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Remarks on the Maximum Principle for Parabolic-Type PDEs

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Abstract

This paper addresses the maximum principle for uniform, parabolic second-order linear differential operators. Specifically, if $u \in C^2(D)$ satisfies the parabolic, uniform differential inequality $L[u] \geq 0$ in the cylinder $D = \Omega \times [a, b] \subset \mathbb{R}^{N+1}$ and there exists $(x_0, t_0) \in D$, such that $u(x_0, t_0) \geq 0$ and $M = u(x_0, t_0) \geq u(x, t)$ for all $(x, t) \in D$ then $u(x, t) \equiv u(x_0, t_0) = M =$ a constant for all (x, t) in the region D bounded by $\Omega \times [a, t_0]$, where $\Omega \subset \mathbb{R}^N$ bounded domain.

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1 Introduction

In elementary calculus courses, it is shown that if a function $u \in C^2(\Omega)$ satisfies inequality $u^{''} > 0$ in an interval $\Omega = [a, b]$, then u attains its maximum at one of the interval's ends. The principle of the maximum represents a generalization of such a fact. In particular, as shown in [2] harmonic functions in a domain $\Omega \subset \mathbb{R}^N$ cannot attain a maximum within Ω unless they are constant, that is, if $\Delta u \geq 0$ in a domain Ω and there exists $x_0 \in \Omega$ such that $u(x_0) = M = \max_{\overline{\Omega}} u$. Then $u \equiv u(x_0) = M=$ a constant in Ω. The PDE that models the heat flow over a thin rod, of length l, made of a homogeneous material is

$L[u] \equiv u_{xx} - u_t = f(x,t)$, where $u = u(x,t)$ represents the rod's temperature at point x and instant t, and f is the heat dissipation rate of the rod. The maximum principle for the heat equation asserts that if $u \in C^1([0,l] \times [0,T])$, $l > 0, T > 0$ and there exist partial derivatives u_t, u_x, u_{xx}, and such derivatives are continuous in the rectangle $R = [0,l] \times [0,T]$, then the maximum value of u over clausure $\bar{R}$ should occur on one of the three sides B, S_1, S_2, where $B = \{(x,0) : 0 \leq x \leq l\}$, $S_1 = \{(0,t) : 0 \leq t \leq T\}$, $S_2 = \{(l,t) : 0 \leq t \leq T\}$ under the condition that $L[u] \equiv u_{xx} - u_t \geq 0$ in $(0,l) \times (0,T)$. Generally, u cannot attain a local maximum if $L[u] \equiv \Delta u - u_t > 0$ because if u has a local maximum at an interior point within the cylinder D then, at such a point $\Delta u \leq 0$ and $u_t = 0$. If $L[u] \equiv \Delta u - u_t \geq 0$ the maximum principle asserts that the maximum of u within the clausure of $D \cup \partial D$ should occur on the boundary of the cylinder D, that is, on $\Omega \times \{0\}$ or else $\partial\Omega \times [0,T]$. In [2] it is shown that if $\Omega \subset \mathbb{R}^N$, $u \in C^2(\Omega) \cap C(\overline{\Omega})$ satisfies $L[u](x) \equiv \sum_{i=1}^N \sum_{j=1}^N a_{ij}(x) u_{x_i x_j} + \sum_{i=1}^N b_i(x) u_{x_i} + c(x)u \geq 0$ in a domain Ω where L is uniformly elliptical, the coefficients of L are uniformly bounded continuous functions, $c(x,t) \leq 0$ and $a_{ij}(x) \equiv a_{ji}(x)$ in Ω for all $1 \leq i,j \leq N$, and there exists $x_0 \in \Omega$, such that $0 < M = u(x_0) \geq u(x)$ for all $x \in \Omega$, then $u(x) \equiv u(x_0) = M=$ a constant in Ω. Hereinafter, Ω represents a bounded domain of $\mathbb{R}^N$, for N being a positive integer, $-\infty \leq a < b \leq \infty$, $I = (a,b)$, $D = \Omega \times I$. The second order differential operator given by

$$L[u] \equiv \left(\sum_{i=1}^N \sum_{j=1}^N a_{ij}(x,t) \frac{\partial^2 u}{\partial x_i x_j} + \sum_{i=1}^N b_i(x,t) \frac{\partial u}{\partial x_i} + c(x,t)u \right) - \frac{\partial u}{\partial t} \quad (1)$$

is considered for all $(x,t) \in D$ as well as for all $u \in C^2(D)$, where the coefficients of (1) are functions defined in the cylinder D, for $1 \leq i,j \leq N$. The operator (1) is called parabolic in $(x,t) = (x_1, x_2, \ldots, x_N, t)$ if, for every t fixed, the operator consisting of the first terms of the sum is elliptic in (x,t), this is to say that (1) is parabolic if there exists a constant $m > 0$ with the following property: for all column vectors $\xi = (\xi_1, \xi_2, \ldots, \xi_N)^T \neq 0$ the following inequality holds:

$$\sum_{i=1}^N \sum_{j=1}^N a_{ij}(x,t) \xi_i \xi_j \geq m \sum_{i=1}^N \xi_i^2. \quad (2)$$

The operator (1) is uniformly parabolic in D (see [4]) if (2) holds for the same $m > 0$ for all $(x,t) \in D = \bar{\Omega} \times (a,b)$. Here it is assumed that the operator (1) is uniformly parabolic in D, the coefficients of (1) are continuous functions in D, $c(x,t) \leq 0$ y $a_{ij}(x,t) \equiv a_{ji}(x,t)$ within D for all $1 \leq i,j \leq N$.

The following section shows the maximum principle for the parabolic case, analogous to the elliptical case [2]. The proof presented herein makes use of a

slight variation with respect to that presented in [4] and also to the method used by Hopf applied to elliptical-type operators (see [2])
Before listing and proving the maximum principle, an approximation theorem for such a principle is presented. This theorem is also known as the strong maximum principle (see [2] and [4])

Theorem 1 *Suppose* $u \in C^{2,1}(D)$, $L[u] \geq 0$ *in* D, *where* L *is the operator defined in* (1) *and there exists* $(x_0, t_0) \in D$ *such that* $u(x_0, t_0) = M \geq u(x, t)$ *for all* $(x, t) \in D$ *and* $u(x_0, t_0) > 0$ *then* $u \equiv u(x_0, t_0) = M$ = *a constant in* $\Omega \times [a, t_0]$

In proving this theorem, four lemmas will be used. The first Lemma asserts that a function $u \in C^2(D)$ that satisfies the strict differential inequality $L[u] > 0$ in a domain D cannot attain a positive maximum in D.

Lemma 1 *If* $L[u] > 0$ *in* D *then* u *cannot attain a positive maximum in* D

Proof. Suppose that u attains a positive maximum in $P_0 = (x_0, t_0) \in D$. In this case, by using an appropriate change of variable (see [2]), it can be proved that

$$\sum_{i=1}^{N}\sum_{j=1}^{N} a_{ij}(x_0, t_0)\frac{\partial^2 u}{\partial x_i x_j}(x_0, t_0) \leq 0 \tag{3}$$

Furthermore, since $u(P_0)$ is a maximum in D, the first partial derivatives of u in (x_0, t_0) are zero, that is

$$\frac{\partial u}{\partial x_i}(x_0, t_0) = 0, \quad \frac{\partial u}{\partial t}(x_0, t_0) = 0 \tag{4}$$

$u(x_0, t_0) > 0$ and $c(x_0, t_0) \leq 0$ and due to (3) (4) then

$$L[u](x_0, t_0) = \sum_{i=1}^{N}\sum_{j=1}^{N} a_{ij}(x_0, t_0)\frac{\partial^2 u}{\partial x_i x_j}(x_0, t_0) + c(x_0, t_0)u(x_0, t_0) \leq 0. \tag{5}$$

This contradiction proves the Lemma. ■
The following Lemma asserts that if the maximum of u in D is $M > 0$ and $u < M$ within an appropriate ellipsoid $E \subset D$, and $u = M$ at a unique point on the boundary of E then the hyper-plane tangent to E, at such a point, is parallel the the x axis, that is, from a geometrical view-point, the maximum of u should be located at either the north pole or the south pole of E.

Lemma 2 *If* $u \in C^{2,1}(D)$, $L[u] \geq 0$ *in* D, u *has a positive maximum* M *in* D, $(x_1^*, x_2^*, \ldots, x_N^*, t^*) \in D$, *and there exist* $\lambda_i > 0$, $i = 0, 1, \ldots, N$, $R > 0$ *such that, the solid ellipsoid given by* $E = \{(x_1, x_2, \ldots, x_N, t) : \sum_{i=0}^{N} \lambda_i (x_i - x_i^*)^2 + \lambda_0 (t - t^*)^2 \leq R^2\} \subset D$, $u < M$ *within* E, *and there exists* $(\bar{x}, \bar{t}) \in \partial E$ *such that* $u(\bar{x}, \bar{t}) = M$, *then* $x^* = \bar{x}$.

Proof. Suppose that $\bar{x} \neq x^*$. It can be assumed that $\bar{P} = (\bar{x}, \bar{t})$ is the unique point in ∂E such that $u(\bar{P}) = M$. Let us construct $0 < r < |\bar{x} - x^*|$ such that the closed ball $\overline{B_r}(\bar{P}) \subset D$. Let $\bar{C} = \partial B_r(\bar{P})$, $C_1 = \bar{C} \cap E$, $C_2 = \bar{C} - C_1$. Since C_1 is a compact set, there exist $\delta > 0$ such that $u < M - \delta$ in C_1. Let $h(x,t)$ be the following function $h(x,t) = \exp\left(-\alpha\left[\sum_{i=0}^N \lambda_i(x_i - x_i^*)^2 + \lambda_0(t - t^*)^2\right]\right) - \exp\{-\alpha R^2\}$, where $\alpha > 0$ is a constant that will be determined as convenient. By definition $h > 0$ within E, $h = 0$ in ∂E, $h < 0$ outside E. Using a simple calculation it is known that:

$$\begin{aligned} L[h](x,t) &= \left(\sum_{i=1}^N \sum_{j=1}^N a_{ij}(x,t)\frac{\partial^2 h}{\partial x_i x_j} + \sum_{i=1}^N b_i(x,t)\frac{\partial h}{\partial x_i} + c(x,t)h\right) - \frac{\partial h}{\partial t} \\ &= \exp\left(-\alpha\left[\sum_{i=0}^N \lambda_i(x_i - x_i^*)^2 + \lambda_0(t-t^*)^2\right]\right)\{4\alpha^2 \sum_{i=1}^N\sum_{j=1}^N a_{ij}\lambda_i\lambda_j(x_i - x_i^*)(x_j - x_j^*) \\ &\quad - 2\alpha\left[\sum_{i=1}^N a_{ii}\lambda_i + \sum_{i=1}^N b_i\lambda_i(x_i - x_i^*) - \lambda_0(t-t^*)\right] + c\} - c\exp(-\alpha R^2) \end{aligned} \quad (6)$$

Since L is uniformly parabolic in D, by (2), there exists $m > 0$ such that

$$\sum_{i=1}^N\sum_{j=1}^N a_{ij}\lambda_i\lambda_j(x_i - x_i^*)(x_j - x_j^*) \geq m\sum_{i=1}^N \lambda_i^2(x_i - x_i^*)^2 \quad (7)$$

Also, for $(x,t) \in \overline{B_r(\bar{P})}$, it can be stated that $|x^* - \bar{x}| \leq |x - \bar{x}| + |x - x^*|$, then $|x - x^*| \geq |x^* - \bar{x}| - \underline{|x - \bar{x}|} > |x^* - \bar{x}| - r > 0$, so for sufficiently large α, $L[h] > 0$ for all $(x,t) \in \overline{B_r(\bar{P})}$. Let us define function $v(x,t) = u(x,t) + \epsilon h(x,t)$ in $\overline{B_r(\bar{P})}$, where $\epsilon > 0$. Since $u < M - \delta$ for $(x,t) \in C_1$, $\epsilon > 0$ can be chosen sufficiently small $\epsilon < \delta/h$ such that $v < M$ in C_1. Also, since $h < 0$ and $u \leq M$ in C_2, then $v < M$ in C_2 and therefore $v < M$ over the entire bound $\bar{C} = \partial B_r(\bar{P})$; furthermore, $v(\bar{P}) = u(\bar{P}) + \epsilon h(\bar{P}) = M$. Due to the continuity of v and the compactness of $\bar{B}$ then v has a positive maximum within $B_r(\bar{P})$; additionally, $L[v] = L[u] + \epsilon L[h] > 0$ in $B_r(\bar{P})$. This fact contradicts the Lemma 1 ■

The following shows that if $L[u] \geq 0$ in D and u has a positive maximum M at point $(x_0, t_0) \in D$ then $u \equiv M$ on the copy of Ω on the hyper-plane $t = t_0$ denoted as $\Omega \times \{t_0\}$, where $a \leq t_0 \leq b$.

Lemma 3 *If $L[u] \geq 0$ in D and u has a positive maximum at point $P_0 = (x_0, t_0) \in D$, then $u \equiv u(P_0) = M$ = a constant in $\Omega \times \{t_0\}$*

Proof. Suppose $u(P_0) = M > 0$ and there exists a point $P_1 = (x_1, t_0) \in \Omega \times \{t_0\}$ such that $u(P_1) < u(P_0) = M$. Since Ω is a connected set, there exists

a continuous function $\gamma : [0,1] \to \Omega \times \{t_0\}$ such that $\gamma(0) = P_1$, $\gamma(1) = P_0$. Let $G = \{s \in [0,1] : u(\gamma(s)) < M\}$; since G is bounded, there exists $s^* \in (0,1)$ such that $s^* = \sup G$, $P^* = \gamma(s^*)$, that is $P^* = (x^*, t_0) \in \gamma([0,1])$ such that $u(P^*) = u(P_0) = M$, $u(\gamma(s)) < u(P_0)$ for all $0 \leq s < s^*$.
Over $\gamma([0,1])$ let us take a point $\bar{P} = (\bar{x}, t_0)$ between P_1 and P^* such that $dist(\bar{P}, P^*) < d/2$, where $0 < d < dist(\gamma([0,1]), \partial\Omega \times t_0)$. Since $u(\bar{P}) < u(P^*)$, there exists $\epsilon > 0$ such that $u(P) < u(P^*)$ for all P within the segment $\sigma = \{\bar{x}\} \times [t_0 - \epsilon, t_0 + \epsilon]$. Now consider the family of ellipsoids E_λ: $|x - \bar{x}|^2 + \lambda(t - t_0)^2 \leq \lambda\epsilon^2$. It can be easily observed that the ends of σ are located over E_λ and that E_λ approximates σ as λ tends to zero. As a result, there exists $\lambda = \tilde{\lambda} > 0$ with the following properties: $u(P) < u(P_0)$ for all P within $E_{\tilde{\lambda}}$ and there exists a point $Q = (y, t) \in \partial E_{\tilde{\lambda}}$ such that $u(Q) = u(P_0)$. Since $u(P) < u(P^*)$ for all $P \in \sigma$, Q is not in σ, that is $y \neq \bar{x}$. ■
The following Lemma asserts that if $L[u] \geq 0$ in D and there exists (x_0, t_0) such that $u(x_0, t_0) = M \geq u(x, t)$ for all (x, t) in a closed rectangle $S \in D$, $(x_0, t_0) \in S$ and $M > 0$, then $u \equiv u(x_0, t_0) = M =$ a constant in S

Lemma 4 *Suppose* $(x_0, t_0) = (x_{01}, x_{02}, \ldots, x_{0N}, t_0) \in D$, $a_0, a_1, \ldots, a_N$ *are positive, real numbers and the rectangle* $S = \{(x_1, x_2, \ldots, x_N, t) \in \mathbb{R}^{N+1} : x_{0i} - a_i \leq x_i \leq x_{0i} + a_i, t_0 - a_0 \leq t \leq t_0 + a_0, i = 1, 2, \cdots, N\} \subset D$. *Si* $L[u] \geq 0$ *in* D *and* $u(x_0, t_0) = M \geq u(P)$ *for all* $P \in S$, *then* $u(x,t) \equiv u(x_0, t_0) = M =$ *a constant for all* $(x, t) \in S$ *where* $t_0 - a_0 \leq t \leq t_0 + a_0$, *under the condition that* $u(x_0, t_0) = M > 0$.

Proof. Suppose there is $Q = (x^*, t^*) \in S$ such that $u(Q) < u(P_0)$, then it is possible to assume that $t^* < t_0$. Over the segment γ that joins points Q and P_0 there exists a point $P_1 = (x_1, t_1)$ such that $u(P_1) = u(P_0)$ and $u(P) < u(P_0)$ for every point P over the segment γ between Q and P_1, $x_1 = (x_{11}, x_{21}, \ldots, x_{N1})$. Le us suppose that $P_1 = P_0$, $t^* = t_1 - \tilde{a_1}$, for some real number $\tilde{a_1} > 0$. Since P_1 lies within S, there exist N real numbers $b_1 > 0, b_2 > 0, \ldots, b_N$, such that the rectangle $S_1 = \{(x_1, x_2, \ldots x_N, t) \in \mathbb{R}^{N+1} : x_{i1} - b_i \leq x_i \leq x_{i1} + b_i, t_1 - \tilde{a_1} \leq t \leq t_1, i = 1, 2, \cdots, N\} \subset S$. By Lemma 3, if $P = (x, t) \in S_1$ y $t < t_1$, then $u < u(P_1)$ in $\Omega \times \{t\}$.
Consider the following function: $h(x, t) = (t_1 - t) - k|x - x_1|^2$, $k > 0$. Then it can be assumed that the paraboloid given by $M = \{(x, t) \in \mathbb{R}^{N+1} : t_1 - t = k|x - x_1|^2\} = \{(x_1, x_2, \ldots x_N, t) : -\sum_{i=1}^{N} k[(x_i - x_{i1})^2 + (t - t_1)] = 0\}$ intersects the set $(\Omega \times \{t^*\}) \cap S_1$. By the definition of function h, $h \equiv 0$ in M, $h < 0$ in the set above the paraboloid M and $h > 0$ in the subset bellow M. Furthermore,

$$L[h](x,t) = 1 - 2k\sum_{i=1}^{N} a_{ii} - 2k\sum_{i=1}^{N} b_i(x_i - x_{i1}) + c(x,t)[(t_1 - t) - k|x - x_1|^2] > 0$$

in S_1 if the dimensions of S_1 are allowed to be sufficiently small, and $k > 0$ is such that $4k\sum_{i=1}^{N} a_{ii} < 1$ in S_1. Let R' be the set bounded by M and

$\Omega \times \{t^*\}$ and with $B' = \partial R' \cap M$, $B'' = \partial R' - B'$. In B'', $u < u(P_0) - \delta$ for $\delta > 0$ sufficiently small, then there exists $\epsilon > 0$ such that $v \equiv u + \epsilon h < 0$ in B'', also $v \equiv u$ in $B' - \{P_1\}$ and $v(P_1) = u(P_1)$. Since $L[h] > 0$ in S_1, $L[v] = L[u] + \epsilon L[h] > 0$ in S_1. From this last inequality and by Lemma 1 then: $\max_{\overline{R'}} v = \max_{\partial R'} v = v(P_1) = u(P_1) = u(P_0)$, it can be concluded that $0 \leq \frac{\partial v}{\partial t}(P_1) = \frac{\partial u}{\partial t}(P_1) + \epsilon \frac{\partial h}{\partial t}(P_1) = -\epsilon + \frac{\partial u}{\partial t}(P_1)$, $\frac{\partial u}{\partial t}(P_1) \geq \epsilon > 0$. Since $u(P_1)$ is maximum in S, it is true that $\frac{\partial u}{\partial t}(P_1) = 0$. This contradiction proves the Lemma. ■

Proof Theorem. Suppose there exists $\bar{P} = (\bar{x}, \bar{t}) \in D$ such that $u(\bar{P}) < u(P_0)$, $\bar{t} \leq t_0$, by Lemma 3, $\bar{t} < t_0$. Since D is a connected set, there exists a continuous function $\gamma : [0, 1] \to D$ such that $\gamma(0) = \bar{P}$ and $\gamma(1) = P_0$, and exists $P_1 = (x_{11}, x_{21}, \ldots, x_{N1}) = \gamma(s^*) \in \gamma([0, 1])$ such that $u(P_1) = u(P_0)$ and $u(\gamma(s)) < u(P_0)$ for all $0 \leq s < s^*$. Since D is an open set, there exists $a > 0$ such that the rectangle $S = \{(x_1, x_2, \ldots x_N, t) \in \mathbb{R}^{N+1} : x_{i1} - a \leq x_i \leq x_{i1} + a, t_1 - a \leq t \leq t_1 + a, i = 1, 2, \ldots, N\} \in D$. By Lemma 4 $u = u(P_0)$ in $S^* = \{(x, t) \in S : t \leq t_1\}$ and $S^* \cap \gamma([0, s^*])$ is non-empty. This fact leads to a contradiction. ■

Theorem 2 *Suppose $u \in C^{2,1}(D)$, $L[u] \geq 0$ in D. If u attains a non-negative maximum at point $P_0 = (x_0, t_0) \in D$, then u is constant in $\Omega \times [a, t_0]$*

The proof is exactly the same to that of Theorem 1

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