Geometric Separation Problems

Ivo Vigan

Dissertation Proposal

Outline

Motivation: Sensor Networks

- Full Coverage: Historically the topic of interested in Sensor Networks

- Barrier Coverage: Recently, sensors used to provide *barriers* as a defense mechanism against intruders at buildings, estates, national borders etc.

General Setup

Given a set of points and a set of separating objects (line segments, disks), select the minimum number of separators such that every path between two points is intersected.

Outline

Separating two Points using Line Segments

Problem (2-Cells-Separation Problem; Alt, Cabello et al. 12)

Given a set of n line segments and two points s and t, select the minimum number of segments one needs to retain so that any s-t path intersects some of the retained segments.

Theorem (Alt, Cabello et al. 12)

Can be computed in time $O(nk + n^2)$,
k being the number of segment intersections.

Separating two Points using Circles

Problem (Circle 2-Cells-Separation Problem)

Given a set of n Circle and two points s and t, select the minimum number of circles one needs to retain so that any s-t path intersects some of the retained segments.

Theorem (Cabello et al., SoCG13)

Can be computed in time $O(nk + n^2 \log n)$,
k being the number of Circle intersections.

Proposed Problem

Generalize this Problem from 2 to k points. Is this problem hard? (Later)

Connecting two Points

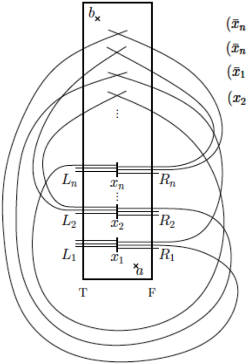
Problem (2-Cells-Connection Problem; Alt, Cabello et al. 12)

Given a set of line segments and two points s and t, compute the minimum number of segments one needs to remove so that there is a path connecting s and t that does not intersect any of the remaining segments.

Theorem (Alt, Cabello et al. 12)

The 2-Cells-Connection Problem is NP*-hard.*

NP-hardness proof outline



$(\bar{x}_n \vee \bar{x}_1)$

$(\bar{x}_n \vee x_2)$

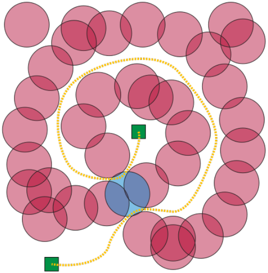
$(\bar{x}_1 \vee \bar{x}_2)$

$(x_2 \vee x_n)$

The 2-Cells-Connection Problem for Disks

Proposed Problem

What is the complexity of the 2-Cells-Connection Problem in the setting of (unit) disks?



- In 2009, Kirkpatrick presents a 3-approximation algorithm for *unit* disks, by observing that no (Euclidean) shortest s-t path that intersects a fixed number of distinct sensors intersects any one sensor more than three times.
- In 2014, Kornman et al: FPT algorithm for fat sensors

Connecting all Points

Problem (All-Cells-Connection Problem; Alt, Cabello et al. 12)

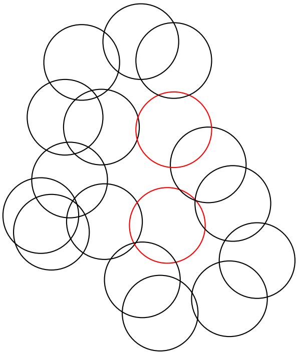
Compute the minimum number of segments one needs to remove so that the arrangement induced by the remaining segments has a single cell.

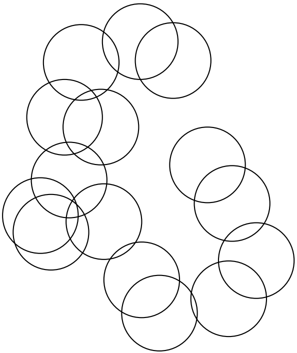
NP-hard by a reduction from the planar feedback vertex set:

All Cell Connection Problem for Disks

Proposed Problem

Given a set of unit disks embedded in the plane, find a minimum cardinality subset s.t. the remaining disks induce an arrangement consisting of only a single cell. Is this problem NP-hard?

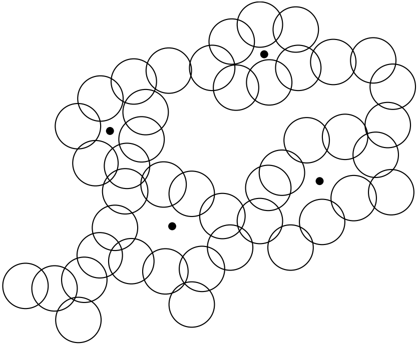




Separating Points using Disks

Problem (Point Isolation Problem [1])

Given a set S of k points in the plane and a collection $\mathcal{D}$ of n unit disks embedded in the plane, no disk contains a point of S.
Find a minimum cardinality subset $\mathcal{D}' \subseteq \mathcal{D}$, s.t. every path between any two points in S is intersected by at least one disk in $\mathcal{D}'$.



[1] Matt Gibson, Gaurav Kanade and Kasturi Varadarajan,
On Isolating Points using Disks ESA'11 2011

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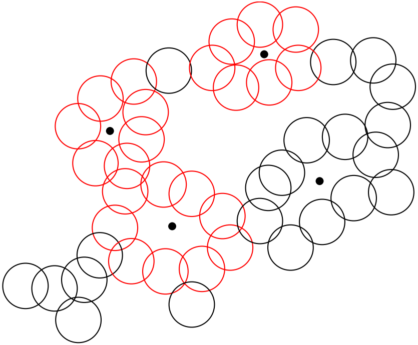

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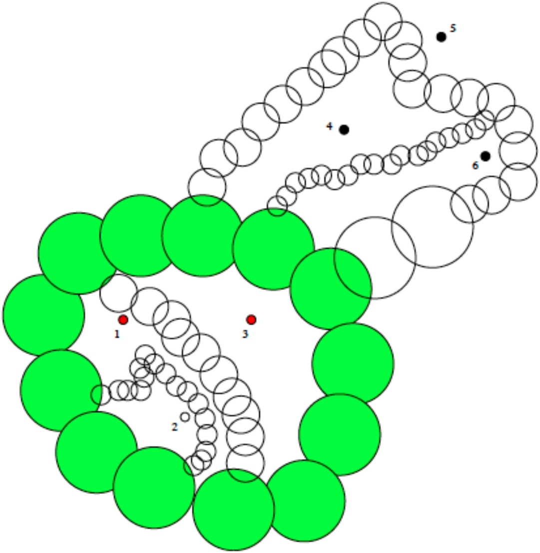
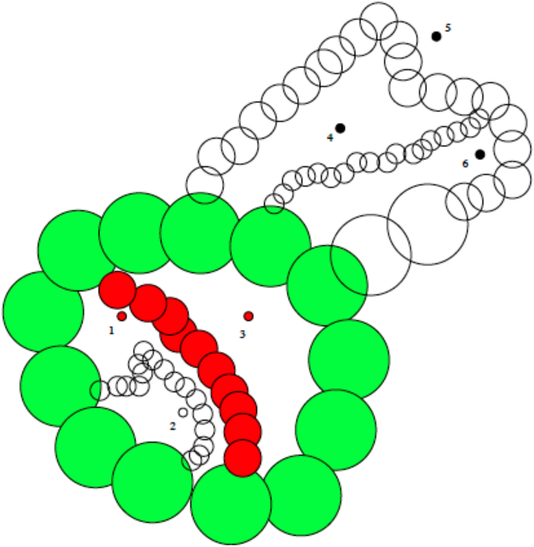
A recursive approximation Algorithm for k-points



A recursive approximation Algorithm for k-points



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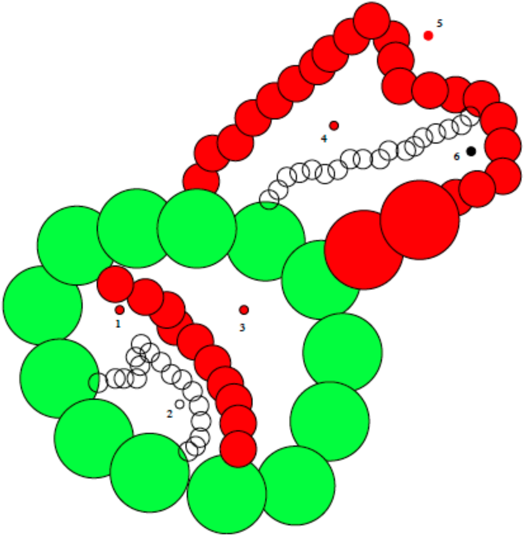

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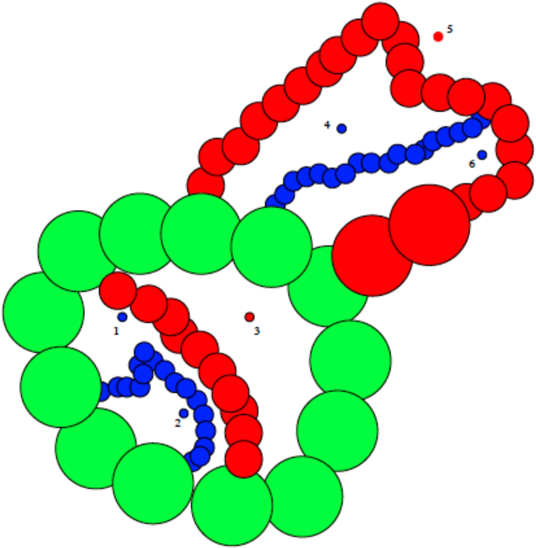


Problem to Investigate

Proposed Problem

What is the approximation ration of this algorithm?

Problem to Investigate

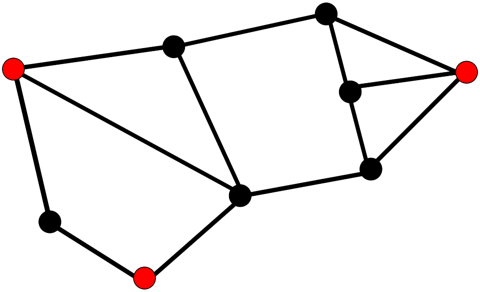
Proposed Problem

What is the computational complexity of the Point Isolation Problem?

Multiterminal Cut Problem

Problem (Multiterminal Cut Problem)

Given a graph $G = (V, E)$ *and a set* $S \subseteq V$ *of* k *terminals, find the minimum cardinality set* $E' \subseteq E$ *such that in* $G' = (V, E \setminus E')$ *there is no path between any two nodes in* S*.*



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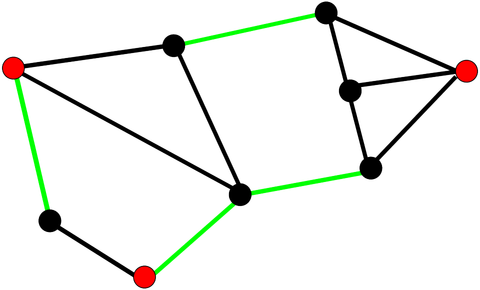


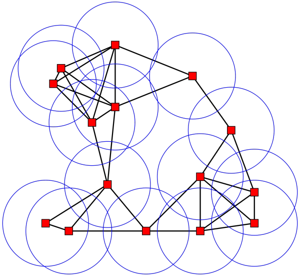
Theorem (Papadimitriou, Seymour, Yannakakis,... 1994)

Planar Multiterminal Cut Problem NP*-complete if k is not fixed.*

Complexity of Multiterminal Cut Problem on Unit Disk Graphs

Proposed Problem

Is the Multiterminal Cut Problem on Unit Disk Graphs NP*-hard on Unit Disk Graphs?*



Outline

Separating Points using Lines

Problem

Given a set of points, build from scratch a minimum cardinality set of Separating lines.

Theorem (Freimer, Mitchell, 91)

Separating Points from scratch using lines is NP*-complete.*

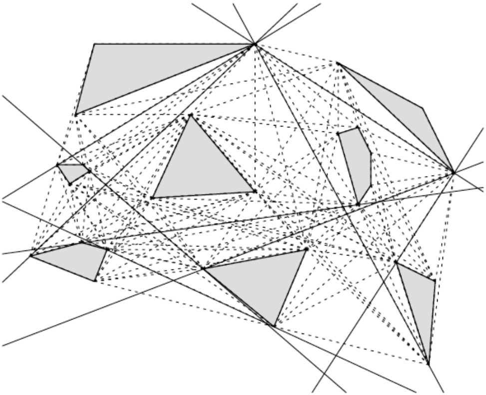
Theorem (Freimer, Mitchell, 91)

Separating Points from scratch using only horizontal and vertical lines NP*-complete.*

Approximating Solution

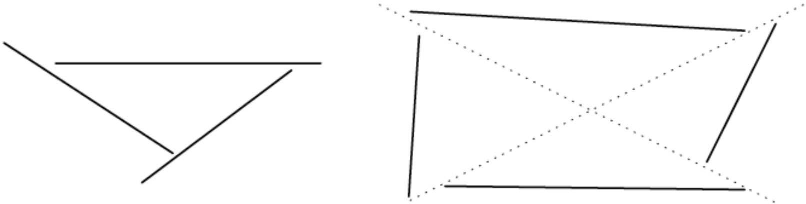
As with all geometric Separating problems, polytime $O(\log n)$-approximable through the set cover lens $(X, \mathcal{R})$:

1. $X =$ edges of visibility graph on S
2. $\mathcal{R} = \{\{\text{edges of } S \text{ stabbed by } h\} | h \text{ a candidate hyperplane } \}$



Optimization is hard, what about deciding if separable?

Freimer and Mitchell present a polynomial time algorithms for testing separability of polygons with lines.

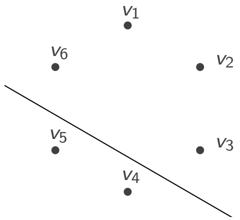


A Combinatorial Problem

Definition (Bipartition)

For a set S of n points in the plane, a *bipartition* P of S is a set $\{U, S \setminus U\}$ consisting of two disjoint nonempty subsets of S which respectively are fully contained in the two open half-planes bounded by some line.

Example



$$S = \{v_1, v_2, v_3, v_4, v_5, v_6\} \subseteq \mathbb{R}^2$$

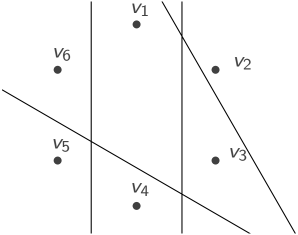
$$P = \{\{v_1, v_2, v_3, v_6\}, \{v_4, v_5\}\}$$

Separating Family

Definition (Separating Family)

A set $\mathcal{P}$ of bipartitions is called a *separating family* for S if for every distinct $p, q \in S$ there is a $P = \{U, S \setminus U\}$ in $\mathcal{P}$, s.t. $p \in U$ and $q \in S \setminus U$.

Example



$\{\{v_1, v_2, v_3, v_4\}, \{v_5, v_6\}\}$

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$\{\{v_1, v_3, v_4, v_5, v_6\}, \{v_2\}\}$

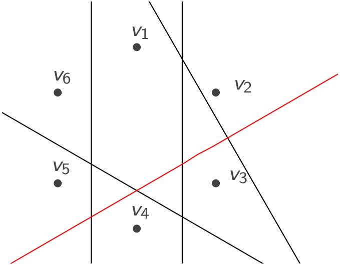
$\{\{v_1, v_4, v_5, v_6\}, \{v_2, v_3\}\}$

Minimal Separating Family

Definition

A separating family $\mathcal{P}$ for S is called *minimal*, if no proper subset of $\mathcal{P}$ is a separating family for S.

Example



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$\{\{v_1, v_2, v_5, v_6\}, \{v_3, v_4\}\}$

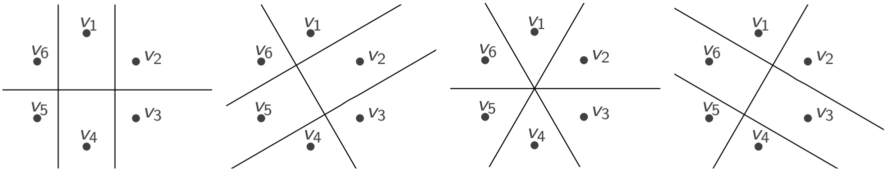
Planed Research

Proposed Problem

Given n convex points in the plane, how many minimal separating families of size k exist?"

Example

How many minimal separating families of size 3 exist for 6 points in convex position?



Previous Results on Sets [1]

Definition (Bipartitions)

A *bipartition* of a set S is $\{S\}$ or an unordered pair $\{U, V\}$ of nonempty subsets of S such that $U \cap V = \emptyset$ and $U \cup V = S$.

Definition (Separating Family)

A collection of bipartitions of S is a *separating family* for S if every two elements in S are separated by some bipartition in the collection, that is, they are contained in the different components of some bipartition.

[1] Joint work with T. Toda, Discrete Mathematics, 313 (3), 2013

Separating Family Example

Example

Let $S = \{1, 2, 3, 4\}$, $\{P_1, P_2\}, \{Q_1, Q_2, Q_3\}$

$$P_1 = \{\{1, 2\}, \{3, 4\}\}, \qquad Q_1 = \{\{1\}, \{2, 3, 4\}\},$$
$$P_2 = \{\{1, 3\}, \{2, 4\}\}, \qquad Q_2 = \{\{1, 2\}, \{3, 4\}\},$$
$$Q_3 = \{\{1, 2, 3\}, \{4\}\}.$$

are two separating families.

Main Result of [1]

Theorem

The number $\tau_{n,k}$ of separating families of size k for an n-element set with $2 \leq n$ and $1 \leq k \leq 2^{n-1}$ is

$$\tau_{n,k} = \frac{(n-1)!}{k!} \sum_{i=1}^{k} (-1)^{k-i} \begin{bmatrix} k \\ i \end{bmatrix} \binom{2^i - 1}{n-1}.$$

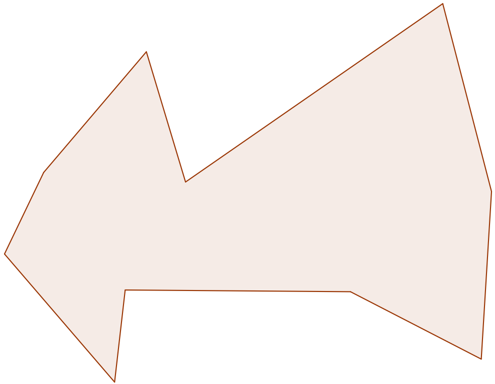
with $\begin{bmatrix} k \\ i \end{bmatrix}$, being the Stirling number of the first kind, which count the number of permutations of k elements with i disjoint cycles.

Outline

Problem Setting

Proposed Problem

Given a region, bounded by a piecewise linear closed border, such as a fence, place few guards inside the fenced region, such that wherever an intruder cuts through the fence, the closest guard is at most a distance one away.



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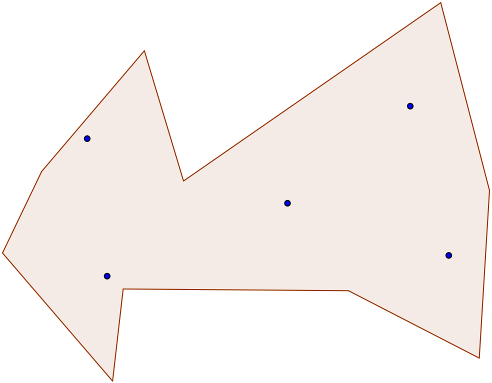
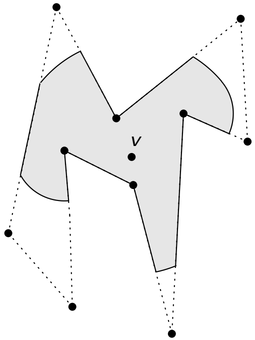


Geodesic Unit Disk

Definition

A *geodesic unit disk* centered at a point v in a polygon P is the set of all points in P whose shortest path distance to v is at most 1.



Problem (Boundary Coverage)

Given a simple polygon, cover its boundary with the minimum number of Geodesic Unit Disks.

Computational Complexity

Theorem (I.V.)

Boundary Coverage is NP*-hard in polygons with holes.*

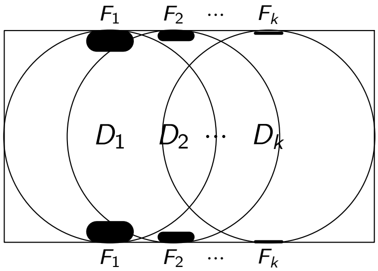
Proposed Problem

Is the Boundary Coverage Problem NP*-hard in simple polygons?*

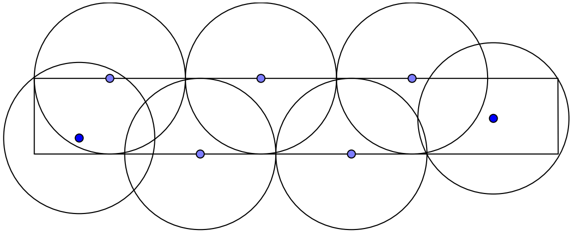
Proposed Problem

Come up with a constant factor approximation algorithm for the Boundary Coverage Problem.

Greedily cover the longest uncovered boundary portion $\Rightarrow \Omega(\log n)$ approximation



Contiguously cover the longest uncovered boundary portion $\Rightarrow \geq 2$ approximation



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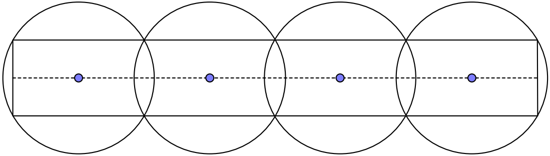


Large Perimeter Boundary

Proposed Problem (Efrat)

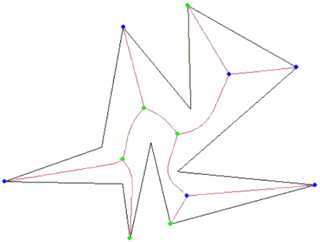
If the perimeter is much larger than n, can we obtain a better approximation ratio for the Boundary Coverage Problem?

Covering Long Perimeter Polygons (Preliminary Result)

Theorem

If the polygon perimeter L is at least $n^{1+\delta}$, with $\delta > 0$, a simple linear time algorithm achieves an approximation ratio which goes to one as L/n goes to infinity.

Idea: Cover long corridors of polygon almost optimally.



Covering with Euclidean Disks

Problem (Euclidean Boundary Coverage)

Given a polygon, possibly with holes, cover its boundary using the minimum number of Euclidean Unit disks.

Proposed Problem (Line Segment Coverage)

Given a set of line segments cover them using the minimum number of Euclidean Unit disks.

Preliminary Result:

Theorem (Mark de Berg, I.V.)

One can compute in $O(n)$ time a 6-approximation for the number of Euclidean unit disk needed to cover a set of n line segments.

Related Work:
Gupta et al. 2010, present a PTAS for intersecting axis aligned segments by disks.

Thank You