1 = 1$
(b) knowing $(\boldsymbol{u}_h^{n-1}, \lambda_h^n)$, compute $(\boldsymbol{u}_h^n, p_h^n, \lambda_h^{n+1})$ solution of (80) or (88).

The stopping criteria for iteration is

$$\|\boldsymbol{u}_h^n - \boldsymbol{u}_h^{n-1}\| \leq 10^{-6}$$

Let us consider

$$\begin{cases} u_1(x,y) = 20x^2(1-x)^2 y(1-2y) \\ u_2(x,y) = -20x(1-x)(1-2x)(1-y)^2 y^2 \\ p(x,y) = (2x-1)(2y-1) \end{cases} \tag{93}$$

5.1. Numerical examples for Stokes problem (1)-(5). $(\boldsymbol{u}, p)$ defined by (93) turns out to be the solution of the problem (1)-(5) under the appropriate choice g where $\nu = 1$ and $\boldsymbol{f}(= \boldsymbol{f}_{stokes})$ is given by

$$\begin{cases} f_1(x,y) = 80x^2(1-x)^2 - 20(2+12x^2-12x)y(1-2y) + 2(2y-1); \\ f_2(x,y) = 20(12x-6)y^2(1-y)^2 + 20x(1-2x)(1-x)(2+12y^2-12y) + 2(2x-1) \end{cases} \tag{94}$$

It is easy to verify that the solution $\boldsymbol{u}$ satisfies $\boldsymbol{u} = \boldsymbol{0}$ on Γ, $\boldsymbol{u} \cdot \boldsymbol{n} = u_2 = 0, \;\; u_1 \neq 0$ on S. By direct computations, we have

$$\begin{aligned} \boldsymbol{\sigma_\tau} &= -60x^2(1-x)^2 \qquad \text{on} \quad S \\ \boldsymbol{u_\tau} &= 20x^2(1-x)^2 \qquad \text{on} \quad S \end{aligned} \tag{95}$$

and

$$\max_S |\boldsymbol{\sigma_\tau}| = 3.75. \tag{96}$$

On the other hand, from the slip boundary conditions (5), we have

$$|\boldsymbol{\sigma_\tau}| \leq g + k|\boldsymbol{u_\tau}| \qquad \text{on} \quad S \tag{97}$$

then we find from (97) that with g constant:

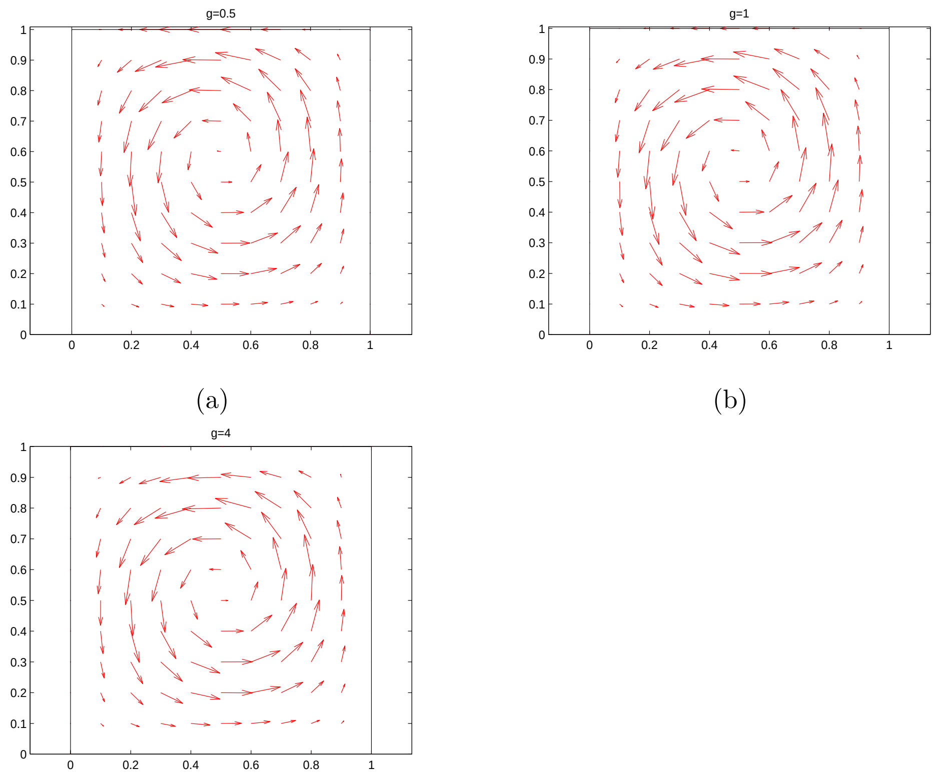
$g + k|\boldsymbol{u_\tau}| \;\; \geq \;\; 3.75 \Rightarrow$ (93) remains a solution.

$g + k|\boldsymbol{u_\tau}| \;\; < \;\; 3.75 \Rightarrow$ (93) is no longer a solution and a non-trivial slip occurs.

Indeed it is observable in figures 1, slip and non-slip condition on the boundary. In fact in figure 1-a and figure 1-b, $g + k|\boldsymbol{u_\tau}| < 3.75$ and we see the manifestation of the slip due to the adherence of the flow at the boundary, whereas in figure 1-c, $g + k|\boldsymbol{u_\tau}| \geq 3.75$ and no slip occurs. In addition, we find that

(a) as the threshold g of tangential stress increases, the more difficult it becomes for a non-trivial slip to occur,
(b) the smaller the threshold g of tangential stress becomes, the more easier it becomes for a non-trivial slip to occur,

which is in agreement with the predicted outcome.
For all the numerical results here, we set $\nu = 1$, $k = 10^{-1}$, $\rho = 0.5$ and g is indicated on the pictures.



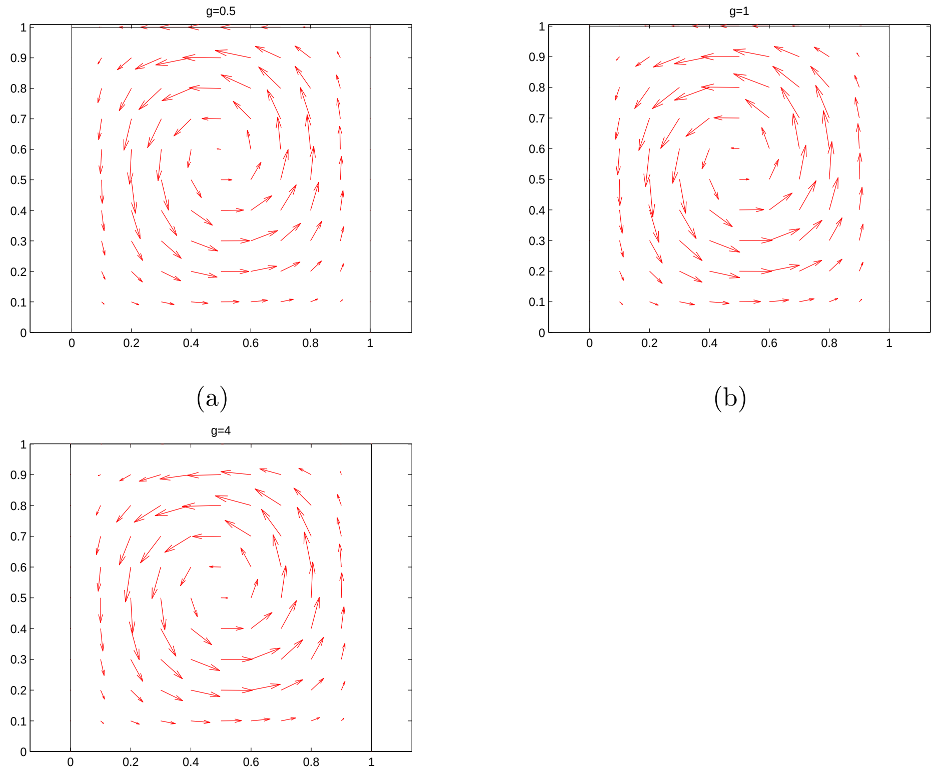
(a) (b) (c)

FIGURE 1. Velocity field respectively for $g = 0.5$, $g = 1$, $g = 4$

5.2. Numerical examples for Navier-Stokes problem (2)–(5),(9). For Navier-Stokes problem, we consider the same solution (93) as in Stokes problem with appropriate choice of g , and $\boldsymbol{f}$ given by

$$\boldsymbol{f} = \boldsymbol{f}_{stokes} + (\boldsymbol{u} \cdot \nabla)\boldsymbol{u}$$

We observe similar pattern as commented for figure 1. In our computations we did not observe a major difference between Stokes and Navier-Stokes as far as the driven cavity is concerned. Of course as it was expected, the simulations with Navier Stokes system is more time involve than the one of Stokes equations.



(a) (b)

(c)

FIGURE 2. Velocity field respectively for $g = 0.5$, $g = 1$, $g = 4$

5.3. Numerical accuracy check. We evaluate the error between approximate solutions and exact ones as the division number N increased. Since we do not know the explicit exact solution when $g = 1$, we employ the approximate solutions with $N = 60$ as the reference solutions $(\boldsymbol{u}_{ref}, p_{ref})$, and we compute the H^1-norm and L^2-norm respectively for velocity and pressure of the difference of the reference solution and the approximate solution $(\boldsymbol{u}_h, p_h)$. The results are presented in Table 1 for Stokes problem and Table 2 for Navier-Stokes problem.

TABLE 1. convergence results for Stokes problem

h	$\|\boldsymbol{u}_{ref} - \boldsymbol{u}_h\|_1$	rate H^1	$\|p - p_h\|$	rate L^2
1/6	1.150e-3		1.308e-2	
1/10	7.814e-4	0.756	7.071e-3	1.204
1/12	6.863e-4	0.711	5.657e-3	1.223
1/15	5.783e-4	0.767	4.243e-3	1.289
1/20	4.790e-4	0.654	2.928e-3	1.289
1/30	6.185e-4	0.630	1.814e-3	1.180

TABLE 2. convergence results for Navier-Stokes problem

h	$\|\boldsymbol{u}_{ref} - \boldsymbol{u}_h\|_1$	rate H^1	$\|p - p_h\|$	rate L^2
1/6	1.103e-2		1.208e-2	
1/10	8.262e-3	0.566	7.171e-3	1.021
1/12	7.499e-3	0.531	5.957e-3	1.017
1/15	6.714e-3	0.495	4.743e-3	1.021
1/20	7.896e-3	0.563	3.528e-3	1.028
1/30	6.446e-3	0.500	2.414e-3	0.935

6. Conclusions

The purpose of this work was to introduce a threshold slip boundary conditions for the numerical analysis of the Stokes and Navier Stokes equations. The resulting variational inequalities obtained are analyzed by the means of fixed approach, and a priori error estimates are derived using sufficient conditions for existence of solutions. Next we have formulated and established the convergence of the Uzawa's algorithm associated to the finite element equations for both the Stokes and Navier Stokes equations. Finally, some numerical simulations which confirm the predictions of the theory presented are shown. We intend to actively continue to work in the research directions presented in this paper.

Acknowledgments

The authors appreciate the contribution of the anonymous reviewers whose extensive and thoughtful suggestions have led to significant improvement of this work.

References

[1] C. Le Roux, Steady Stokes flows with threshold slip boundary conditions, *Mathematical Models and Methods in Applied Sciences*, **15**, 1141–1168, 2005.
[2] C. Le Roux, & A. Tani, Steady solutions of the Navier-Stokes equations with threshold slip boundary conditions, *Mathematical Methods in Applied Sciences*, **30**, 595–624, 2007.
[3] G. Duvaut, & J.L. Lions, *Inequalities in Mechanics and Physics*, Grundlehren der Mathematischen Wissenschaften, Vol. 219, Springer-Verlag, Berlin, 1976.
[4] R. Glowinski, J.L. Lions, & R.Trémolières, *Numerical Analysis of Variational Inequalities*, Studies in Mathematics and its Applications, Vol. 8, North-Holland, Amsterdam, 1981.
[5] D. Kinderlehrer, & G. Stampacchia,*An introduction to variational inequalities and their applications.* Academic Press 1980.
[6] N. Kikuchi, & J.T. Oden, *Contact Problems in Elasticity*, SIAM, Philadelphia, 1988.
[7] M. Sofonea, W. Han, & M. Shillor, *Analysis and approximation of contact problems with adhesion or damage*, Pure and Applied Mathematics, Vol. 276, Chapman & Hall/CRC, Boca Raton, FL, 2006.
[8] M. Shillor, M. Sofonea, & J.J. Telega, *Models and Analysis of Quasistatic Contact:Variational Methods*, Lectures Notes in Physics, **vol** 655, Springer Berlin Heidelberg, 2004.
[9] H. Weimin, & B.D. Reddy, *Plasticity Mathematical theory and numerical analysis*, Second edition. Interdisciplinary Applied Mathematics, 9. Springer, New York, 2013.
[10] J. K. Djoko, *On the time approximation of the Stokes equations with slip boundary conditions of friction type*, to appear in *International Journal of Numerical Analysis and Modeling.*
[11] J. K. Djoko, *Discontinuous Galerkin Finite element discretization for steady Stokes flows with threshold slip boundary conditions*, to appear in *Questiones Mathematicae.*
[12] J. K. Djoko, & M. Mbehou, *Finite element analysis of the stationary power-law Stokes equations driven by friction boundary conditions*, *in review.*
[13] Y. Li, & K. Li, *Penalty finite element method for Stokes problem with nonlinear slip boundary conditions*, *Applied Mathematics and Computation*, **204**, 216–226, 2008.
[14] Y. Li, & K. Li, *Uzawa iteration method for Stokes type variational inequality of the second kind*, *Acta Mathematicae Applicatae Sinica, English Series*, **27**, 303–316, 2011.

[15] Y. Li, & R. An, *Semi discrete stabilized finite element methods for Navier Stokes equations with nonlinear slip boundary conditions based on regularization procedure*, *Numerishe Mathematik*, **117**, 1–36, 2011.
[16] Y. Li, & R. An, *Penalty finite element method for Navier Stokes equations with nonlinear slip boundary conditions. International Journal for Numerical Methods in Fluids*, **69**, 550–566, 2012.
[17] M. Ayadi, K. G. Mohamed, & S. Taoufik, *Mixed formulation for the Stokes problem with Tresca friction. Compts Rendus de L'Academie de Sciences*, Paris, Ser. I, **348**, 1069–1072, 2010.
[18] T. Kashiwabara, *On a finite element approximation of the Stokes system under nonhomoheneous slip type boundary conditions. Japan Journal of Industrial and Applied Mathematics*, **30**, no 1, 227-261, 2013.
[19] H. Fujita, *A mathematical analysis of motions of viscous incompressible fluid under leak or slip boundary conditions*, In: *Mathematical Fluid Mechanics and Modeling (Kyoto, 1994)*, RIMS Kōkyūroko, Vol. 888, Kyoto University, Kyoto, 199–216, 1994.
[20] H. Fujita, & H. Kawarada, *Variational inequalities for the Stokes equation with boundary conditions of friction type*, In: *Recent Developments in Domain Decomposition Methods and Flow Problems (Kyoto, 1996; Anacapri, 1996)*, GAKUTO Internat. Ser. Math. Sci. Appl., Vol. 11, Gakkōtosho, 15–33, 1998.
[21] P. Ciarlet, *Finite Element Method for Elliptic Problems*, North Holland, Amstedam, 1978.
[22] V. Girault, & P.A. Raviart, *Finite Element Approximation of the Navier-Stokes Equations*, Springer-Verlag, New York, 1986.
[23] F. Brezzi, & M. Fortin, *Mixed and Hybrid Finite Element Methods*, Springer Series in Computational Mathematics, **vol** 15, 1991.
[24] H. Brezis, *Functional Analysis, Sobolev Spaces and Partial Differential Equations.* Springer, 2010.
[25] A. Kendall, & H. Weimin,*Theoretical numerical analysis. A functional analysis framework*, Second edition. Texts in Applied Mathematics, 39. Springer, New York, 2005.
[26] Y. He, & J. Li, *Convergence of three iterative methods based on the finite element discretization for the stationary NavierStokes equations*, Computer Methods in Applied Mechanics and Engineering 198.15 (2009): 1351-1359.
[27] J. Alberty, C. Carstensen, S.A. Funken, & R. Klose, *Matlab implementation of the finite element method in elasticity*, *Computing*, **69**, No 3, 239-263, 2002.
[28] J. Koko, *Vectorized Matlab Codes for the Stokes Problem with P1-Bubble/P1 Finite Element.*

Department of Mathematics and Applied Mathematics, University of Pretoria,
Private bag X20, Hatfied 0028, Pretoria, South Africa.
E-mail: `jules.djokokamdem@up.ac.za`

Department of Mathematics and Applied Mathematics, University of Pretoria,
Private bag X20, Hatfied 0028, Pretoria, South Africa.
Department of Mathematics, University of Yaounde I-Cameroon.
E-mail: `mbehoumoh@gmail.com`